LECTURE NOTES ON GENERALIZED HEEGAARD SPLITTINGS

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1. Introduction

These notes grew out of a lecture series given at RIMS in the summer of 2001. The authors were visiting RIMS in conjunction with the Research Project on Low-Dimensional Topology in the Twenty-First Century. They had been invited by Professor Tsuyoshi Kobayashi. The lecture series was first suggested by Professor Hitoshi Murakami.

The lecture series was aimed at a broad audience that included many graduate students. Its purpose lay in familiarizing the audience with the basics of 3-manifold theory and introducing some topics of current research. The first portion of the lecture series was devoted to standard topics in the theory of 3-manifolds. The middle portion was devoted to a brief study of Heegaaard splittings and generalized Heegaard splittings. The latter portion touched on a brand new topic: fork complexes.

During this time Professor Tsuyoshi Kobayashi had raised some interesting questions about the connectivity properties of generalized Heegaard splittings. The latter portion of the lecture series was motivated by these questions. And fork complexes were invented in an effort to illuminate some of the more subtle issues arising in the study of generalized Heegaard splittings.

In the standard schematic diagram for generalized Heegaard splittings, Heegaard splittings are stacked on top of each other in a linear fashion. See Figure 1. This can cause confusion in those cases in which generalized Heegaaard splittings possess interesting connectivity properties. In these cases, some of the topological features of the 3-manifold are captured by the connectivity properties of the generalized Heegaard splitting rather than by the Heegaard splittings of submanifolds into which the generalized Heegaard splitting decomposes the 3-manifold. See Figure 2. Fork complexes provide a means of description in this context.

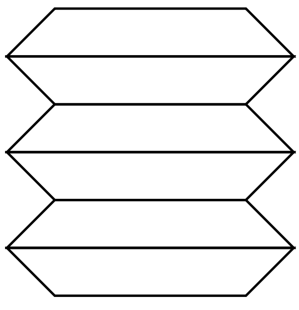

Figure 1. *The standard schematic diagram*

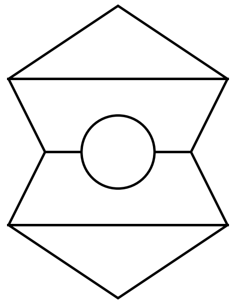

FIGURE 2. *A more informative schematic diagram for a generalized Heegaard splitting for a manifold homeomorphic to* (*a surface*) $\times S^1$

The authors would like to express their appreciation of the hospitality extended to them during their stay at RIMS. They would also like to thank the many people that made their stay at RIMS delightful, illuminating and productive, most notably Professor Hitoshi Murakami, Professor Tsuyoshi Kobayashi, Professor Jun Murakami, Professor Tomotada Ohtsuki, Professor Kyoji Saito, Professor Makoto Sakuma, Professor Kouki Taniyama and Dr. Yo'av Rieck. Finally, they would like to thank Dr. Ryosuke Yamamoto for drawing the fine pictures in these lecture notes.

2. Preliminaries

2.1. **PL 3-manifolds.** Let M be a PL 3-manifold, i.e., M is a union of 3-simplices σ_i^3 $(i = 1, 2, \ldots, t)$ such that $\sigma_i^3 \cap \sigma_j^3$ $(i \neq j)$ is emptyset, a vertex, an edge or a face and that for each vertex v, $\bigcup_{v \in \sigma_j^3} \sigma_j^3$ is a 3-ball (*cf.* [14]). Then the decomposition $\{\sigma_i^3\}_{1 \leq i \leq t}$ of M is called a *triangulation* of M.

Example 2.1.1. (1) The 3-ball B^3 is the simplest PL 3-manifold in a sense that B^3 is homeomorphic to a 3-simplex.
(2) The 3-sphere S^3 is a 3-manifold obtained from two 3-balls by attaching their boundaries. Since S^3 is homeomorphic to the boundary of a 4-simplex, we see that S^3 is a union of five 3-simplices. It is easy to show that this gives a triangulation of S^3.

Exercise 2.1.2. Show that the following 3-manifolds are PL 3-manifolds.
(1) The solid torus $D^2 \times S^1$.
(2) $S^2 \times S^1$.
(3) The lens spaces. Note that a *lens space* is obtained from two solid tori by attaching their boundaries.

Let K be a three dimensional simplicial complex and X a sub-complex of K, that is, X a union of vertices, edges, faces and 3-simplices of K such that X is a simplicial complex. Let K'' be the second barycentric subdivision of K. A *regular neighborhood* of X in K, denoted by $\eta(X; K)$, is a union of the 3-simplices of K'' intersecting X (*cf.* Figure 3).

Proposition 2.1.3. *If X is a PL 1-manifold properly embedded in a PL 3-manifold M (namely, $X \cap \partial M = \partial X$), then $\eta(X; M) \cong X \times B^2$, where X is*

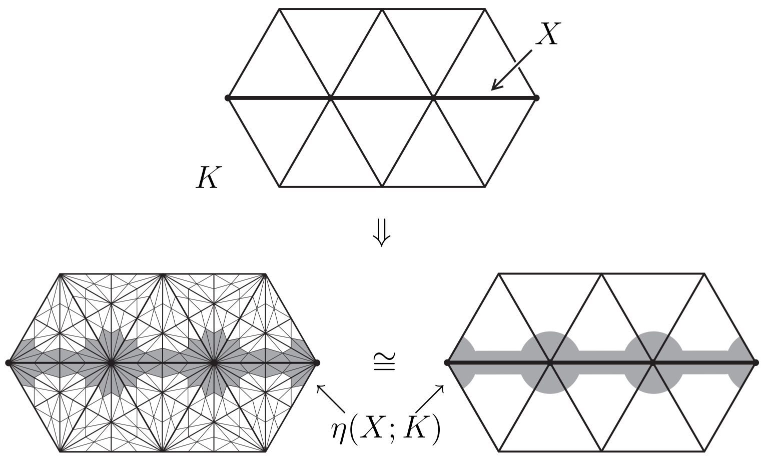

FIGURE 3

identified with $X \times \{$*a center of* $B^2\}$ *and* $\eta(X; M) \cap \partial M$ *is identified with* $\partial X \times B^2$ *(cf. Figure 4).*

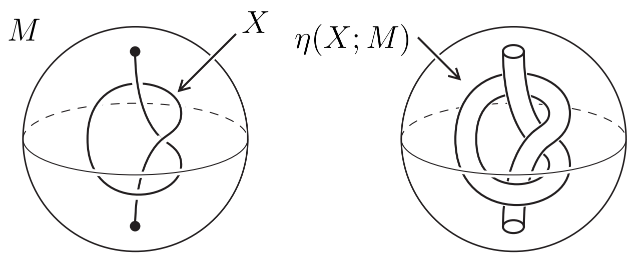

FIGURE 4

Proposition 2.1.4. *Suppose that a PL* 3*-manifold* M *is orientable. If* X *is an orientable PL* 2*-manifold properly embedded in* M *(namely,* $X \cap \partial M = \partial X$*), then* $\eta(X; M) \cong X \times [0, 1]$*, where* X *is identified with* $X \times \{1/2\}$ *and* $\eta(X; M) \cap \partial M$ *is identified with* $\partial X \times [0, 1]$.

Theorem 2.1.5 (Moise [10])**.** *Every compact* 3*-manifold is a PL* 3*-manifold.*

In the remainder of these notes, we work in the PL category unless otherwise specified.

2.2. **Fundamental definitions.** By the term *surface*, we will mean a connected compact 2-manifold.

Let F be a surface. A loop α in F is said to be *inessential* in F if α bounds a disk in F, otherwise α is said to be *essential* in F. An arc γ properly embedded in F is said to be *inessential* in F if γ cuts off a disk from F, otherwise γ is said to be *essential* in F.

Let M be a compact orientable 3-manifold. A disk D properly embedded in M is said to be *inessential* in M if D cuts off a 3-ball from M, otherwise D is said to be *essential* in M. A 2-sphere P properly embedded in M is said to be *inessential* in M if P bounds a 3-ball in M, otherwise P is said to be *essential* in M. Let F be a surface properly embedded in M. We say that F is *∂-parallel* in M if F cuts off a 3-manifold homeomorphic to $F \times [0,1]$ from M. We say that F is *compressible* in M if there is a disk $D \subset M$ such that $D \cap F = \partial D$ and ∂D is an essential loop in F. Such a disk D is called a *compressing disk*. We say that F is *incompressible* in M if F is not compressible in M. The surface F is *∂-compressible* in M if there is a disk $\delta \subset M$ such that $\delta \cap F$ is an arc which is essential in F, say γ, in F and that $\delta \cap \partial M$ is an arc, say γ', with $\gamma' \cup \gamma = \partial\delta$. Otherwise F is said to be *∂-incompressible* in M. Suppose that F is homeomorphic neither to a disk nor to a 2-sphere. The surface F is said to be *essential* in M if F is incompressible in M and is not ∂-parallel in M.

Definition 2.2.1. Let M be a connected compact orientable 3-manifold.

(1) M is said to be *reducible* if there is a 2-sphere in M which does not bound a 3-ball in M. Such a 2-sphere is called a *reducing 2-sphere* of M. M is said to be *irreducible* if M is not reducible.
(2) M is said to be *∂-reducible* if there is a disk properly embedded in M whose boundary is essential in ∂M. Such a disk is called a *∂-reducing disk*.

3. Heegaard splittings

3.1. **Definitions and fundamental properties.**

Definition 3.1.1. A 3-manifold C is called a *compression body* if there exists a closed surface F such that C is obtained from $F \times [0,1]$ by attaching 2-handles along mutually disjoint loops in $S \times \{1\}$ and filling in some resulting 2-sphere boundary components with 3-handles (*cf.* Figure 5). We denote $F \times \{0\}$ by $\partial_+ C$ and $\partial C \setminus \partial_+ C$ by $\partial_- C$. A compression body C is called a *handlebody* if $\partial_- C = \emptyset$. A compression body C is said to be *trivial* if $C \cong F \times [0,1]$.

Definition 3.1.2. For a compression body C, an essential disk in C is called a *meridian disk* of C. A union Δ of mutually disjoint meridian disks of C is called a *complete meridian system* if the manifold obtained from C by cutting along Δ are the union of $\partial_- C \times [0,1]$ and (possibly empty) 3-balls. A complete meridian system Δ of C is *minimal* if the number of the components of Δ is minimal among all complete meridian system of C.

Remark 3.1.3. The following properties are known for compression bodies.

(1) A compression body C is reducible if and only if $\partial_- C$ contains a 2-sphere component.
(2) A minimal complete meridian system Δ of a compression body C cuts C into $\partial_- C \times [0,1]$ if $\partial_- C \neq \emptyset$, and Δ cuts C into a 3-ball if $\partial_- C = \emptyset$ (hence C is a handlebody).
(3) By extending the cores of the 2-handles in the definition of the compression body C vertically to $F \times [0,1]$, we obtain a complete meridian system Δ of C such that the manifold obtained by cutting C along Δ is homeomorphic to a union of $\partial_- C \times [0,1]$ and some (possibly empty) 3-balls. This gives a *dual description* of compression bodies. That is, a compression body C is obtained from $\partial_- C \times [0,1]$ and some (possibly empty) 3-balls by attaching

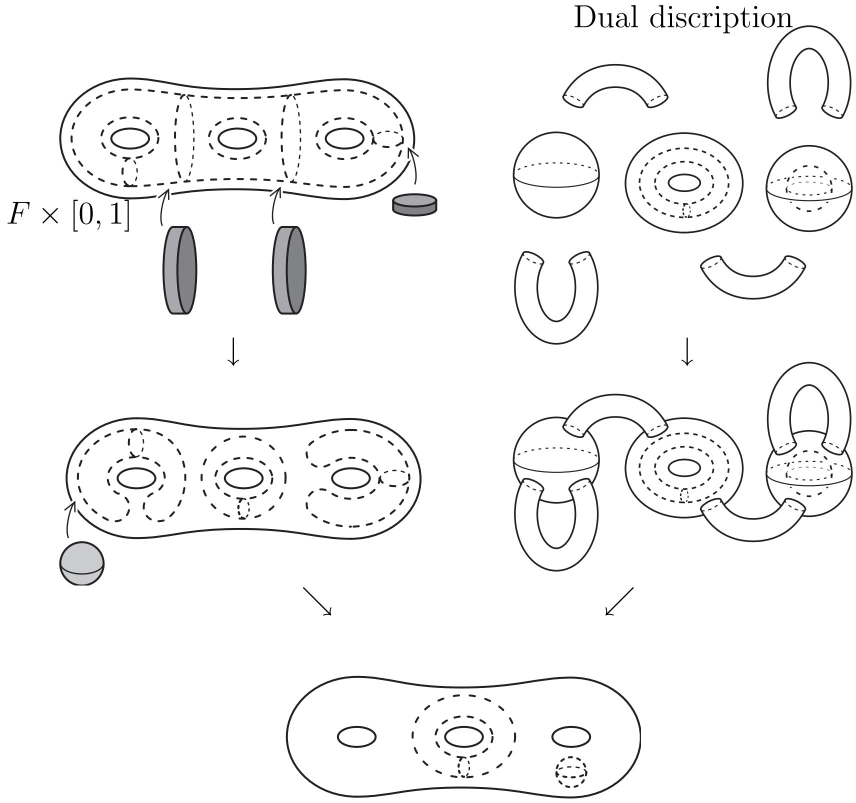

FIGURE 5

some 1-handles to $\partial_- C \times \{1\}$ and the boundary of the 3-balls (*cf.* Figure 5).

(4) For any compression body C, $\partial_- C$ is incompressible in C.
(5) Let C and C' be compression bodies. Suppose that C'' is obtained from C and C' by identifying a component of $\partial_- C$ and $\partial_+ C'$. Then C'' is a compression body.
(6) Let D be a meridian disk of a compression body C. Then there is a complete meridian system Δ of C such that D is a component of Δ. Any component obtained by cutting C along D is a compression body.

Exercise 3.1.4. Show Remark 3.1.3.

An annulus A properly embedded in a compression body C is called a *spanning annulus* if A is incompressible in C and a component of ∂A is contained in $\partial_+ C$ and the other is contained in $\partial_- C$.

Lemma 3.1.5. *Let C be a non-trivial compression body. Let A be a spanning annulus in C. Then there is a meridian disk D of C with $D \cap A = \emptyset$.*

Proof. Since C is non-trivial, there is a meridian disk of C. We choose a meridian disk D of C such that D intersects A transversely and $|D \cap A|$ is minimal among all such meridian disks. Note that $A \cap \partial_- C$ is an essential loop in the component of $\partial_- C$ containing $A \cap \partial_- C$. We shall prove that $D \cap A = \emptyset$. To this end, we suppose $D \cap A \neq \emptyset$.

Claim 1. There are no loop components of $D \cap A$.

Proof. Suppose that $D \cap A$ has a loop component which is inessential in A. Let α be a loop component of $D \cap A$ which is *innermost* in A, that is, α cuts off a disk δ_α from A such that the interior of δ_α is disjoint from D. Such a disk δ_α is called an *innermost disk* for α. We remark that α is not necessarily innermost in D. Note that α also bounds a disk in D, say δ'_α. Then we obtain a disk D' by applying *cut and paste operation* on D with using δ_α and δ'_α, i.e., D' is obtained from D by removing the interior of δ'_α and then attaching δ_α (*cf.* Figure 6). Note that D' is a meridian disk of C. Moreover, we can isotope the interior of D' slightly so that $|D' \cap A| < |D \cap A|$, a contradiction. (Such an argument as above is called an *innermost disk argument.*)

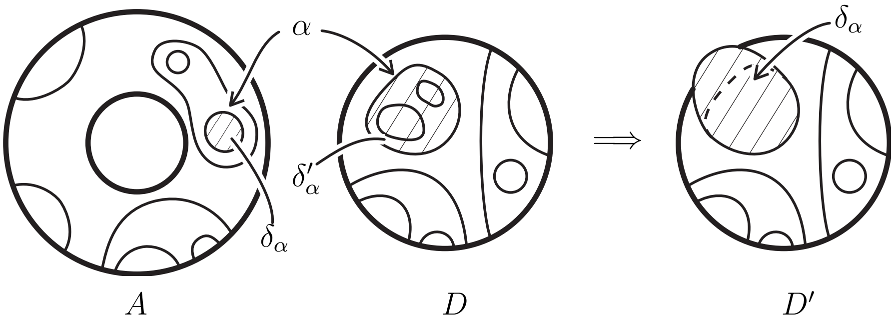

Figure 6

Hence if $D \cap A$ has a loop component, we may assume that the loop is essential in A. Let α' be a loop component of $D \cap A$ which is innermost in D, and let $\delta_{\alpha'}$ be the innermost disk in D with $\partial \delta_{\alpha'} = \alpha'$. Then α' cuts A into two annuli, and let A' be the component obtained by cutting A along α' such that A' is adjacent to $\partial_- C$. Set $D'' = A' \cup \delta_{\alpha'}$. Then $D''(\subset C)$ is a compressing disk of $\partial_- C$, contradicting (4) of Remark 3.1.3. Hence we have Claim 1.

Claim 2. There are no arc components of $D \cap A$.

Proof. Suppose that there is an arc component of $D \cap A$. Note that $\partial D \subset \partial_+ C$. Hence we may assume that each component of $D \cap A$ is an inessential arc in A whose endpoints are contained in $\partial_+ C$. Let γ be an arc component of $D \cap A$ which is *outermost* in A, that is, γ cuts off a disk δ_γ from A such that the interior of δ_γ is disjoint from D. Such a disk δ_γ is called an *outermost disk* for γ. Note that γ cuts D into two disks $\bar{\delta}_\gamma$ and $\bar{\delta}'_\gamma$ (*cf.* Figure 7).

If both $\bar{\delta}_\gamma \cup \delta_\gamma$ and $\bar{\delta}'_\gamma \cup \delta_\gamma$ are inessential in C, then D is also inessential in C, a contradiction. So we may assume that $\bar{D} = \bar{\delta}_\gamma \cup \delta_\gamma$ is essential in C. Then we can isotope $\bar{D}$ slightly so that $|\bar{D} \cap A| < |D \cap A|$, a contradiction. Hence we have Claim 2. (Such an argument as above is called an *outermost disk argument.*)

Hence it follows from Claims 1 and 2 that $D \cap A = \emptyset$, and this completes the proof of Lemma 3.1.5. □

Remark 3.1.6. Let A be a spanning annulus in a non-trivial compression body C.

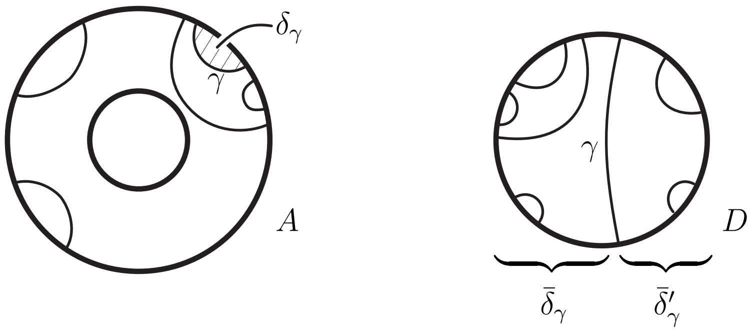

FIGURE 7

(1) By using the arguments of the proof of Lemma 3.1.5, we can show that there is a complete meridian system Δ of C with $\Delta \cap A = \emptyset$.
(2) It follows from (1) above that there is a meridian disk E of C such that $E \cap A = \emptyset$ and E cuts off a 3-manifold which is homeomorphic to (a closed surface) $\times$ $[0, 1]$ containing A.

Exercise 3.1.7. Show Remark 3.1.6.

Let $\bar{\alpha} = \alpha_1 \cup \cdots \cup \alpha_p$ be a union of mutually disjoint arcs in a compression body C. We say that $\bar{\alpha}$ is *vertical* if there is a union of mutually disjoint spanning annuli $A_1 \cup \cdots \cup A_p$ in C such that $\alpha_i \cap A_j = \emptyset$ $(i \neq j)$ and α_i is an essential arc properly embedded in A_i $(i = 1, 2, \ldots, p)$.

Lemma 3.1.8. *Suppose that $\bar{\alpha} = \alpha_1 \cup \cdots \cup \alpha_p$ is vertical in C. Let D be a meridian disk of C. Then there is a meridian disk D' of C with $D' \cap \bar{\alpha} = \emptyset$ which is obtained by cut-and-paste operation on D. Particularly, if C is irreducible, then D is ambient isotopic such that $D \cap \bar{\alpha} = \emptyset$.*

Proof. Let $\bar{A} = A_1 \cup \cdots \cup A_p$ be a union of annuli for $\bar{\alpha}$ as above. By using innermost disk arguments, we see that there is a meridian disk D' such that no components of $D' \cap \bar{A}$ are loops which are inessential in $\bar{A}$. We remark that D' is ambient isotopic to D if C is irreducible. Note that each component of $\bar{A}$ is incompressible in C. Hence no components of $D' \cap \bar{A}$ are loops which are essential in $\bar{A}$. Hence each component of $D' \cap \bar{A}$ is an arc; moreover since ∂D is contained in $\partial_+ C$, the endpoints of the arc components of $D' \cap \bar{A}$ are contained in $\partial_+ C \cap \bar{A}$. Then it is easy to see that there exists an arc $\beta_i (\subset A_i)$ such that β_i is essential in A_i and $\beta_i \cap D' = \emptyset$. Take an ambient isotopy h_t $(0 \leq t \leq 1)$ of C such that $h_0(\beta_i) = \beta_i$, $h_t(\bar{A}) = \bar{A}$ and $h_1(\beta_i) = \alpha_i$ $(i = 1, 2, \ldots, p)$ (*cf.* Figure 8). Then the ambient isotopy h_t assures that D' is isotoped so that D' is disjoint from $\bar{\alpha}$. □

In the remainder of these notes, let M be a connected compact orientable 3-manifold.

Definition 3.1.9. Let $(\partial_1 M, \partial_2 M)$ be a partition of ∂-components of M. A triplet $(C_1, C_2; S)$ is called a *Heegaard splitting* of $(M; \partial_1 M, \partial_2 M)$ if C_1 and C_2 are compression bodies with $C_1 \cup C_2 = M$, $\partial_- C_1 = \partial_1 M$, $\partial_- C_2 = \partial_2 M$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = S$. The surface S is called a *Heegaard surface* and the *genus* of a Heegaard splitting is defined by the genus of the Heegaard surface.

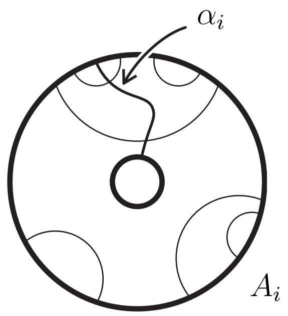

FIGURE 8

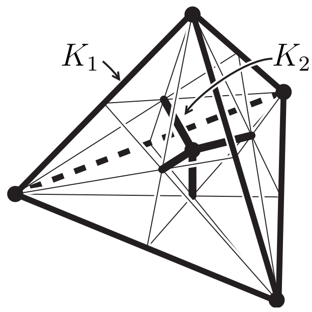

FIGURE 9

Theorem 3.1.10. *For any partition* $(\partial_1 M, \partial_2 M)$ *of the boundary components of* M*, there is a Heegaard splitting of* $(M; \partial_1 M, \partial_2 M)$.

Proof. It follows from Theorem 2.1.5 that M is triangulated, that is, there is a finite simplicial complex K which is homeomorphic to M. Let K' be a barycentric subdivision of K and K_1 the 1-skeleton of K. Here, a 1-*skeleton* of K is a union of the vertices and edges of K. Let $K_2 \subset K'$ be the dual 1-skeleton (see Figure 9). Then each of K_i $(i = 1, 2)$ is a finite graph in M.

Case 1. $\partial M = \emptyset$.

Recall that K_1 consists of 0-simplices and 1-simplices. Set $C_1 = \eta(K_1; M)$ and $C_2 = \eta(K_2; M)$. Note that a regular neighborhood of a 0-simplex corresponds to a 0-handle and that a regular neighborhood of a 1-simplex corresponds to a 1-handle. Hence C_1 is a handlebody. Similarly, we see that C_2 is also a handlebody. Then we see that $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial C_1 = \partial C_2$. Hence $(C_1, C_2; S)$ is a Heegaard splitting of M with $S = C_1 \cap C_2$.

Case 2. $\partial M \neq \emptyset$.

In this case, we first take the barycentric subdivision of K and use the same notation K. Recall that K' is the barycentric subdivision of K. Note that no 3-simplices of K intersect both $\partial_1 M$ and $\partial_2 M$. Let $N(\partial_2 M)$ be a union of the 3-simplices in K' intersecting $\partial_2 M$. Then $N(\partial_2 M)$ is homeomorphic to $\partial_2 M \times [0, 1]$, where $\partial_2 M \times \{0\}$ is identified with $\partial_2 M$. Set $\partial_2' M = \partial_2 M \times \{1\}$. Let $\bar{K}_1$ ($\bar{K}_2$

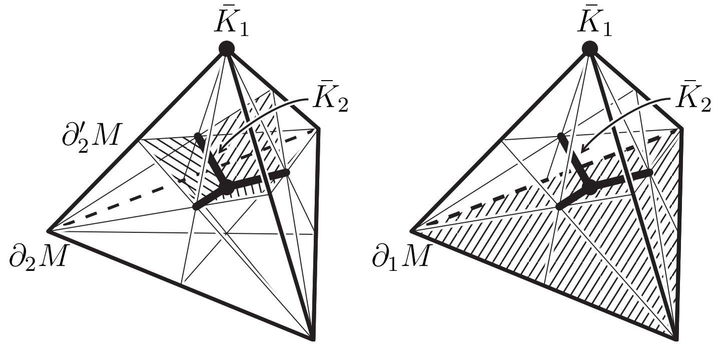

FIGURE 10

resp.) be the maximal sub-complex of K_1 (K_2 resp.) such that $\bar{K}_1$ ($\bar{K}_2$ resp.) is disjoint from $\partial_2' M$ ($\partial_1 M$ resp.) (*cf.* Figure 10).

Set $C_1 = \eta(\partial_1 M \cup \bar{K}_1; M)$. Note that $C_1 = \eta(\partial_1 M; M) \cup \eta(\bar{K}_1; M)$. Note again that a regular neighborhood of a 0-simplex corresponds to a 0-handle and that a regular neighborhood of a 1-simplex corresponds to a 1-handle. Hence C_1 is obtained from $\partial_1 M \times [0,1]$ by attaching 0-handles and 1-handles and therefore C_1 is a compression body with $\partial_- C_1 = \partial_1 M$. Set $C_2 = \eta(N(\partial_2 M) \cup \bar{K}_2; M)$. By the same argument, we can see that C_2 is a compression body with $\partial_- C_2 = \partial_2 M$. Note that $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial C_1 = \partial C_2$. Hence $(C_1, C_2; S)$ is a Heegaard splitting of M with $S = C_1 \cap C_2$. □

We now introduce alternative viewpoints to Heegaard splittings as remarks below.

Definition 3.1.11. Let C be a compression body. A finite graph Σ in C is called a *spine* of C if $C \setminus (\partial_- C \cup \Sigma) \cong \partial_+ C \times [0,1)$ and every vertex of valence one is in $\partial_- C$ (*cf.* Figure 11).

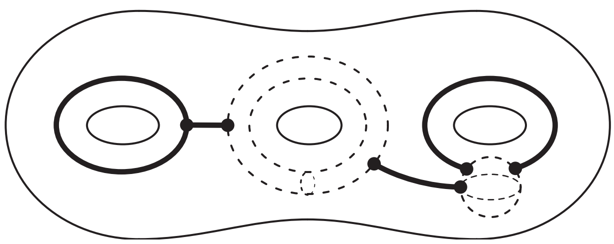

FIGURE 11

Remark 3.1.12. Let $(C_1, C_2; S)$ be a Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$. Let Σ_i be a spine of C_i, and set $\Sigma_i' = \partial_i M \cup \Sigma_i$ $(i = 1, 2)$. Then

$$M \setminus (\Sigma_1' \cup \Sigma_2') = (C_1 \setminus \Sigma_1') \cup_S (C_2 \setminus \Sigma_2') \cong S \times (0,1).$$

Hence there is a continuous function $f : M \to [0,1]$ such that $f^{-1}(0) = \Sigma_1'$, $f^{-1}(1) = \Sigma_2'$ and $f^{-1}(t) \cong S$ $(0 < t < 1)$. This is called a *sweep-out picture*.

Remark 3.1.13. Let $(C_1, C_2; S)$ be a Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$. By a dual description of C_1, we see that C_1 is obtained from $\partial_1 M \times [0,1]$ and 0-handles $\mathcal{H}^0$ by attaching 1-handles $\mathcal{H}^1$. By Definition 3.1.1, C_2 is obtained from $S \times [0,1]$ by attaching 2-handles $\mathcal{H}^2$ and filling some 2-sphere boundary components with 3-handles $\mathcal{H}^3$. Hence we obtain the following decomposition of M:

$$M = \partial_1 M \times [0,1] \cup \mathcal{H}^0 \cup \mathcal{H}^1 \cup S \times [0,1] \cup \mathcal{H}^2 \cup \mathcal{H}^3.$$

By collapsing $S \times [0,1]$ to S, we have:

$$M = \partial_1 M \times [0,1] \cup \mathcal{H}^0 \cup \mathcal{H}^1 \cup_S \mathcal{H}^2 \cup \mathcal{H}^3.$$

This is called a *handle decomposition of M induced from $(C_1, C_2; S)$*.

Definition 3.1.14. Let $(C_1, C_2; S)$ be a Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$.

(1) The splitting $(C_1, C_2; S)$ is said to be *reducible* if there are meridian disks D_i $(i = 1, 2)$ of C_i with $\partial D_1 = \partial D_2$. The splitting $(C_1, C_2; S)$ is said to be *irreducible* if $(C_1, C_2; S)$ is not reducible.

(2) The splitting $(C_1, C_2; S)$ is said to be *weakly reducible* if there are meridian disks D_i $(i = 1, 2)$ of C_i with $\partial D_1 \cap \partial D_2 = \emptyset$. The splitting $(C_1, C_2; S)$ is said to be *strongly irreducible* if $(C_1, C_2; S)$ is not *weakly reducible*.

(3) The splitting $(C_1, C_2; S)$ is said to be *∂-reducible* if there is a disk D properly embedded in M such that $D \cap S$ is an essential loop in S. Such a disk D is called a *∂-reducing disk* for $(C_1, C_2; S)$.

(4) The splitting $(C_1, C_2; S)$ is said to be *stabilized* if there are meridian disks D_i $(i = 1, 2)$ of C_i such that ∂D_1 and ∂D_2 intersect transversely in a single point. Such a pair of disks is called a *cancelling pair of disks* for $(C_1, C_2; S)$.

Example 3.1.15. Let $(C_1, C_2; S)$ be a Heegaard splitting such that each of $\partial_- C_i$ $(i = 1, 2)$ consists of two 2-spheres and that S is a 2-sphere. Note that there does not exist an essential disk in C_i. Hence $(C_1, C_2; S)$ is strongly irreducible.

Suppose that $(C_1, C_2; S)$ is stabilized, and let D_i $(i = 1, 2)$ be disks as in (4) of Definition 3.1.14. Note that since ∂D_1 intersects ∂D_2 transversely in a single point, we see that each of ∂D_i $(i = 1, 2)$ is non-separating in S and hence each of D_i $(i = 1, 2)$ is non-separating in C_i. Set $C_1' = \mathrm{cl}(C_1 \setminus \eta(D_1; C_1))$ and $C_2' = C_2 \cup \eta(D_1; C_1)$. Then each of C_i' $(i = 1, 2)$ is a compression body with $\partial_+ C_1' = \partial_+ C_2'$ (*cf.* (6) of Remark 3.1.3). Set $S' = \partial_+ C_1' (= \partial_+ C_2')$. Then we obtain the Heegaard splitting $(C_1', C_2'; S')$ of M with $genus(S') = genus(S) - 1$. Conversely, $(C_1, C_2; S)$ is obtained from $(C_1', C_2'; S')$ by adding a trivial handle. We say that $(C_1, C_2; S)$ is obtained from $(C_1', C_2'; S')$ by *stabilization*.

Observation 3.1.16. *Every reducible Heegaard splitting is weakly reducible.*

Lemma 3.1.17. *Let $(C_1, C_2; S)$ be a Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$ with $genus(S) \geq 2$. If $(C_1, C_2; S)$ is stabilized, then $(C_1, C_2; S)$ is reducible.*

Proof. Suppose that $(C_1, C_2; S)$ is stabilized, and let D_i $(i = 1, 2)$ be meridian disks of C_i such that ∂D_1 intersects ∂D_2 transversely in a single point. Then $\partial \eta(\partial D_1 \cup \partial D_2; S)$ bounds a disk D_i' in C_i for each $i = 1$ and 2. In fact, D_1' (D_2' resp.) is obtained from two parallel copies of D_1 (D_2 resp.) by adding a band along $\partial D_2 \setminus$ (the product region between the parallel disks) ($\partial D_1 \setminus$ (the product region between the parallel disks) resp.) (*cf.* Figure 12).

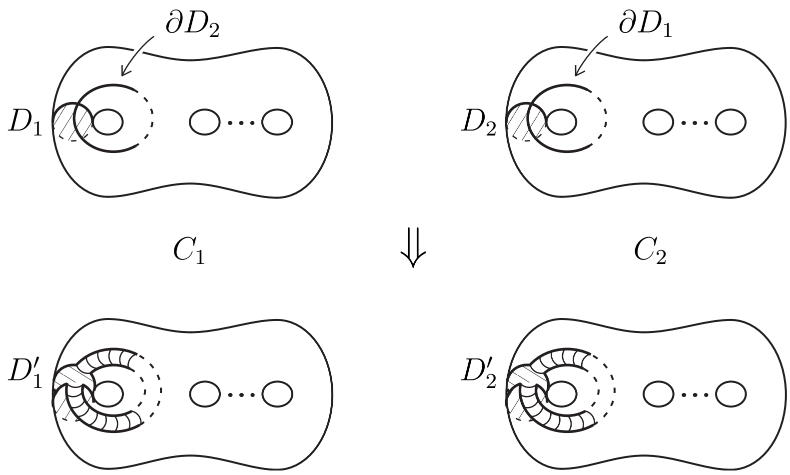

FIGURE 12

Note that $\partial D_1' = \partial D_2'$ cuts S into a torus with a single hole and the other surface S'. Since $genus(S) \geq 2$, we see that $genus(S') \geq 1$. Hence $\partial D_1' = \partial D_2'$ is essential in S and therefore $(C_1, C_2; S)$ is reducible. □

Definition 3.1.18. Let $(C_1, C_2; S)$ be a Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$.

(1) Suppose that $M \cong S^3$. We call $(C_1, C_2; S)$ a *trivial splitting* if both C_1 and C_2 are 3-balls.
(2) Suppose that $M \not\cong S^3$. We call $(C_1, C_2; S)$ a *trivial splitting* if C_i is a trivial handlebody for $i = 1$ or 2.

Remark 3.1.19. Suppose that $M \not\cong S^3$. If $(M; \partial_1 M, \partial_2 M)$ admits a trivial splitting $(C_1, C_2; S)$, then it is easy to see that M is a compression body. Particularly, if C_2 (C_1 resp.) is trivial, then $\partial_- M = \partial_1 M$ and $\partial_+ M = \partial_2 M$ ($\partial_- M = \partial_2 M$ and $\partial_+ M = \partial_1 M$ resp.).

Lemma 3.1.20. *Let $(C_1, C_2; S)$ be a non-trivial Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$. If $(C_1, C_2; S)$ is ∂-reducible, then $(C_1, C_2; S)$ is weakly reducible.*

Proof. Let D be a ∂-reducing disk for $(C_1, C_2; S)$. (Hence $D \cap S$ is an essential loop in S.) Set $D_1 = D \cap C_1$ and $A_2 = D \cap C_2$. By exchanging subscripts, if necessary, we may suppose that D_1 is a meridian disk of C_1 and A_2 is a spanning annulus in C_2. Note that $A_2 \cap \partial_- C_2$ is an essential loop in the component of $\partial_- C_2$ containing $A_2 \cap \partial_- C_2$. Since C_2 is non-trivial, there is a meridian disk of C_2. It follows from Lemma 3.1.5 that we can choose a meridian disk D_2 of C_2 with $D_2 \cap A_2 = \emptyset$. This implies that $D_1 \cap D_2 = \emptyset$. Hence $(C_1, C_2; S)$ is weakly reducible. □

3.2. **Haken's theorem.** In this subsection, we prove the following.

Theorem 3.2.1. *Let $(C_1, C_2; S)$ be a Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$.*

(1) *If M is reducible, then $(C_1, C_2; S)$ is reducible or C_i is reducible for $i = 1$ or 2.*
(2) *If M is ∂-reducible, then $(C_1, C_2; S)$ is ∂-reducible.*

Note that the statement (1) of Theorem 3.2.1 is called Haken's theorem and proved by Haken [4], and the statement (2) of Theorem 3.2.1 is proved by Casson and Gordon [1].

We first prove the following proposition, whose statement is weaker than that of Theorem 3.2.1, after showing some lemmas.

Proposition 3.2.2. *If M is reducible or ∂-reducible, then $(C_1, C_2; S)$ is reducible, ∂-reducible, or C_i is reducible for $i = 1$ or 2.*

We give a proof of Proposition 3.2.2 by using Otal's idea (*cf.* [11]) of viewing the Heegaard splittings as a graph in the three dimensional space.

Edge slides of graphs. Let Γ be a finite graph in a 3-manifold M. Choose an edge σ of Γ. Let p_1 and p_2 be the vertices of Γ incident to σ. Set $\bar{\Gamma} = \Gamma \setminus \sigma$. Here, we may suppose that $\sigma \cap \partial\eta(\bar{\Gamma}; M)$ consists of two points, say $\bar{p}_1$ and $\bar{p}_2$, and that $\mathrm{cl}(\sigma \setminus (p_1 \cup p_2))$ consists of α_0, α_1 and α_2 with $\partial\alpha_0 = \bar{p}_1 \cup \bar{p}_2$, $\partial\alpha_1 = p_1 \cup \bar{p}_1$ and $\partial\alpha_2 = p_2 \cup \bar{p}_2$ (*cf.* Figure 13).

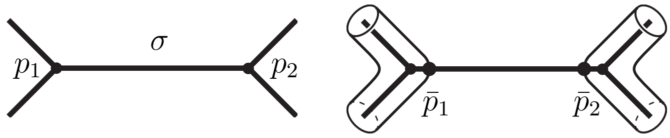

FIGURE 13

Take a path γ on $\partial\eta(\bar{\Gamma}; M)$ with $\partial\gamma \ni \bar{p}_1$. Let $\bar{\sigma}$ be an arc obtained from $\gamma \cup \alpha_0 \cup \alpha_2$ by adding a 'straight short arc' in $\eta(\bar{\Gamma}; M)$ connecting the endpoint of γ other than $\bar{p}_1$ and a point p_1' in the interior of an edge of $\bar{\Gamma}$ (*cf.* Figure 14). Let Γ' be a graph obtained from $\bar{\Gamma} \cup \bar{\sigma}$ by adding p_1' as a vertex. Then we say that Γ' is obtained from Γ by an *edge slide* on σ.

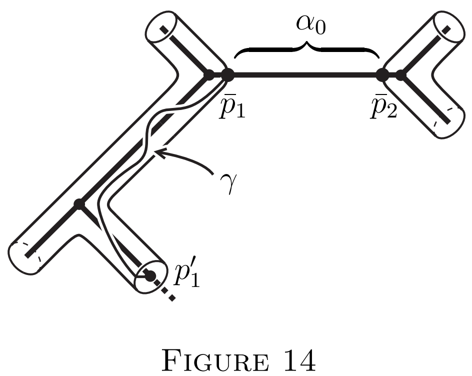

FIGURE 14

If p_1 is a trivalent vertex, then it is natural for us not to regard p_1 as a vertex of Γ'. Particularly, the deformation of Γ which is depicted as in Figure 15 is realized by an edge slide and an isotopy. This deformation is called a *Whitehead move*.

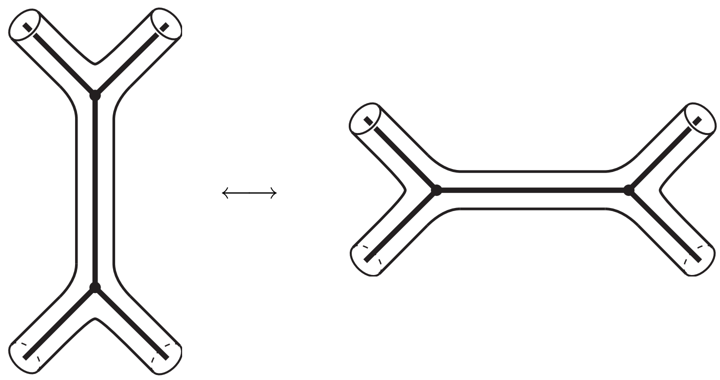

FIGURE 15

A Proof of Proposition 3.2.2. Let Σ be a spine of C_1. Note that $\eta(\partial_- C_1 \cup \Sigma; M)$ is obtained from regular neighborhoods of $\partial_- C_1$ and the vertices of Σ by attaching 1-handles corresponding to the edges of Σ. Set $\Sigma_\eta = \eta(\Sigma; M)$. The notation h_v^0, called a *vertex* of Σ_η, means a regular neighborhood of a vertex v of Σ. Also, the notation h_σ^1, called an *edge* of Σ_η, means a 1-handle corresponding to an edge σ of Σ. Let $\Delta = D_1 \cup \cdots \cup D_k$ be a minimal complete meridian system of C_2.

Let P be a reducing 2-sphere or a ∂-reducing disk of M. If P is a ∂-reducing disk, we may assume that $\partial P \subset \partial_- C_2$ by changing subscripts. We may assume that P intersects Σ and Δ transversely. Set $\Gamma = P \cap (\Sigma_\eta \cup \Delta)$. We note that Γ is a union of disks $P \cap \Sigma_\eta$ and a union of arcs and loops $P \cap \Delta$ in P. We choose P, Σ and Δ so that the pair $(|P \cap \Sigma|, |P \cap \Delta|)$ is minimal with respect to lexicographic order.

Lemma 3.2.3. *Each component of $P \cap \Delta$ is an arc.*

Proof. For some disk component, say D_1, of Δ, suppose that $P \cap D_1$ has a loop component. Let α be a loop component of $P \cap D_1$ which is innermost in D_1, and let δ_α be an innermost disk for α. Let δ'_α be a disk in P with $\partial \delta'_\alpha = \alpha$. Set $P' = (P \backslash \delta'_\alpha) \cup \delta_\alpha$ if P is a ∂-reducing disk, or set $P' = (P \backslash \delta'_\alpha) \cup \delta_\alpha$ and $P'' = \delta'_\alpha \cup \delta_\alpha$ if P is a reducing 2-sphere. If P is a ∂-reducing disk, then P' is also a ∂-reducing disk. If P is a reducing 2-sphere, then either P' or P'', say P', is a reducing 2-sphere. Moreover, we can isotope P' so that $(|P' \cap \Sigma|, |P' \cap \Delta|) < (|P \cap \Sigma|, |P \cap \Delta|)$. This contradicts the minimality of $(|P \cap \Sigma|, |P \cap \Delta|)$. □

By Lemma 3.2.3, we can regard Γ as a graph in P which consists of *fat-vertices* $P \cap \Sigma_\eta$ and edges $P \cap \Delta$. An edge of the graph Γ is called a *loop* if the edge joins a fat-vertex of Γ to itself, and a loop is said to be *inessential* if the loop cuts off a disk from $\mathrm{cl}(P \setminus \Sigma_\eta)$ whose interior is disjoint from $\Gamma \cap \Sigma_\eta$.

Lemma 3.2.4. *Γ does not contain an inessential loop.*

Proof. Suppose that Γ contains an inessential loop μ. Then μ cuts off a disk δ_μ from $\mathrm{cl}(P \setminus \Sigma_\eta)$ such that the interior of δ_μ is disjoint from $\Gamma \cap \Sigma_\eta$ (*cf.* Figure 16).

We may assume that $\delta_\mu \cap \Delta = \delta_\mu \cap D_1$. Then μ cuts D_1 into two disks D'_1 and D''_1 (*cf.* Figure 17).

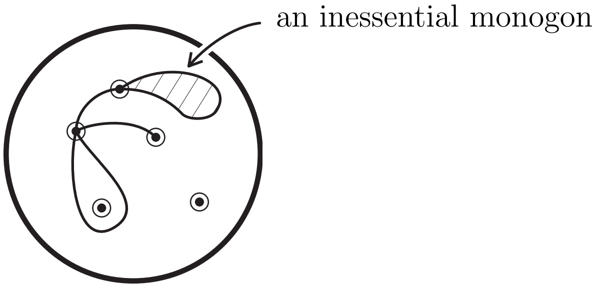

FIGURE 16

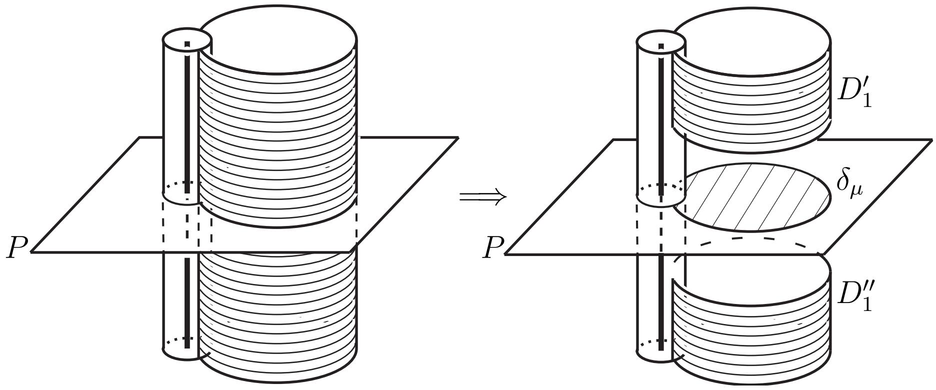

FIGURE 17

Let C_2' be the component, which is obtained by cutting C_2 along Δ, such that C_2' contains δ_μ. Let D_1^+ be the copy of D_1 in C_2' with $D_1^+ \cap \delta_\mu \neq \emptyset$ and D_1^- the other copy of D_1. Note that C_2' is a 3-ball or a (a component of $\partial_- C_2)\times[0,1]$. This shows that there is a disk δ_μ' in $\partial_+ C_2'$ such that $\partial\delta_\mu = \partial\delta_\mu'$ and $\partial\delta_\mu \cup \partial\delta_\mu'$ bounds a 3-ball in C_2'. Note that $\delta_\mu' \cap D_1^+ \neq \emptyset$. By changing superscripts, if necessary, we may assume that $\delta_\mu' \supset D_1'$ (*cf.* Figure 18).

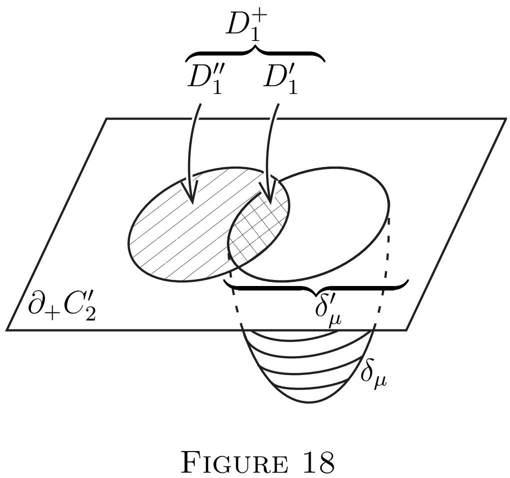

FIGURE 18

Set $D_0 = \delta_\mu \cup D_1'$ if $\delta_\mu' \cap D_1^- \neq \emptyset$, and $D_0 = \delta_\mu \cup D_1''$ if $\delta_\mu' \cap D_1^- = \emptyset$. We may regard D_0 as a disk properly embedded in C_2. Set $\Delta' = D_0 \cup D_2 \cup \cdots \cup D_k$. Then we see that Δ' is a minimal complete meridian system of C_2. We can further isotope D_0 slightly so that $|P \cap \Delta'| < |P \cap \Delta|$. This contradicts the minimality of $(|P \cap \Sigma|, |P \cap \Delta|)$. □

A fat-vertex of Γ is said to be *isolated* if there are no edges of Γ adjacent to the fat-vertex (*cf.* Figure 19).

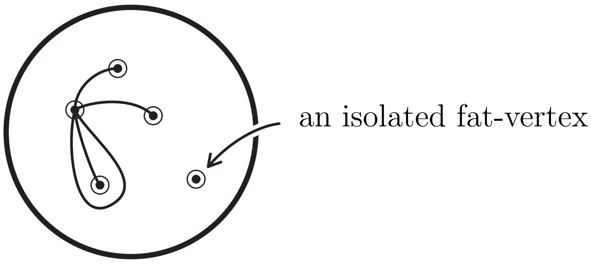

FIGURE 19

Lemma 3.2.5. *If* Γ *has an isolated fat-vertex, then* $(C_1, C_2; S)$ *is reducible or* ∂*-reducible.*

Proof. Suppose that there is an isolated fat-vertex D_v of Γ. Recall that D_v is a component of $P \cap \Sigma_\eta$ which is a meridian disk of C_1. Note that D_v is disjoint from Δ (*cf.* Figure 20).

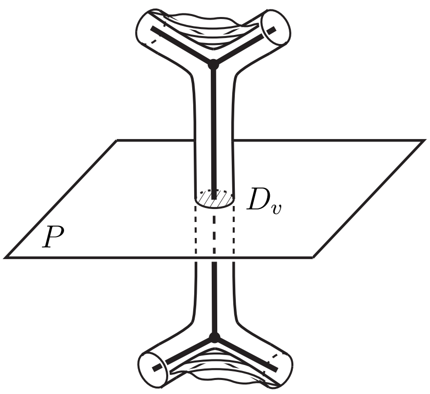

FIGURE 20

Let C_2' be the component obtained by cutting C_2 along Δ such that $\partial C_2'$ contains ∂D_v. If ∂D_v bounds a disk D_v' in C_2', then D_v and D_v' indicates the reducibility of $(C_1, C_2; S)$. Otherwise, C_2' is a (a closed orientable surface) $\times [0, 1]$, and ∂D_v is a boundary component of a spanning annulus in C_2' (and hence C_2). Hence we see that $(C_1, C_2; S)$ is ∂-reducible. □

Lemma 3.2.6. *Suppose that no fat-vertices of* Γ *are isolated. Then each fat-vertex of* Γ *is a base of a loop.*

Proof. Suppose that there is a fat-vertex D_w of Γ which is not a base of a loop. Since no fat-vartices of Γ are isolated, there is an edge of Γ adjacent to D_w. Let σ be the edge of Σ with $h^1_\sigma \supset D_w$. (Recall that h^1_σ is a 1-handle of Σ_η corresponding to σ.) Let D be a component of Δ with $\partial D \cap h^1_\sigma \neq \emptyset$. Let C_w be a union of the arc components of $D \cap P$ which are adjacent to D_w. Let γ be an arc component of C_w which is outermost among the components of C_w. We call such an arc γ an *outermost edge for* D_w *of* Γ. Let $\delta_\gamma \subset D$ be a disk obtained by cutting D along γ whose interior is disjoint from the edges incident to D_w. We call such a disk δ_γ an *outermost disk for* (D_w, γ). (Note that δ_γ may intersects P transversely (*cf.* Figure 21).) Let $D_{w'} (\neq D_w)$ be the fat-vertex of Γ attached to γ. Then we have the following three cases.

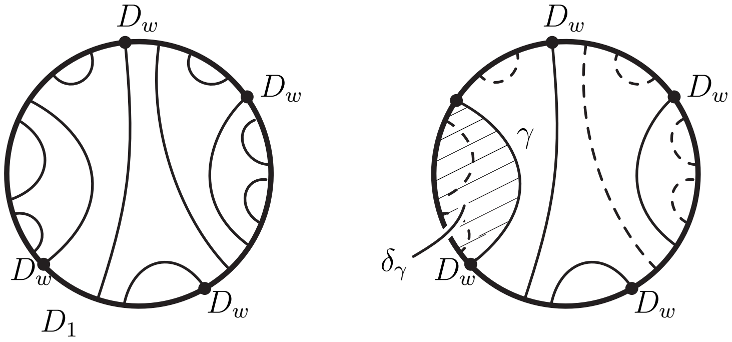

FIGURE 21

Case 1. $(\partial \delta_\gamma \setminus \gamma) \subseteq (h^1_\sigma \cap D)$.

In this case, we can isotope σ along δ_γ to reduce $|P \cap \Sigma|$ (*cf.* Figure 22).

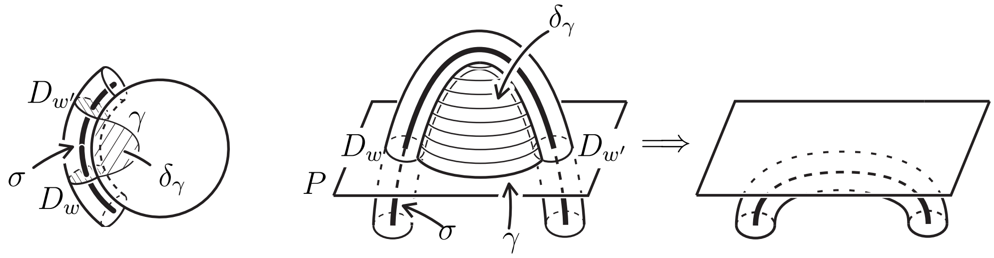

FIGURE 22

Case 2. $(\partial \delta_\gamma \setminus \gamma) \not\subseteq (h^1_\sigma \cap D)$ and $D_{w'} \not\subset (h^1_\sigma \cap D)$.

Let p be the vertex of Σ such that $p \cap \sigma \neq \emptyset$ and $h^0_p \cap \delta_\gamma \neq \emptyset$. Let β be the component of $\mathrm{cl}(\sigma \setminus D_w)$ which satisfies $\beta \cap p \neq \emptyset$. Then we can slide β along δ_γ so that β contains γ (*cf.* Figure 23). We can further isotope β slightly to reduce $|P \cap \Sigma|$, a contradiction.

Case 3. $(\partial \delta_\gamma \setminus \gamma) \not\subseteq (h^1_\sigma \cap D)$ and $D_{w'} \subset (h^1_\sigma \cap D)$.

Let p and p' be the endpoints of σ. Let β and β' be the components of $\mathrm{cl}(\sigma \setminus (D_w \cup D_{w'}))$ which satisfy $p \cap \beta \neq \emptyset$ and $p' \cap \beta' \neq \emptyset$. Suppose first that $p \neq p'$. Then we can slide β along δ_γ so that β contains γ (*cf.* Figure 24). We can further isotope β slightly to reduce $|P \cap \Sigma|$, a contradiction.

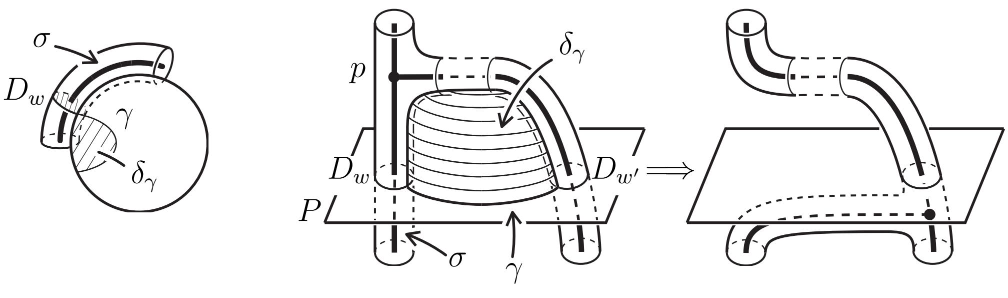

Figure 23

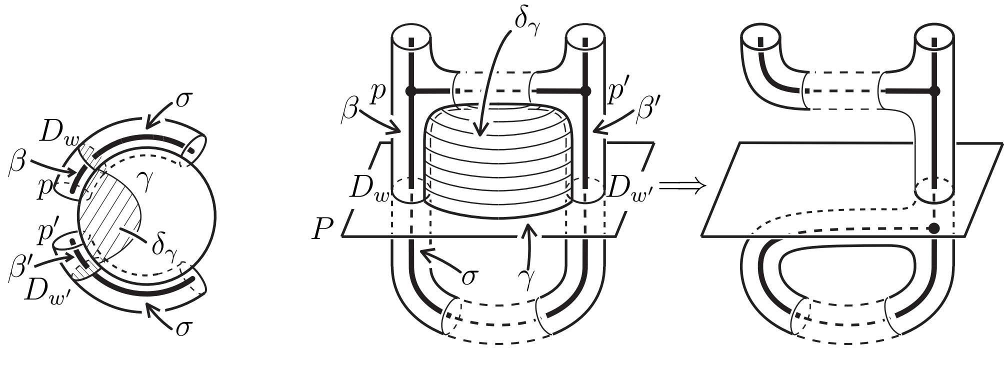

Figure 24

Suppose next that $p = p'$. In this case, we perform the following operation which is called a *broken edge slide*.

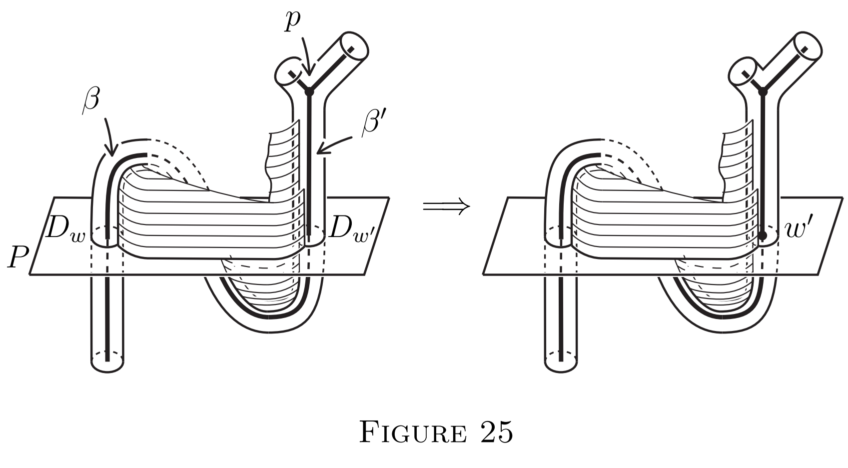

Figure 25

We first add $w' = D_{w'} \cap \Sigma$ as a vertex of Σ. Then w' cuts σ into two edges β' and $\mathrm{cl}(\sigma \setminus \beta')$. Since γ is an outermost edge for D_w of Γ, we see that $\beta' \subset \beta$ (*cf.* Figure 25). Hence we can slide $\mathrm{cl}(\beta \setminus \beta')$ along δ_γ so that $\mathrm{cl}(\beta \setminus \beta')$ contains γ. We now remove the verterx w' of Σ, that is, we regard a union of β' and $\mathrm{cl}(\sigma \setminus \beta')$ as an edge of Σ again. Then we can isotope $\mathrm{cl}(\sigma \setminus \beta')$ slightly to reduce $|P \cap \Sigma|$, a contradiction (*cf.* Figure 26).

□

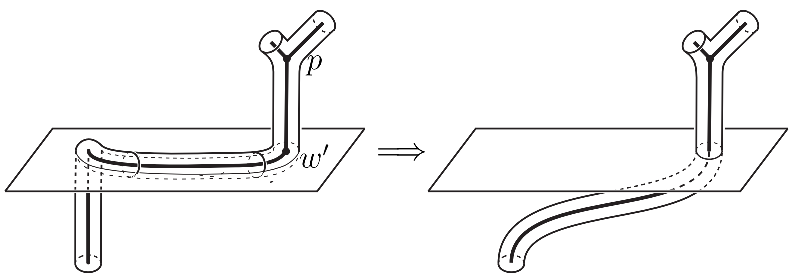

FIGURE 26

Proof of Proposition 3.2.2. By Lemma 3.2.5, if there is an isolated fat-vertex of Γ, then we have the conclusion of Proposition 3.2.2. Hence we suppose that no fat-vertices of Γ are isolated. Then it follows from Lemma 3.2.6 that each fat-vertex of Γ is a base of a loop. Let μ be a loop which is innermost in P. Then μ cuts a disk δ_μ from $\mathrm{cl}(P \setminus \Sigma_\eta)$. Since μ is essential (*cf.* Lemma 3.2.4), we see that δ_μ contains a fat-vertex of Γ. But since μ is innermost, such a fat-vertex is not a base of any loop. Hence such a fat-vertex is isolated, a contradiction. This completes the proof of Proposition 3.2.2. □

Proof of (1) in Theorem 3.2.1. Suppose that M is reducible. Then by Proposition 3.2.2, we see that $(C_1, C_2; S)$ is reducible or ∂-reducible, or C_i is reducible for $i = 1$ or 2. If $(C_1, C_2; S)$ is reducible or C_i is reducible for $i = 1$ or 2, then we are done. So we may assume that C_1 and C_2 are irreducible and that $(C_1, C_2; S)$ is ∂-reducible. By induction on the genus of the Heegaard surface S, we prove that $(C_1, C_2; S)$ is reducible.

Suppose that $genus(S) = 0$. Since C_i $(i = 1, 2)$ are irreducible, we see that each of C_i $(i = 1, 2)$ is a 3-ball. Hence M is the 3-sphere and therefore M is irreducible, a contradiction. So we may assume that $genus(S) > 0$. Let P be a ∂-reducing disk of M with $|P \cap S| = 1$. By changing subs cripts, if necessary, we may assume that $P \cap C_1 = D$ is a disk and $P \cap C_2 = A$ is a spanning annulus.

Suppose that $genus(S) = 1$. Since C_i $(i = 1, 2)$ are irreducible, we see that ∂C_i contain no 2-sphere components. Since C_1 contains an essential disk D, we see that $C_1 \cong D^2 \times S^1$. Since C_2 contains a spanning annulus A, we see that $C_2 \cong T^2 \times [0, 1]$. It follows that $M \cong D^2 \times S^1$ and hence M is irreducible, a contradiction.

Suppose that $genus(S) > 1$. Let C_1' (C_2' resp.) be the manifold obtained from C_1 (C_2 resp.) by cutting along D (A resp.), and let A^+ and A^- be copies of A in $\partial C_2'$. Then we see that C_1' consists of either a compression body or a union of two compression bodies (*cf.* (6) of Remark 3.1.3). Let C_2'' be the manifold obtained from C_2' by attaching 2-handles along A^+ and A^-. It follows from Remark 3.1.6 that C_2'' consists of either a compression body or a union of two compression bodies.

Suppose that C_1' consists of a compression body. This implies that C_2'' consists of a compression body (*cf.* Figure 27). We can naturally obtain a homeomorphism $\partial_+ C_1' \to \partial_+ C_2''$ from the homeomorphism $\partial_+ C_1 \to \partial_+ C_2$. Set $\partial_+ C_1' = \partial_+ C_2'' = S'$. Then $(C_1', C_2''; S')$ is a Heegaard splitting of the 3-manifolds M' obtained by cutting M along P. Note that $genus(S') = genus(S) - 1$. Moreover, by using innermost disk arguments, we see that M' is also reducible.

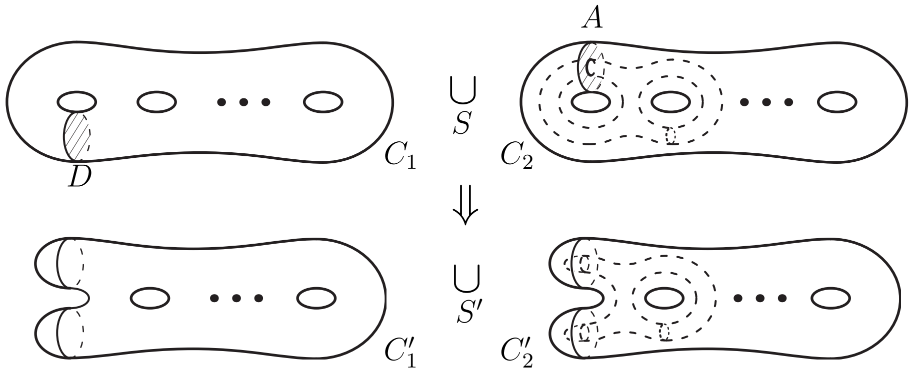

FIGURE 27

Claim.

(1) If C_1' is reducible, then C_1 is reducible.
(2) If C_2'' is reducible, then one of the following holds.
   (a) C_2 is reducible.
   (b) The component of $\partial_- C_2$ intersecting A is a torus, say T.

Proof. Exercise 3.2.7.

Recall that we assume that C_i $(i = 1, 2)$ are irreducible. Hence it follows from (1) of the claim that C_1' is irreducible. Also it follows from (2) of the claim that either (I) C_2'' is irreducible or (II) C_2'' is reducible and the condition (b) of (2) in Claim 1 holds.

Suppose that the condition (I) holds. Then by induction on the genus of a Heegaard surface, $(C_1', C_2''; S')$ is reducible, i.e., there are meridian disks D_1 and D_2 of C_1' and C_2'' respectively with $\partial D_1 = \partial D_2$. Note that this implies that C_i $(i = 1, 2)$ are non-trivial. Let α^+ and α^- be the co-cores of the 2-handles attached to C_2''. Then we see that $\alpha^+ \cup \alpha^-$ is vertical in C_2''. It follows from Lemma 3.1.8 that we may assume that $D_2 \cap (\alpha^+ \cup \alpha^-) = \emptyset$, i.e., D_2 is disjoint from the 2-handles. Hence the pair of disks D_1 and D_2 survives when we restore C_1 and C_2 from C_1' and C_2'' respectively. This implies that $(C_1, C_2; S)$ is reducible and hence we obtain the conclusion (1) of Theorem 3.2.1.

Suppose that the condition (II) holds. Then it follows from (2) of Remark 3.1.6 that there is a separating disk E_2 in C_2 such that E_2 is disjoint from A and that E_2 cuts off $T^2 \times [0, 1]$ from C_2 with $T^2 \times \{0\} = T$. Let ℓ be a loop in $S \cap (T^2 \times \{1\})$ which intersects $A \cap S (= \partial A \cap S = \partial D)$ in a single point. Let E_1 be a disk properly embedded in C_1 which is obtained from two parallel copies of D by adding a band along $\ell \setminus$ (the product region between the parallel disks). Since $genus(S) > 1$, we see that E_1 is a separating meridian disk of C_1. Since ∂E_1 is isotopic to ∂E_2, we see that $(C_1, C_2; S)$ is reducible. Hence we obtain the conclusion (1) of Theorem 3.2.1.

The case that C_1' is a union of two compression bodies is treated analogously, and we leave the proof for this case to the reader (Exercise 3.2.8). □

Exercise 3.2.7. Show the claim in the proof of Theorem 3.2.1.

Exercise 3.2.8. Prove that the conclusion (1) of Theorem 3.2.1 holds in case that C_1' consists of two compression bodies.

Proof of (2) *in Theorem 3.2.1.* Suppose that M is ∂-reducible. If $(C_1, C_2; S)$ is ∂-reducible, then we are done. Let $\widehat{C}_i$ be the compression body obtained by attaching 3-balls to the 2-sphere boundary components of C_i $(i = 1, 2)$. Set $\widehat{M} = \widehat{C}_1 \cup \widehat{C}_2$. Then $\widehat{M}$ is also ∂-reducible. Then it follows from (1) of Remark 3.1.3 and Proposition 3.2.2 that $(\widehat{C}_1, \widehat{C}_2; S)$ is reducible or ∂-reducible. If $(\widehat{C}_1, \widehat{C}_2; S)$ is ∂-reducible, then we see that $(C_1, C_2; S)$ is also ∂-reducible. Hence we may assume that $(\widehat{C}_1, \widehat{C}_2; S)$ is reducible. By induction on the genus of a Heegaard surface, we prove that $(C_1, C_2; S)$ is ∂-reducible. Let P' be a reducing 2-sphere of $\widehat{M}$ with $|P' \cap S| = 1$. For each $i = 1$ and 2, set $D_i = P' \cap \widehat{C}_i$, and let $\widehat{C}_i'$ be the manifold obtained by cutting $\widehat{C}_i$ along D_i, and let D_i^+ and D_i^- be copies of D_i in $\partial \widehat{C}_i'$. Then each of $\widehat{C}_i'$ $(i = 1, 2)$ is either (1) a compression body if D_i is non-separating in $\widehat{C}_i$ or (2) a union of two compression bodies if D_i is separating in $\widehat{C}_i$. Note that we can naturally obtain a homeomorphism $\partial_+ \widehat{C}_1' \to \partial_+ \widehat{C}_2'$ from the homeomorphism $\partial_+ \widehat{C}_1 \to \partial_+ \widehat{C}_2$. Set $\widehat{M}' = \widehat{C}_1' \cup \widehat{C}_2'$ and $\partial_+ \widehat{C}_1' = \partial_+ \widehat{C}_2' = S'$. Then $(\widehat{C}_1', \widehat{C}_2'; S')$ is either (1) a Heegaard splitting or (2) a union of two Heegaard splittings (*cf.* Figure 28).

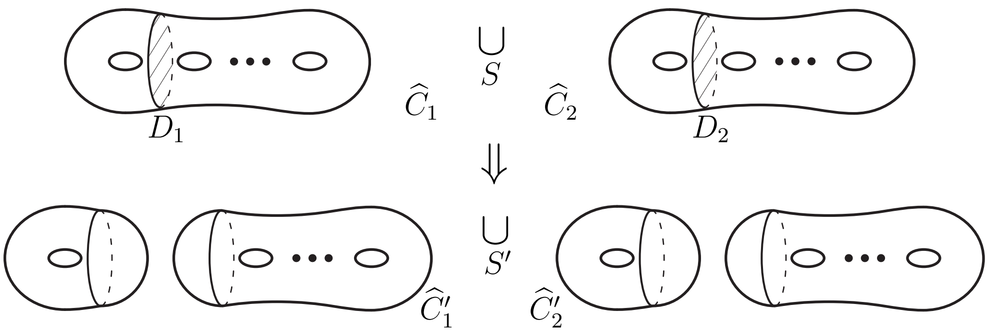

FIGURE 28

By innermost disk arguments, we see that there is a ∂-reducing disk of $\widehat{M}$ disjoint from P'. This implies that a component of $\widehat{M}'$ is ∂-reducible and hence one of the Heegaard splittings of $(\widehat{C}_1', \widehat{C}_2'; S')$ is ∂-reducible. By induction on the genus of a Heegaard surface, we see that $(\widehat{C}_1, \widehat{C}_2; S)$ is ∂-reducible. Therefore $(C_1, C_2; S)$ is also ∂-reducible and hence we have (2) of Theorem 3.2.1. □

3.3. **Waldhausen's theorem.** We devote this subsection to a simplified proof of the following theorem originally due to Waldhausen [21]. To prove the theorem, we exploit Gabai's idea of "thin position" (*cf.* [3]), Johannson's technique (*cf.* [6]) and Otal's idea (*cf.* [11]) of viewing the Heegaard splittings as a graph in the three dimensional space.

Theorem 3.3.1 (Waldhausen)**.** *Any Heegaard splitting of S^3 is standard, i.e., is obtained from the trivial Heegaard splitting by stabilization.*

Thin position of graphs in the 3-sphere. Let $\Gamma \subset S^3$ be a finite graph in which all vertices are of valence three. Let $h : S^3 \to [-1, 1]$ be a height function such that $h^{-1}(t) = P(t) \cong S^2$ for $t \in (-1, 1)$, $h^{-1}(-1) =$ (the south pole of S^3), and $h^{-1}(1) =$ (the north pole of S^3). Let $\mathcal{V}$ denote the set of vertices of Γ.

Definition 3.3.2. The graph Γ is in *Morse position* with respect to h if the following conditions are satisfied.

(1) $h|_{\Gamma\setminus\mathcal{V}}$ has finitely many non-degenerate critical points.
(2) The height of critical points of $h|_{\Gamma\setminus\mathcal{V}}$ and the vertices $\mathcal{V}$ are mutually different.

A set of the *critical heights* for Γ is the set of height at which there is either a critical point of $h|_{\Gamma\setminus\mathcal{V}}$ or a component of $\mathcal{V}$. We can deform Γ by an isotopy so that a regular neighborhood of each vertex v of Γ is either of *Type-y* (i.e., two edges incident to v is above v and the remaining edge is below v) or of *Type-λ* (i.e., two edges incident to v is below v and the remaining edge is above v). Such a graph is said to be in *normal form.* We call a vertex v a *y-vertex* (a *λ-vertex* resp.) if $\eta(v; \Gamma)$ is of Type-y (Type-λ resp.).

Suppose that Γ is in Morse position and in normal form. Note that $\eta(\Gamma; S^3)$ can be regarded as the union of 0-handles corresponding to the regular neighborhood of the vertices and 1-handles corresponding to the regular neighborhood of the edges. A simple loop α in $\partial\eta(\Gamma; S^3)$ is in *normal form* if the following conditions are satisfied.

(a) For each 1-handle ($\cong D^2 \times [0, 1]$), each component of $\alpha \cap (\partial D^2 \times [0, 1])$ is an essential arc in the annulus $\partial D^2 \times [0, 1]$.
(b) For each 0-handle ($\cong B^3$), each component of $\alpha \cap \partial B^3$ is an arc which is essential in the 2-sphere with three holes $\mathrm{cl}(\partial B^3 \setminus$(the 1-handles incident to B^3)).

Let D be a disk properly embedded in $\mathrm{cl}(S^3 \setminus \eta(\Gamma; S^3))$. We say that D is in *normal form* if the following conditions are satisfied.

(1) ∂D is in normal form.
(2) Each critical point of $h|_{\mathrm{int}(D)}$ is non-degenerate.
(3) No critical points of $h|_{\mathrm{int}(D)}$ occur at critical heights of Γ.
(4) No two critical points of $h|_{\mathrm{int}(D)}$ occur at the same height.
(5) $h|_{\partial D}$ is a Morse function on ∂D satisfying the following (*cf.* Figure 29).
    (a) Each minimum of $h|_{\partial D}$ occurs either at a y-vertex in "half-center" singularity or at a minimum of Γ in "half-center" singularity.
    (b) Each maximum of $h|_{\partial D}$ occurs either at a λ-vertex in "half-center" singularity or at a maximum of Γ in "half-center" singularity.

By Morse theory (*cf.* [9]), it is known that D can be put in normal form.

Recall that $h : S^3 \to [-1, 1]$ is a height function such that $h^{-1}(t) = P(t) \cong S^2$ for $t \in (-1, 1)$, $h^{-1}(-1) =$ (the south pole of S^3), and $h^{-1}(1) =$ (the north pole of S^3). We isotope Γ to be in Morse position and in normal form. For $t \in (-1, 1)$, set $w_\Gamma(t) = |P(t) \cap \Gamma|$. Note that $w_\Gamma(t)$ is constant on each component of $(-1, 1) \setminus$(the critical heights of Γ). Set $W_\Gamma = \max\{w_\Gamma(t) | t \in (-1, 1)\}$ (*cf.* Figure 30).

Let n_Γ be the number of the components of $(-1, 1) \setminus$ (the critical heights of Γ) on which the value W_Γ is attained.

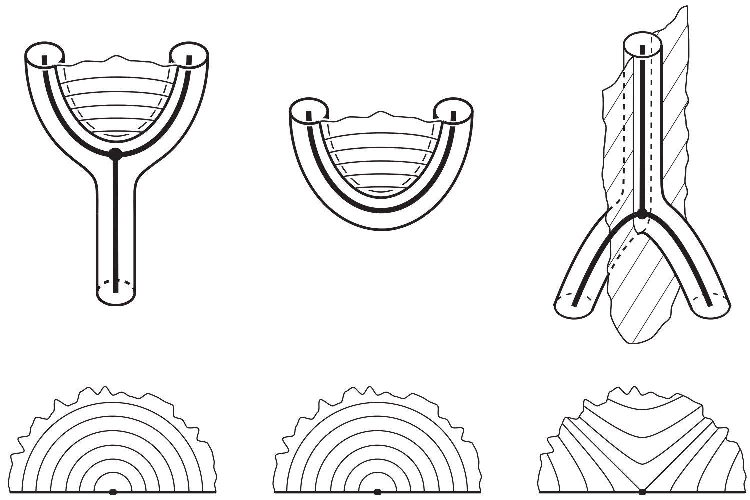

FIGURE 29

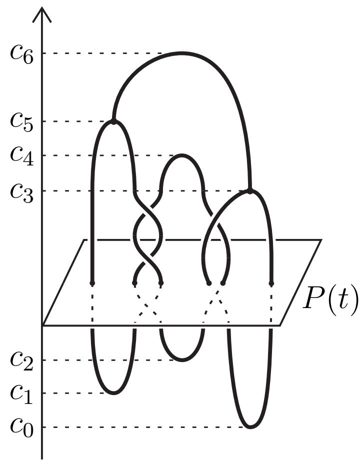

FIGURE 30

Definition 3.3.3. A graph $\Gamma \subset S^3$ is said to be in *thin position* if (W_Γ, n_Γ) is minimal with respect to lexicographic order among all graphs which are obtained from Γ by ambient isotopies and edge slides and are in Morse position and in normal form.

A proof of Theorem 3.3.1. Let $(C_1, C_2; S)$ be a genus $g > 0$ Heegaard splitting of S^3. Let Σ be a trivalent spine of C_1. Note that $\eta(\partial_- C_1 \cup \Sigma; M)$ is obtained from regular neighborhoods of $\partial_- C_1$ and the vertices of Σ by attaching 1-handles corresponding to the edges of Σ. Set $\Sigma_\eta = \eta(\Sigma; M)$. As in Section 3.2, the notation h^0_v, called a *vertex* of Σ_η, means a regular neighborhood of a vertex v of Σ. Also, the notation h^1_σ, called an *edge* of Σ_η, means a 1-handle corresponding to an edge σ of Σ. Let Δ_1 (Δ_2 resp.) be a complete meridian system of C_1 (C_2 resp.).

Proposition 3.3.4. *There is an edge of* Σ_η *which is disjoint from* Δ_2*, or* Σ *is modified by edge slides so that the modified graph contains an unknotted cycle (i.e., the modified graph contains a graph* α *so that* α *bounds a disk in* S^3*).*

Proof of Theorem 3.3.1 via Proposition 3.3.4. We prove Theorem 3.3.1 by induction on the genus of a Heegaard surface. If $genus(S) = 0$, then $(C_1, C_2; S)$ is standard (*cf.* Definition 3.1.18). So we may assume that $genus(S) > 0$ for a Heegaard splitting $(C_1, C_2; S)$.

Suppose first that Σ has an unknotted cycle α. Then $\eta(\alpha; C_1)$ is a standard solid torus in S^3, that is, the exterior of $\eta(\alpha; C_1)$ is a solid torus. Since $C_1^- = \mathrm{cl}(C_1 \setminus \eta(\alpha; C_1))$ is a compression body, we see that $(C_1^-, C_2; S)$ is a Heegaard splitting of the solid torus $\mathrm{cl}(S^3 \setminus \eta(\alpha; C_1))$. Since a solid torus is ∂-reducible, $(C_1^-, C_2; S)$ is ∂-reducible by Theorem 3.2.1, that is, there is a ∂-reducing disk D_α for $(C_1^-, C_2; S)$ with $|D_\alpha \cap S| = 1$. Since $\eta(\alpha; C_1)$ is a standard solid torus in S^3, D_α intersects a meridian disk D'_α of $\eta(\alpha; C_1)$ transversely in a single point. Set $D_2 = D_\alpha \cap C_2$. Then by extending D'_α, we obtain a meridian disk D_1 of C_1 such that ∂D_1 intersects ∂D_2 transversely in a single point, i.e., D_1 and D_2 give stabilization of $(C_1, C_2; S)$. Hence we obtain a Heegaard splitting $(C'_1, C'_2; S')$ with $genus(S') < genus(S)$ (*cf.* Figure 31). By induction on the genus of a Heegaard surface, we can see that $(C_1, C_2; S)$ is standard.

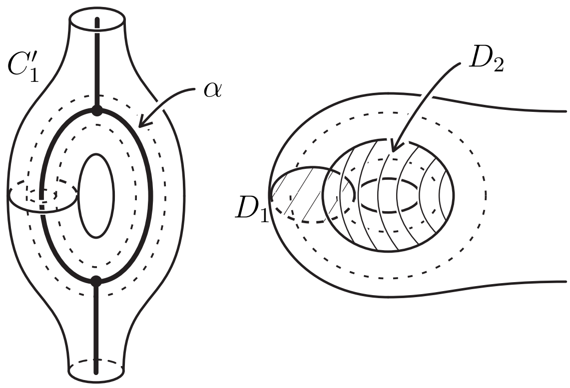

FIGURE 31

Suppose next that there is an edge σ of Σ with $h^1_\sigma \cap \Delta_2 = \emptyset$. Let D_σ be a meridian disk of C_1 which is co-core of the 1-handle h^1_σ. Note that $D_\sigma \cap \Delta_2 = \emptyset$. Cutting C_2 along Δ_2, we obtain a union of 3-balls and hence we see that ∂D_σ bounds a disk, say D'_σ, properly embedded in one of the 3-balls. Note that D'_σ corresponds to a meridian disk of C_2. Hence we see that $(C_1, C_2; S)$ is reducible. It follows from a generalized Schönflies theorem that every 2-sphere in S^3 separates it into two 3-balls (*cf.* Section 2.F.5 of [13]). Hence by cutting S^3 along the reducing 2-sphere and capping off 3-balls, we obtain two Heegaard splittings of S^3 such that the genus of each Heegaard surface is less than that of S. Then we see that $(C_1, C_2; S)$ is standard by induction on the genus of a Heegaard surface. □

In the remainder, we prove Proposition 3.3.4. Let $h : S^3 \to [-1, 1]$ be a height function such that $h^{-1}(t) = P(t) \cong S^2$ for $t \in (-1, 1)$, $h^{-1}(-1) =$ (the south pole of S^3), and $h^{-1}(1) =$ (the north pole of S^3). We may assume that Σ is in thin position. We also assume that each component of Δ_2 is in normal form, Δ_1 intersects Δ_2 transversely and $|\Delta_1 \cap \Delta_2|$ is minimal.

For the proof of Proposition 3.3.4, it is enough to show the following; if there are no edges of Σ_η which are disjoint from Δ_2, then Σ is modified by edge slides so that the modified graph contains an unknotted cycle. Hence we suppose that there are no edges of Σ_η which are disjoint from Δ_2. Set $\Lambda(t) = P(t) \cap (\Sigma_\eta \cup \Delta_2)$. We note that $P(t)$, Σ and Δ_2 intersect transversely at a regular height t. In the following, we mainly consider such a regular height t with $\Lambda(t) \neq \emptyset$ unless otherwise denoted. We also note that we may assume that $\Lambda(t)$ does not contain a loop component by an argument similar to the proof of Lemma 3.2.3. Hence $\Lambda(t)$ is regarded as a graph in $P(t)$ which consists of *fat-vertices* $P(t) \cap \Sigma_\eta$ and edges $P(t) \cap \Delta_2$.

Lemma 3.3.5. *If there is a fat-vertex of* $\Lambda(t)$ *with valence less than two, then* Σ *is modified by edge slides so that the modified graph contains an unknotted cycle.*

Proof. Suppose that there is a fat-vertex D_v of $\Lambda(t)$ with valence less than two. Let σ be the edge of Σ with $h^1_\sigma \supset D_v$ and p one of the endpoints of σ. Since we assume that there are no edges of Σ_η which are disjoint from Δ_2, we see that any fat-vertex of $\Lambda(t)$ is of valence greater than zero. Hence D_v is of valence one. Then there is the disk component D of Δ_2 with $h^1_\sigma \cap D \neq \emptyset$. Since ∂D intersects the fat-vertex D_v in a single point and hence ∂D intersects h^1_σ in a single arc, we can perform an edge slide on σ along $\mathrm{cl}(\partial D \setminus h^1_\sigma)$ to obtain a new graph Σ' from Σ (*cf.* Figure 32). Clearly, Σ' contains an unknotted cycle (bounding a disk corresponding to D_2).

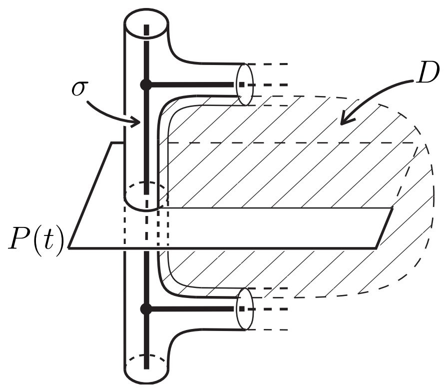

FIGURE 32

□

An edge of a graph $\Lambda(t)$ is said to be *simple* if the edge joins distinct two fat-vertices of $\Lambda(t)$. Recall that an edge of a graph $\Lambda(t)$ is called a *loop* if the edge is not simple.

Lemma 3.3.6. *Suppose that there are no fat-vertices of valence less than two. Then there exists a fat-vertex* D_w *of* $\Lambda(t)$ *such that any outermost edge for* D_w *of* $\Lambda(t)$ *is simple.*

Proof. If $\Lambda(t)$ does not contain a loop, then we are done. So we may assume that $\Lambda(t)$ contains a loop, say μ. Let D_v be a fat-vertex of $\Lambda(t)$ which is a bese of μ. By an argument similar to the proof of Lemma 3.2.4, we can see that μ cuts $\text{cl}(P(t) \setminus D_v)$ into two disks, and each of the two disks contains a fat-vertex of $\Lambda(t)$. Let μ_0 be a loop of $\Lambda(t)$ which is innermost in $P(t)$. Let D_w be a fat-vertex contained in the interior of the innermost disk bounded by μ_0. Note that D_w is not isolated and that every edge contained in the interior of the innermost disk is simple. Hence any outermost edge for D_w of $\Lambda(t)$ is simple. □

Let D_w be a fat-vertex of $\Lambda(t)$ with a simple edge $\gamma(\subset \Lambda(t))$. We may assume that γ is a simple outermost edge for D_w of $\Lambda(t)$ and γ is contained in a disk component D of Δ_2. It follows from Lemma 3.3.6 that we can always find such a fat-vertex D_w and an edge γ if each fat-vertex of $\Lambda(t)$ is of valence greater than one. Let δ_γ be the outermost disk for (D_w, γ). We say an outermost edge γ is *upper* (*lower* resp.) if $\eta(\gamma; \delta_\gamma)$ is above (below resp.) γ with respect to the height function h. Let t_0 be a regular height with $w_\Sigma(t_0) = W_\Sigma$.

Lemma 3.3.7. *Let D_w be a fat-vertex of $\Lambda(t_0)$ with a simple outermost edge for D_w of $\Lambda(t)$. Then we have one of the following.*

(1) *All the simple outermost edges for D_w of $\Lambda(t)$ are either upper or lower.*
(2) *Σ is modified by edge slides so that the modified graph contains an unknotted cycle.*

Proof. Suppose that $\Lambda(t)$ contains simple outermost edges for D_w, say γ and γ', such that γ is upper and γ' is lower. For the proof of Lemma 3.3.7, it is enough to show that Σ is modified by edge slides so that there is an unknotted cycle. Let δ_γ and $\delta_{\gamma'}$ be the outermost disk for (D_w, γ) and $(D_{w'}, \gamma')$ respectively. Let σ be the edge of Σ with $h^1_\sigma \supset D_w$. Let $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) be a union of the components obtained by cutting σ by the two fat-vertices of $\Lambda(t_0)$ incident to γ (γ' resp.) such that a 1-handle correponding to each component intersects $\partial\delta_\gamma \setminus \gamma$ ($\partial\delta_{\gamma'} \setminus \gamma'$ resp.). We note that $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) satisfies one of the following conditions.

(1) $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) consists of an arc such that $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) and σ share a single endpoint.
(2) $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) consists of an arc with $\bar{\gamma} \subset \text{int}(\sigma)$ ($\bar{\gamma}' \subset \text{int}(\sigma)$ resp.).
(3) $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) consists of two subarcs of σ such that each component of $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) and σ share a single endpoint.

In each of the conditions above, corresponding figures are illustrated in Figure 33.

Case (1)-(1). Both $\bar{\gamma}$ and $\bar{\gamma}'$ satisfy the condition (1).

If the endpoints of γ are the same as those of γ', then we can slide $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.) and hence we obtain an unknotted cycle (*cf.* Figure 34). Otherwise, we can perform a Whitehead move on Σ to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 35).

Case (1)-(2). Either $\bar{\gamma}$ or $\bar{\gamma}'$, say $\bar{\gamma}$, satisfies the condition (1) and $\bar{\gamma}'$ satisfies the condition (2).

Then we can slide $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.). Then Σ is further isotoped to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 36).

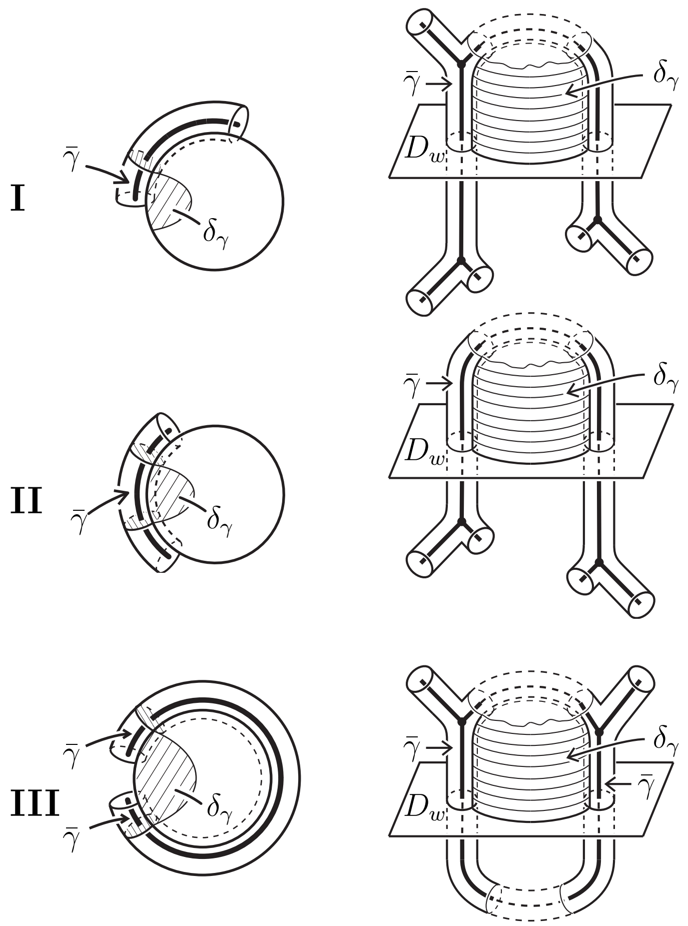

FIGURE 33

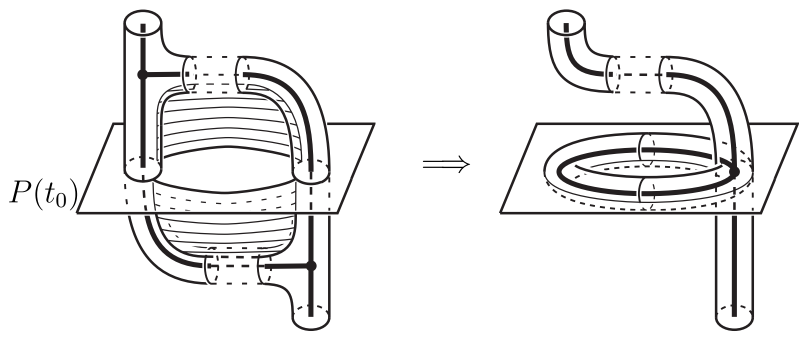

FIGURE 34

Case (1)-(3). Either $\bar{\gamma}$ or $\bar{\gamma}'$, say $\bar{\gamma}$, satisfies the condition (1) and $\bar{\gamma}'$ satisfies the condition (3).

Let $\bar{\gamma}'_1$ and $\bar{\gamma}'_2$ be the components of $\bar{\gamma}'$ with $h^1_{\bar{\gamma}'_1} \supset D_w$. Note that $\bar{\gamma} \supset \bar{\gamma}'_2$ and hence $\text{int}(\bar{\gamma}) \supset \partial\bar{\gamma}'_2$. This implies that $\text{int}(\bar{\gamma}) \cap P(t_0) \neq \emptyset$. Hence we can slide $\bar{\gamma}$

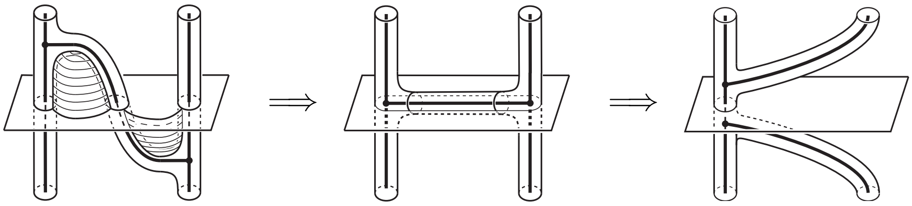

FIGURE 35

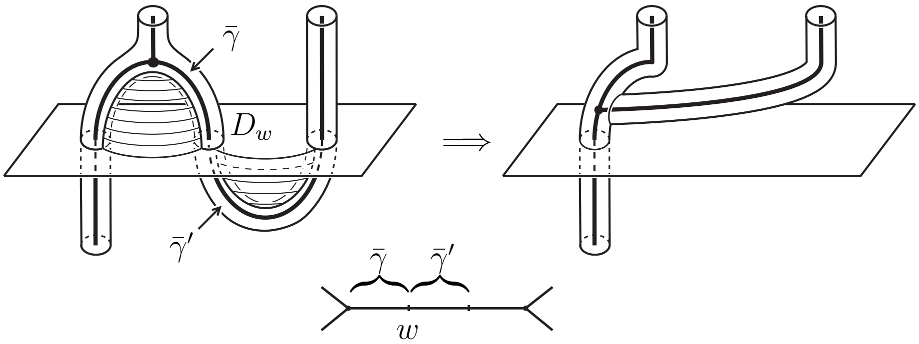

FIGURE 36

to γ along the disk δ_γ. We can further isotope Σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 37).

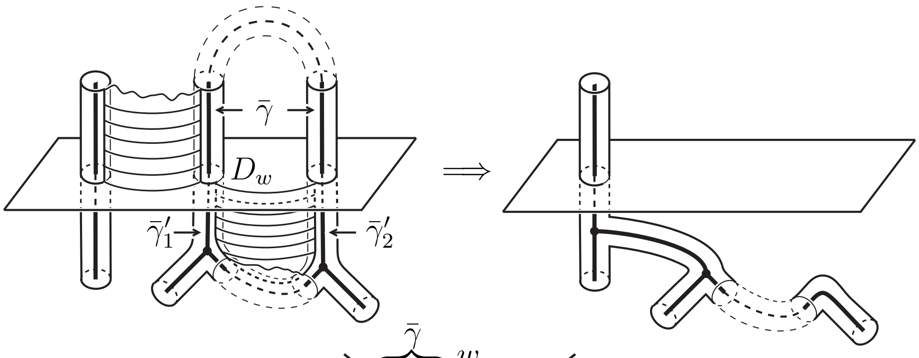

FIGURE 37

Case (2)-(2). Both $\bar{\gamma}$ and $\bar{\gamma}'$ satisfy the condition (2).

Then we can slide $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.). Moreover, we can isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 59).

Case (2)-(3). Either $\bar{\gamma}$ or $\bar{\gamma}'$, say $\bar{\gamma}$, satisfies the condition (2) and $\bar{\gamma}'$ satisfies the condition (3).

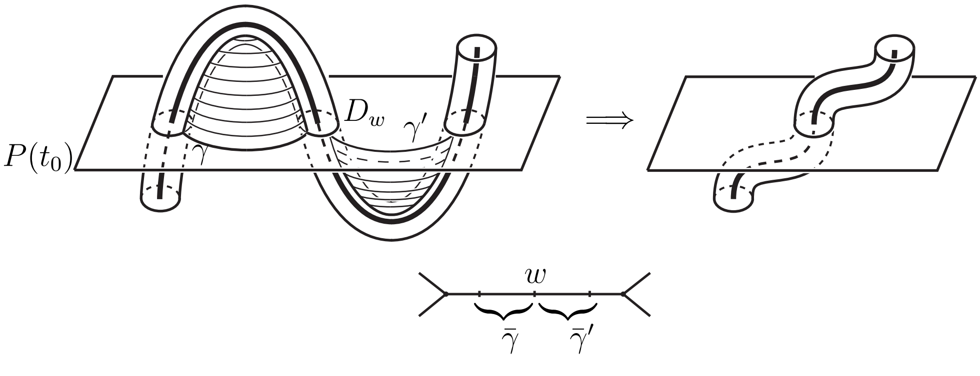

FIGURE 38

Note that $\bar{\gamma}'$ consists of two arcs, say $\bar{\gamma}'_1$ and $\bar{\gamma}'_2$, with $\bar{\gamma}'_1 \cap \bar{\gamma} = D_w$. Then we have the following cases.

(i) $\bar{\gamma}'_2$ is disjoint from $\bar{\gamma}$. In this case, we can slide $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.). Moreover, we can isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 39).

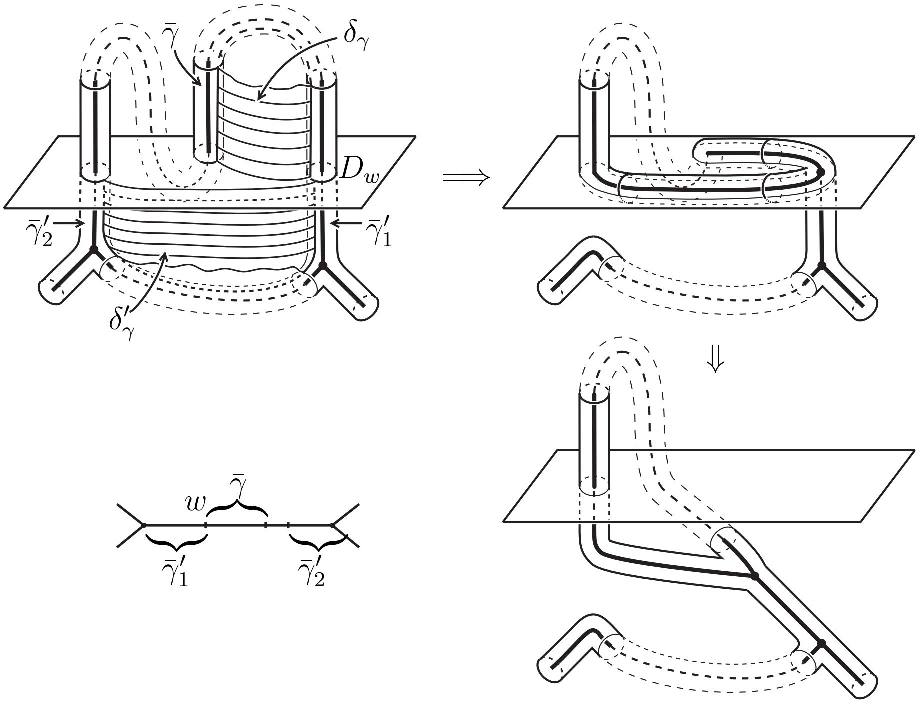

FIGURE 39

(ii) $\bar{\gamma}'_2 \cap \bar{\gamma}$ consists of a point, i.e., $\bar{\gamma}'_2$ and $\bar{\gamma}$ share one endpoint. Note that $\bar{\gamma} \cup \bar{\gamma}' = \sigma$ and $\bar{\gamma} \cap \bar{\gamma}' = \partial\bar{\gamma}$. In this case, we can slide $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.) and hence we obtain an unknotted cycle (*cf.* Figure 62).

(iii) $\bar{\gamma}'_2 \cap \bar{\gamma}$ consists of an arc. In this case, we can slide $\bar{\gamma}$ to γ along the disk δ_γ. Since $\bar{\gamma}'_2 \cap \bar{\gamma}$ consists of an arc, $\bar{\gamma}$ contains at least three critical points. Hence

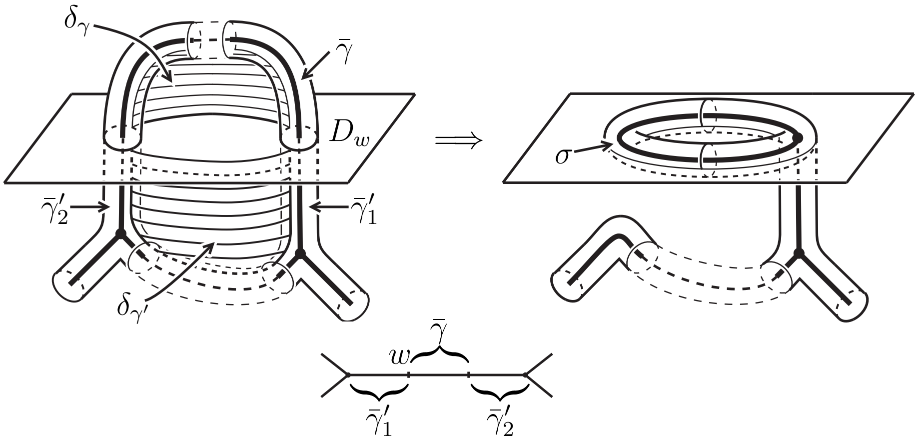

FIGURE 40

we can further isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 41).

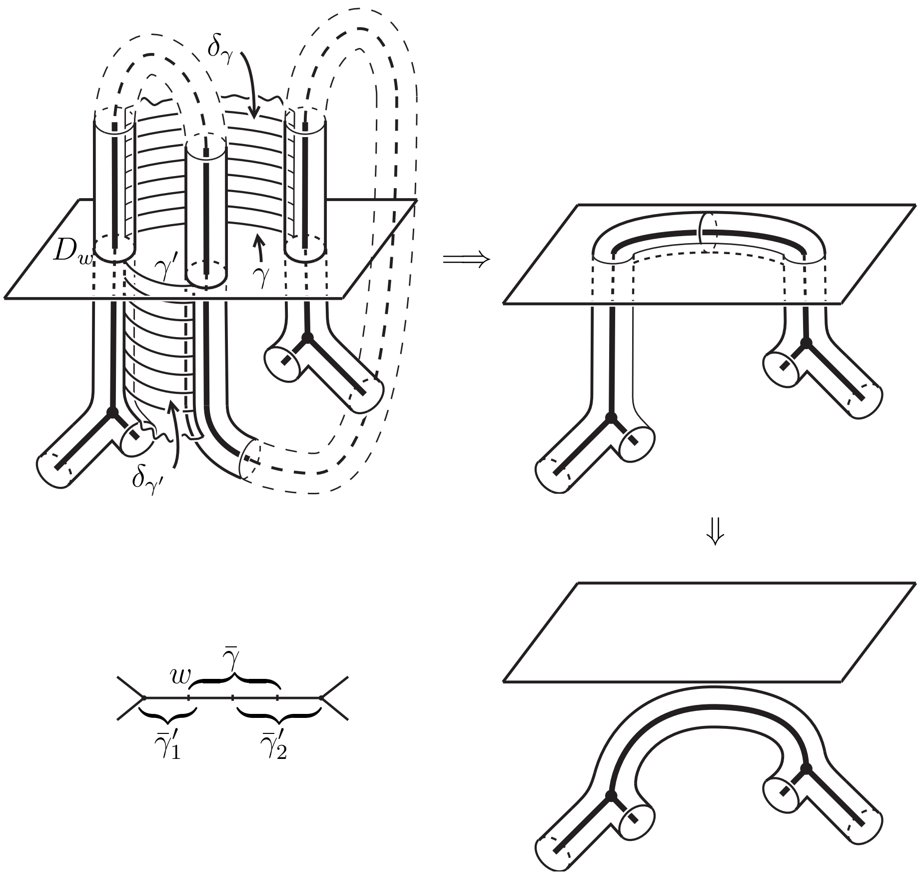

FIGURE 41

Case (3)-(3). Both $\bar{\gamma}$ and $\bar{\gamma}'$ satisfy the condition (3).

Let $\bar{\gamma}_1$ and $\bar{\gamma}_2$ ($\bar{\gamma}'_1$ and $\bar{\gamma}'_2$ resp.) be the components of $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) with $h^1_{\bar{\gamma}_1} \supset D_w$ ($h^1_{\bar{\gamma}'_1} \supset D_w$ resp.). Note that $\bar{\gamma}_1 \supset \bar{\gamma}'_2$ and $\bar{\gamma}'_1 \supset \bar{\gamma}_2$. In this case, we can slide $\bar{\gamma}'_1$ to γ' along the disk $\delta_{\gamma'}$. Since $\bar{\gamma}'_1 \supset \bar{\gamma}_2$, $\bar{\gamma}'_1$ contains at least one critical point. Hence we can further isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 42).

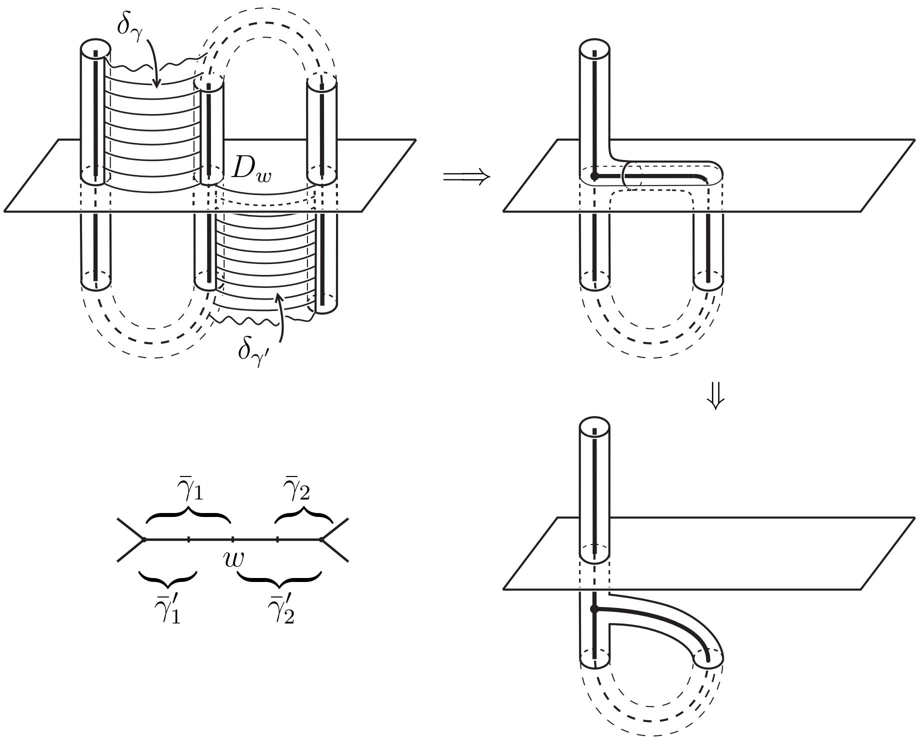

FIGURE 42

□

Suppose that D_w is a fat-vertex of $\Lambda(t_0)$ such that there are no loops based on D_w. It follows from Lemma 3.3.7 that all the simple outermost edges for D_w of $\Lambda(t_0)$ are either upper or lower.

Lemma 3.3.8. *Suppose that all of the simple outermost edges for D_w of $\Lambda(t_0)$ are upper (lower resp.). Then one of the following holds.*

(1) *For each fat-vertex $D_{w'}$ of $\Lambda(t_0)$, every simple outermost edges for $D_{w'}$ of $\Lambda(t_0)$ is upper (lower resp.).*
(2) *Σ is modified by edge slides so that the modified graph contains an unknotted cycle.*

Proof. Since the arguments are symmetric, we may suppose that all the simple outermost edges for D_w of $\Lambda(t_0)$ are upper. Let γ be a simple outermost edge for D_w of $\Lambda(t_0)$. Note that γ is upper. Suppose that there is a fat-vertex $D_{w'}$ such that $\Lambda(t_0)$ contains a lower simple outermost edge γ' for D_w. Let δ_γ ($\delta_{\gamma'}$ resp.) be the outermost disk for (D_w, γ) ($(D_{w'}, \gamma')$ resp.). Let σ (σ' resp.) be the edge of Σ with $h^1_\sigma \supset D_w$ ($h^1_{\sigma'} \supset D_{w'}$ resp.). Let $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) be a union of the components obtained by cutting σ by the two fat-vertices of $\Lambda(t_0)$ incident to γ (γ' resp.) such that a 1-handle correponding to each component intersects $\partial\delta_\gamma \setminus \gamma$ ($\partial\delta_{\gamma'} \setminus \gamma'$ resp.). Then $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) satisfies one of the conditions (1), (2) and (3) in the proof of Lemma 3.3.7. The proof of Lemma 3.3.8 is divided into the following cases.

Case A. $\bar{\gamma} \cap \bar{\gamma}' = \emptyset$.

Then we have the following six cases. In each case, we can slide (a component of) $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.). Moreover, we can isotope σ and σ' slightly to reduce (W_Σ, n_Σ) is reduced, a contradiction.

Case A-(1)-(1). Both $\bar{\gamma}$ and $\bar{\gamma}'$ satisfy the condition (1).

See Figure 43.

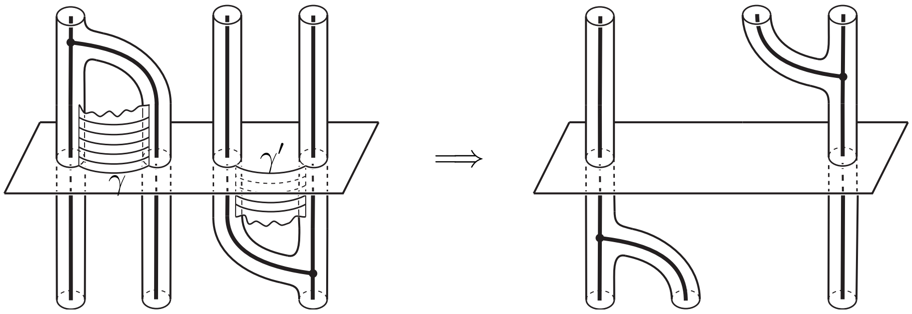

FIGURE 43

Case A-(1)-(2). Either $\bar{\gamma}$ or $\bar{\gamma}'$, say $\bar{\gamma}$, satisfies the condition (1) and $\bar{\gamma}'$ satisfies the condition (2).

See Figure 44.

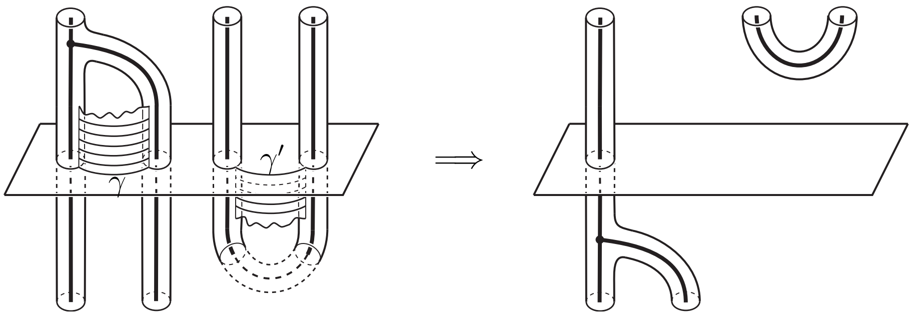

FIGURE 44

Case A-(1)-(3). Either $\bar{\gamma}$ or $\bar{\gamma}'$, say $\bar{\gamma}$, satisfies the condition (1) and $\bar{\gamma}'$ satisfies the condition (3).

See Figure 45.

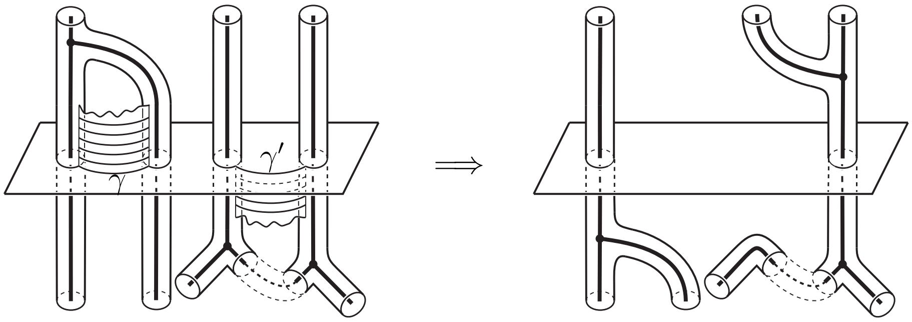

Figure 45

Case A-(2)-(2). Both $\bar{\gamma}$ and $\bar{\gamma}'$ satisfy the condition (2).

See Figure 46.

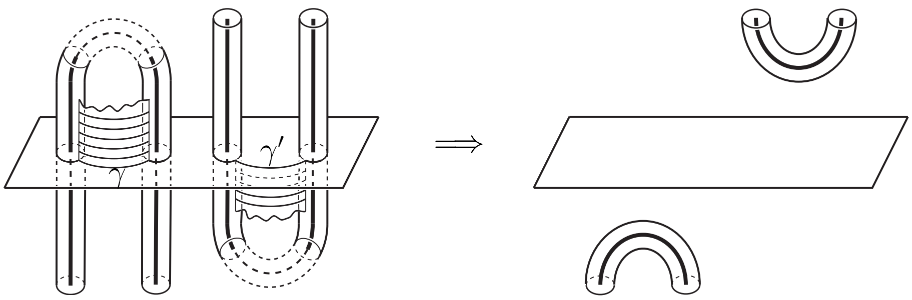

Figure 46

Case A-(2)-(3). Either $\bar{\gamma}$ or $\bar{\gamma}'$, say $\bar{\gamma}$, satisfies the condition (2) and $\bar{\gamma}'$ satisfies the condition (3).

See Figure 47.

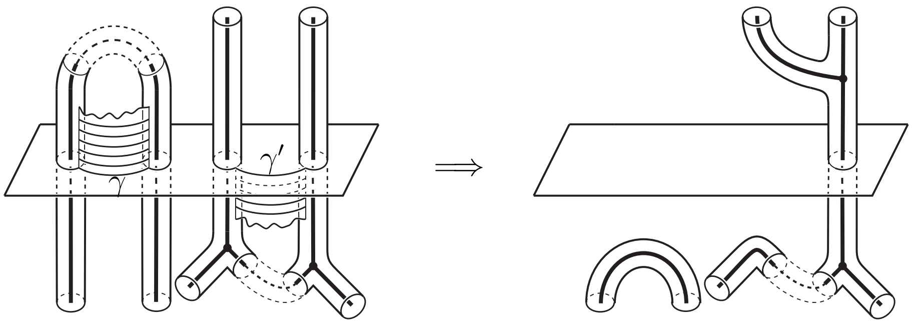

Figure 47

Case A-(3)-(3). Both $\bar{\gamma}$ and $\bar{\gamma}'$ satisfy the condition (3).

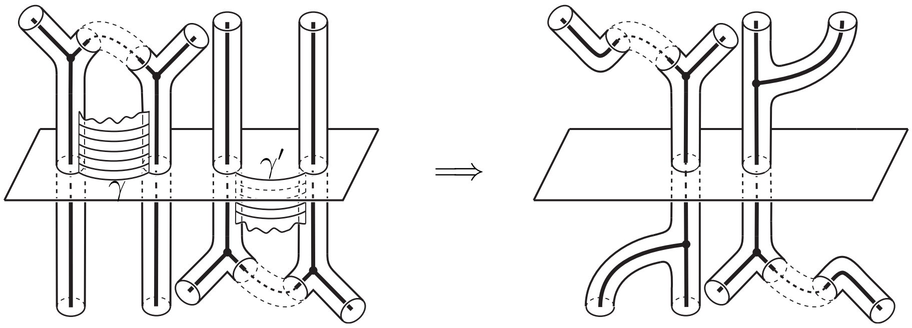

Figure 48

See Figure 48.

Case B. $\bar{\gamma} \cap \bar{\gamma}' \neq \emptyset$.

Case B-(1)-(1). Both $\bar{\gamma}$ and $\bar{\gamma}'$ satisfy the condition (1).

We first suppose that $\mathrm{int}(\bar{\gamma}) \cap \mathrm{int}(\bar{\gamma}') = \emptyset$. Then we can slide $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.). If $\partial\gamma = \partial\gamma' (= \{w, w'\})$, then $\bar{\gamma} \cup \bar{\gamma}'$ composes an unknotted cycle and hence Lemma 3.3.8 holds (*cf.* Figure 49). Otherwise, we can perform a Whitehead move on Σ and hence we can reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 50).

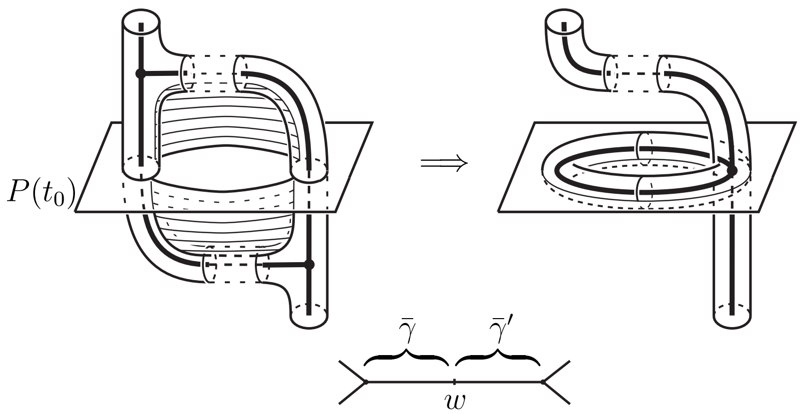

Figure 49

We next suppose that $\mathrm{int}(\bar{\gamma}) \cap \mathrm{int}(\bar{\gamma}') \neq \emptyset$. Then there are two possibilities: (1) $\bar{\gamma} \subset \bar{\gamma}'$ or $\bar{\gamma}' \subset \bar{\gamma}$, say the latter holds and (2) $\bar{\gamma} \not\subset \bar{\gamma}'$ and $\bar{\gamma}' \not\subset \bar{\gamma}$. In each case, we can slide $\bar{\gamma}$ to γ along the disk δ_γ. Moreover, we can isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figures 51 and 52).

Case B-(1)-(2). Either $\bar{\gamma}$ or $\bar{\gamma}'$, say $\bar{\gamma}$, satisfies the condition (1) and $\bar{\gamma}'$ satisfies the condition (2).

We first suppose that $\mathrm{int}(\bar{\gamma}) \cap \mathrm{int}(\bar{\gamma}') = \emptyset$. Then we can slide $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.). Moreover, we can isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 53).

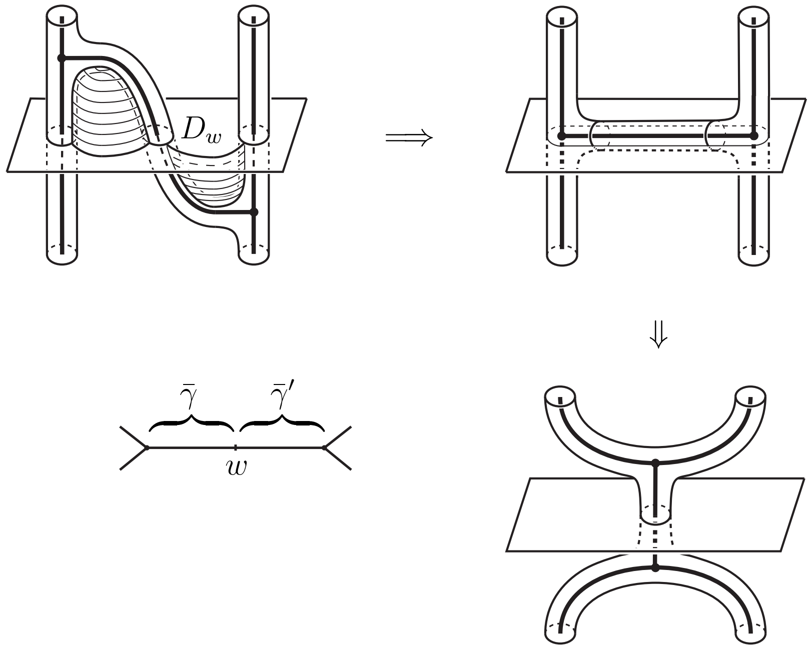

FIGURE 50

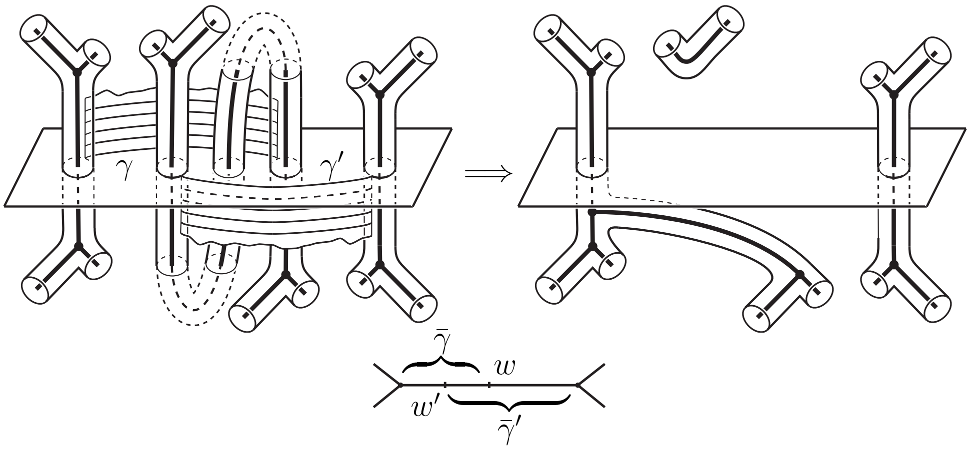

FIGURE 51

We next suppose that $\mathrm{int}(\bar\gamma) \cap \mathrm{int}(\bar\gamma') \neq \emptyset$. Note that it is impossible that $\bar\gamma \subset \bar\gamma'$. Hence there are two possibilities: $\bar\gamma' \subset \bar\gamma$ and $\bar\gamma' \not\subset \bar\gamma$. In each case, we can slide $\bar\gamma$ to γ along the disk δ_γ. Moreover, we can isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figures 54 and 55).

Case B-(1)-(3). Either $\bar\gamma$ or $\bar\gamma'$, say $\bar\gamma$, satisfies the condition (1) and $\bar\gamma'$ satisfies the condition (3).

Let $\bar\gamma'_1$ and $\bar\gamma'_2$ be the components of $\bar\gamma'$ with $h^1_{\bar\gamma'_1} \supset D_{w'}$.

We first suppose that $\bar\gamma \subset \bar\gamma'_1$. Then we can slide $\bar\gamma'_1$ into γ' along the disk $\delta_{\gamma'}$. Moreover, we can isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 56).

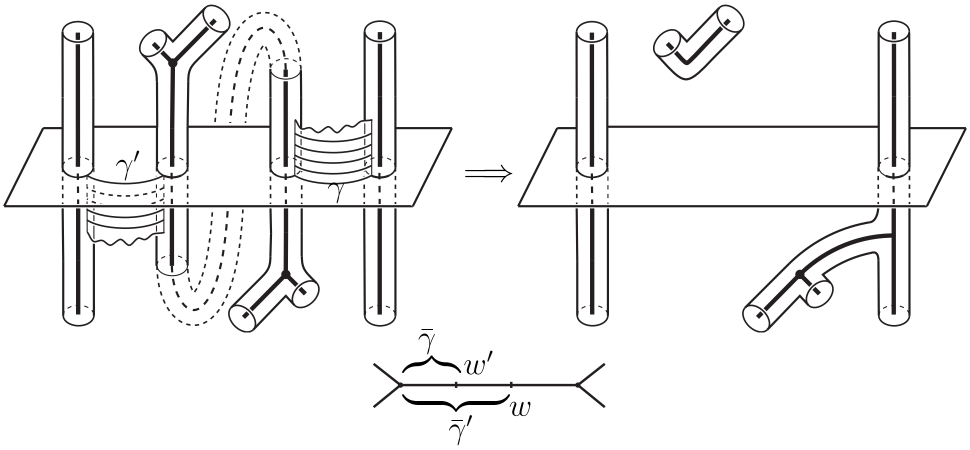

FIGURE 52

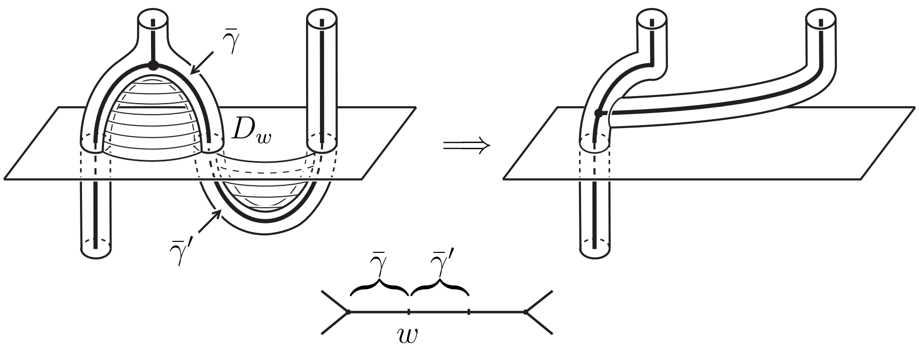

FIGURE 53

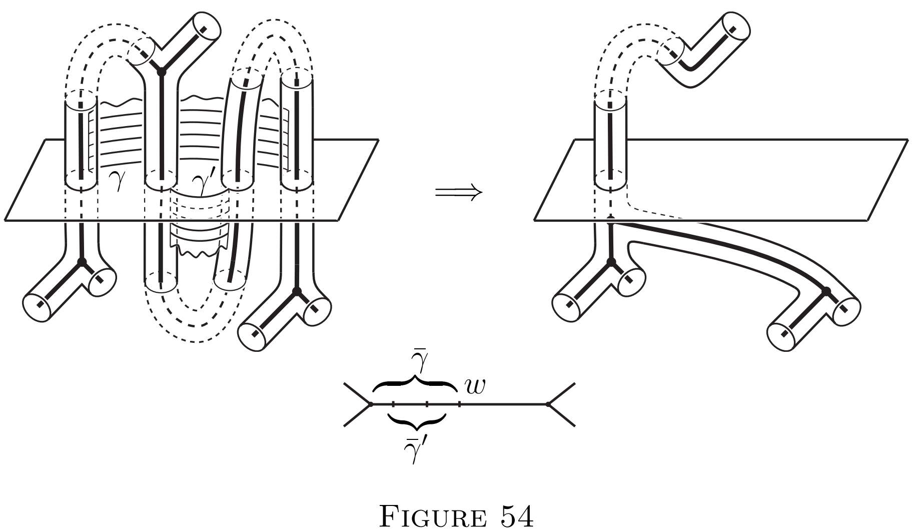

FIGURE 54

We next suppose that $\bar{\gamma}'_1 \subset \bar{\gamma}$. Then there are two possibilities: $\bar{\gamma}'_2 \cap \bar{\gamma} = \emptyset$ and $\bar{\gamma}'_2 \cap \bar{\gamma} \neq \emptyset$. In each case, we can slide $\bar{\gamma}$ to γ along the disk δ_γ. Moreover, we can isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figures 57 and 58).

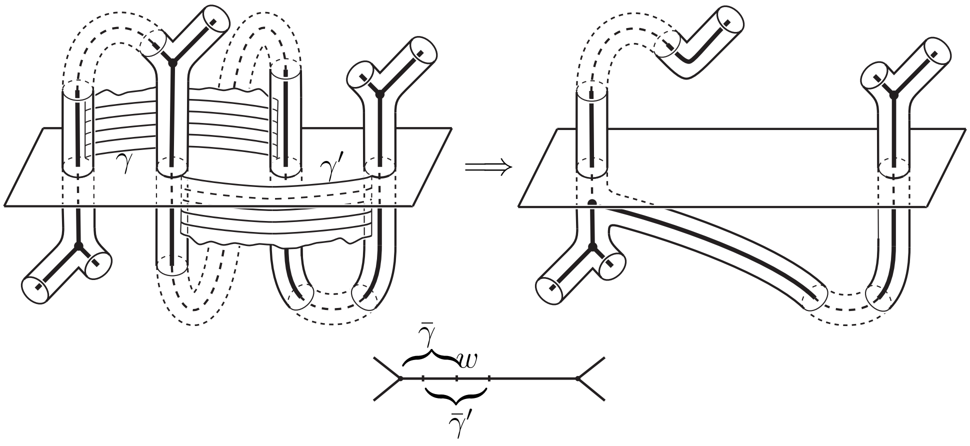

FIGURE 55

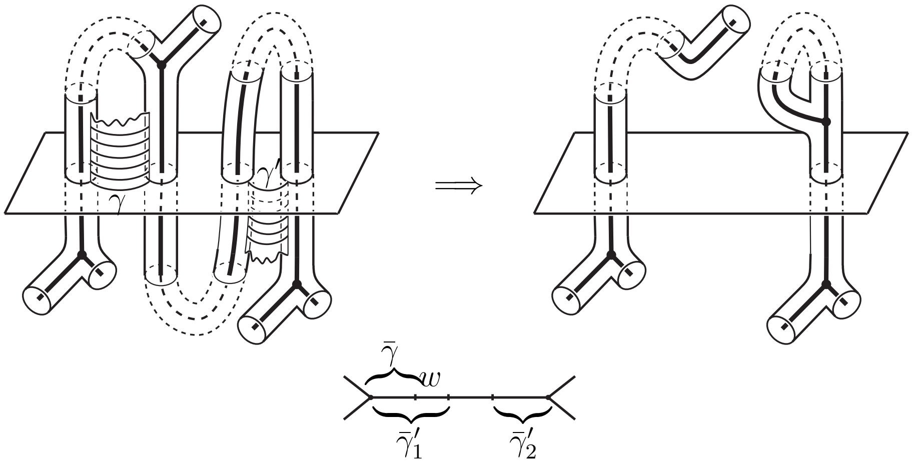

FIGURE 56

Case B-(2)-(2). Both $\bar{\gamma}$ and $\bar{\gamma}'$ satisfy the condition (2).

We first suppose that $\text{int}(\bar{\gamma}) \cap \text{int}(\bar{\gamma}') = \emptyset$. Then we can slide $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.). Moreover, we can isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 59).

We next suppose that $\text{int}(\bar{\gamma}) \cap \text{int}(\bar{\gamma}') \neq \emptyset$. Then there are two possibilities: (1) $\bar{\gamma} \subset \bar{\gamma}'$ or $\bar{\gamma}' \subset \bar{\gamma}$, say the latter holds and (2) $\bar{\gamma} \not\subset \bar{\gamma}'$ and $\bar{\gamma}' \not\subset \bar{\gamma}$. In each case, we can slide $\bar{\gamma}$ to γ along the disk δ_γ. Moreover, we can isotope σ slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figures 60 and 61).

Case B-(2)-(3). Either $\bar{\gamma}$ or $\bar{\gamma}'$, say $\bar{\gamma}$, satisfies the condition (2) and $\bar{\gamma}'$ satisfies the condition (3).

Let $\bar{\gamma}'_1$ and $\bar{\gamma}'_2$ be the components of $\bar{\gamma}'$ with $\partial \bar{\gamma}'_1 \supset D_{w'}$.

We first suppose that $\text{int}(\bar{\gamma}) \cap \text{int}(\bar{\gamma}') = \emptyset$. Since $\bar{\gamma} \cap \bar{\gamma}' \neq \emptyset$, we may suppose that $\bar{\gamma}'_1 \cap \bar{\gamma} (= \partial \bar{\gamma}'_1 \cap \partial \bar{\gamma})$ consists of a single point. Then we can slide $\bar{\gamma}$ ($\bar{\gamma}'_1$ resp.) to γ (γ' resp.) along the disk δ_γ ($\delta_{\gamma'}$ resp.). If $\bar{\gamma}'_2 \cap \bar{\gamma} \neq \emptyset$, then $\bar{\gamma}'_2 \cap \bar{\gamma} = \partial \bar{\gamma}'_2 \cap \bar{\gamma}$ consists of a single point. Hence $\bar{\gamma}'_1 \cup \bar{\gamma}$ composes an unknotted cycle and hence

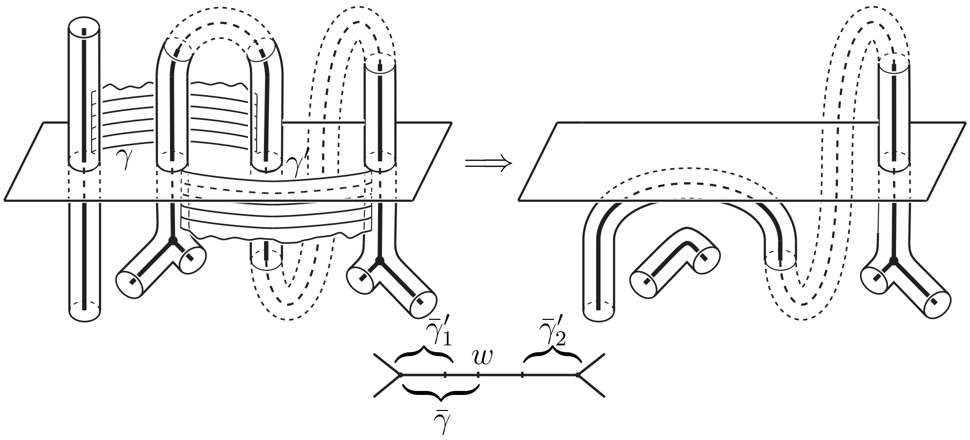

FIGURE 57

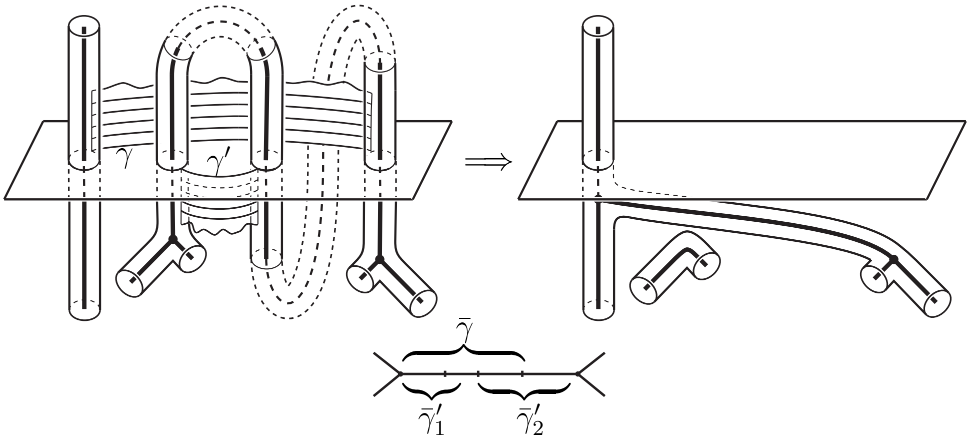

FIGURE 58

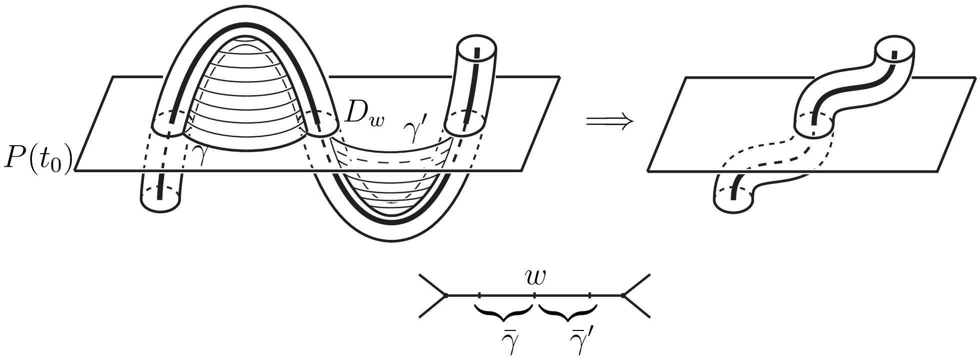

FIGURE 59

Lemma 3.3.8 holds (*cf.* Figure 62). Otherwise, we can further isotope Σ to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 63).

We next suppose that $\mathrm{int}\bar{\gamma} \cap \mathrm{int}\bar{\gamma}' \neq \emptyset$. We may assume that $\mathrm{int}\bar{\gamma} \cap \mathrm{int}\bar{\gamma}'_1 \neq \emptyset$.

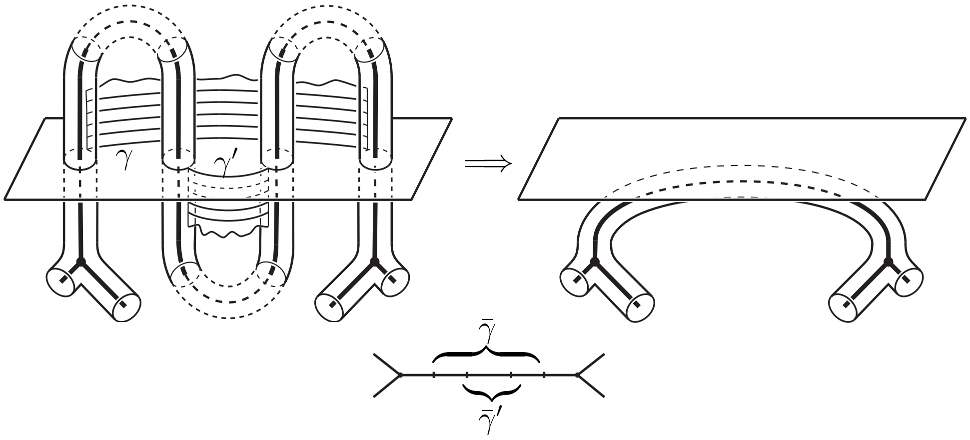

FIGURE 60

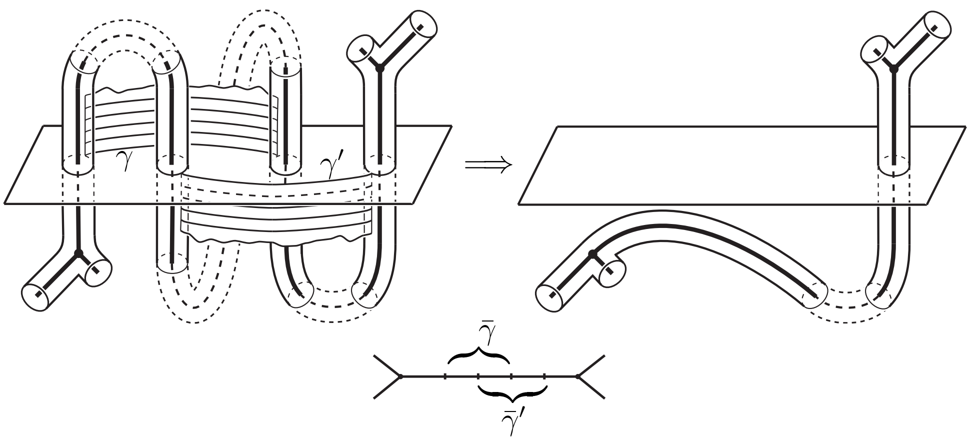

FIGURE 61

Then there are two possibilities: $\text{int}\bar{\gamma} \cap \text{int}\bar{\gamma}'_2 = \emptyset$ and $\text{int}\bar{\gamma} \cap \text{int}\bar{\gamma}'_2 \neq \emptyset$. In each case, we can slide $\bar{\gamma}$ to γ along the disk δ_γ. Moreover, we can isotope σ and σ' slightly to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 64 and Figure 65).

Case B-(3)-(3). Both $\bar{\gamma}$ and $\bar{\gamma}'$ satisfy the condition (3).

Let $\bar{\gamma}_1$ and $\bar{\gamma}_2$ ($\bar{\gamma}'_1$ and $\bar{\gamma}'_2$ resp.) be the components of $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) with $h^1_{\bar{\gamma}_1} \supset D_w$ ($h^1_{\bar{\gamma}'_1} \supset D_{w'}$ resp.). Without loss of generality, we may suppose that $\bar{\gamma}_1 \subset \bar{\gamma}'_1$. Then there are teo possibilities: (1) $\bar{\gamma}_2 \subset \bar{\gamma}'_2$ and (2) $\bar{\gamma}_2 \supset \bar{\gamma}'_2$. In each case, we can slide $\bar{\gamma}'_1$ into γ' along the disk $\delta_{\gamma'}$. Moreover, we can isotope Σ to reduce (W_Σ, n_Σ) is reduced, a contradiction (*cf.* Figure 66 and Figure 67).

□

Let t_0^+ (t_0^- resp.) be the first critical height above t_0 (below t_0 resp.). Since $|P(t_0) \cap \Sigma| = W_\Sigma = \max\{w_\Sigma(t) | t \in (-1, 1)\}$, we see that the critical point of the height t_0^+ (t_0^- resp.) is a maximum or a λ-vertex (a minimum or a y-vertex resp.) (see Figure 68).

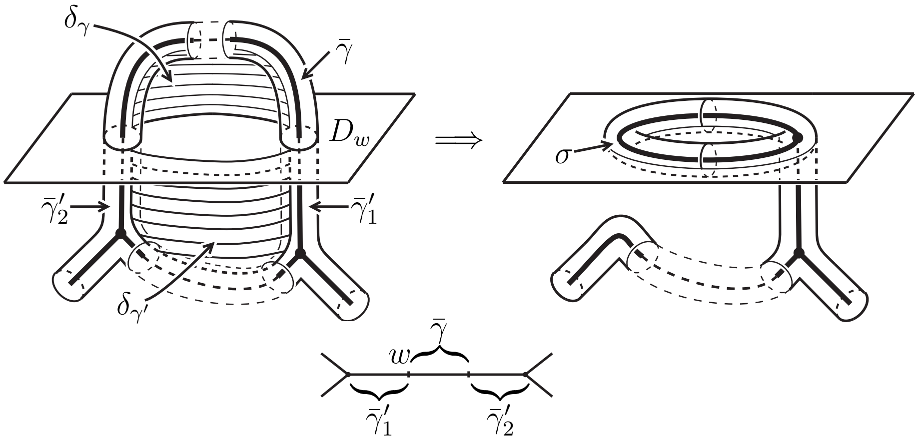

FIGURE 62

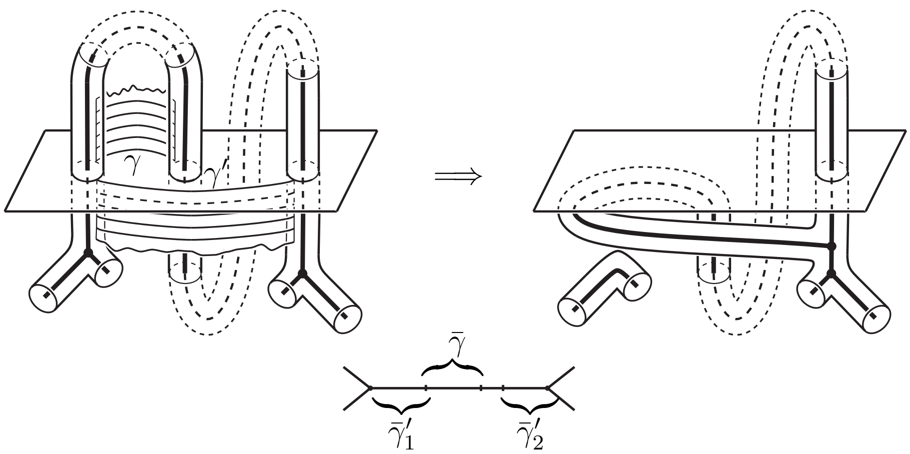

FIGURE 63

Lemma 3.3.9. *The critical height t_0^- is a y-vertex (not a minimum), or Σ is modified by edge slides so that the modified graph contains an unknotted cycle.*

Proof. Suppose that the critical point of the height t_0^- is a minimum. Let t_0^{-+} be a regular height just above t_0^-. Then $\Lambda(t_0^{-+})$ contains a fat-vertex with a lower simple outermost edge for the fat-vertex of $\Lambda(t_0^{-+})$. Hence it follows from Lemma 3.3.8 that every simple outermost edge for each fat-vertex of $\Lambda(t_0^{-+})$ is lower. Similarly, every simple outermost edge for each fat-vertex of $\Lambda(t_0)$ is upper. We now vary t for t_0^{-+} to t_0. Note that for each regular height t, all the simple outermost edges for each fat-vertex of $\Lambda(t)$ are either upper or lower (Lemma 3.3.8); such a regular height t is said to be upper or lower respectively. In these words, t_0^{-+} is lower and t_0 is upper.

Let $c_1, \ldots, c_n$ ($c_1 < \cdots < c_n$) be the critical heights of $h|_{\Delta_2}$ contained in $[t_0^{-+}, t_0]$. Note that the property 'upper' or 'lower' is unchanged at any height of $[t_0^{-+}, t_0] \setminus \{c_1, \ldots, c_n\}$. Hence there exists a critical height c_i such that a height t is changed from lower to upper at c_i. The graph $\Lambda(t)$ is changed as in Figure 69 around the critical height c_i.

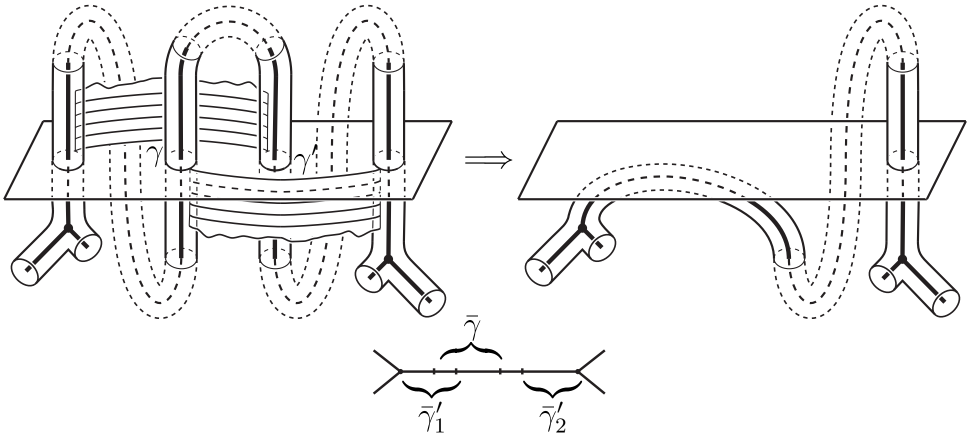

FIGURE 64

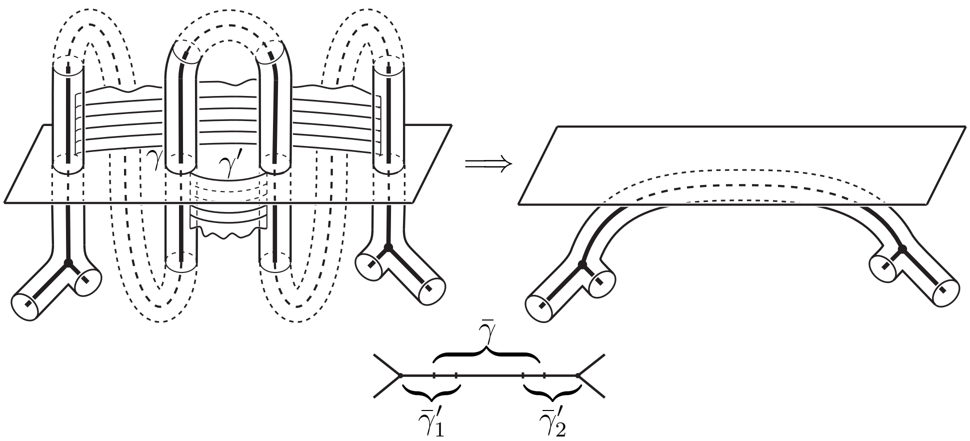

FIGURE 65

Let c_i^+ (c_i^- resp.) be a regular height just above (below resp.) c_i. We note that the lower disk for $\Lambda(c_i^-)$ and the upper disk for $\Lambda(c_i^+)$ in Figure 69 are contained in the same component of Δ_2, say D. We take parallel copies, say D' and D'', of D such that D' is obtained by pushing D into one side and that D'' is obtained by pushing D into the other side (*cf.* Figure 70). Then we may suppose that there is an upper (a lower resp.) simple outermost edge for a fat-vertex in D' (D'' resp.). Hence we can apply the arguments of the proof of Lemma 3.3.7 to modify Σ so that the modified graph contains an unknotted cycle.

□

Let v^- be the y-vertex of Σ at the height t_0^- and t_0^{--} a regular height just below t_0^-. Let v^{--} be the intersection point of the descending edges from v^- in Σ and $P(t_0^{--})$, and let $D_{v^{--}}$ be the fat-vertex of $\Lambda(t_0^{--})$ correponding to v^{--}.

Lemma 3.3.10. *Every simple outermost edge for any fat-vertex of $\Lambda(t_0^{--})$ is lower, or Σ is modified by edge slides so that the modified graph contains an unknotted cycle.*

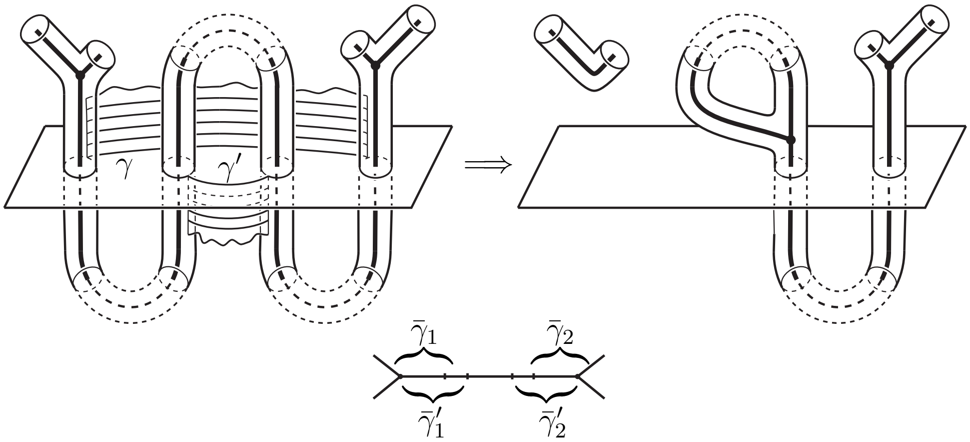

FIGURE 66

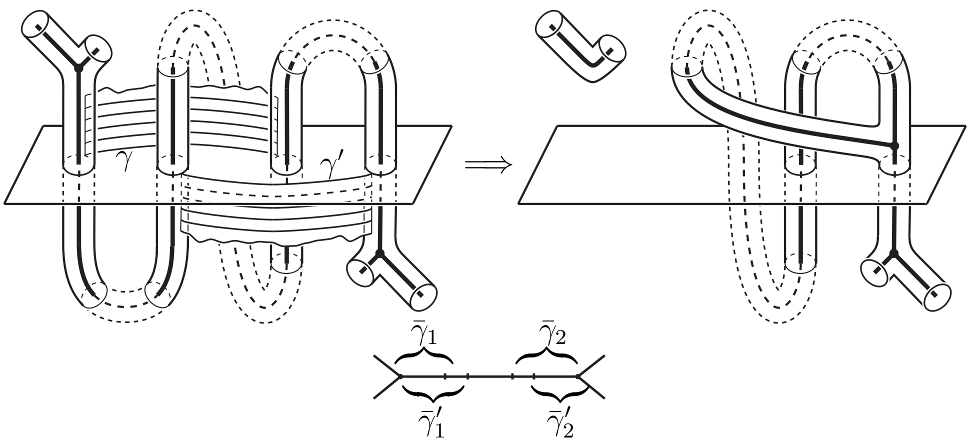

FIGURE 67

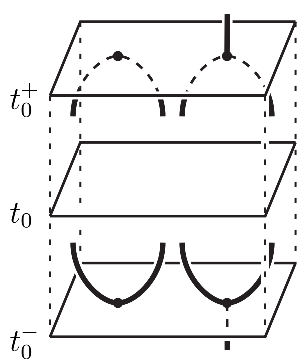

FIGURE 68

Proof. Suppose that there is a fat-vertex D_w of $\Lambda(t_0^{--})$ such that $\Lambda(t_0^{--})$ contains an upper simple outermost edge γ for D_w. Let σ be the edge of Σ with $h^1_\sigma \supset D_w$. Let δ_γ be the outermost disk for (D_w, γ). Let $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) be a union of the components obtained by cutting σ by the two fat-vertices of $\Lambda(t_0)$ incident to γ

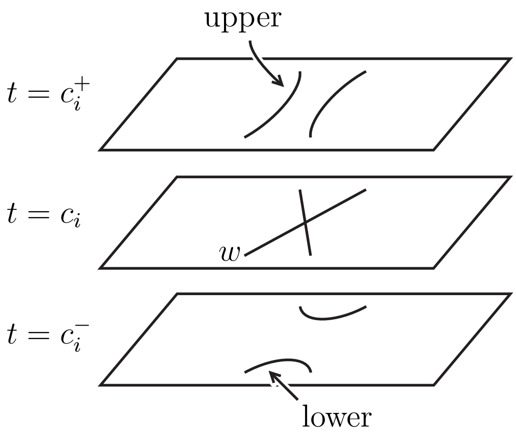

FIGURE 69

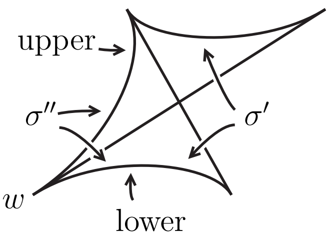

FIGURE 70

(γ' resp.) such that a 1-handle correponding to each component intersects $\partial\delta_\gamma \setminus \gamma$ ($\partial\delta_{\gamma'} \setminus \gamma'$ resp.). Then $\bar{\gamma}$ ($\bar{\gamma}'$ resp.) satisfies one of the conditions (1), (2) and (3) in the proof of Lemma 3.3.7.

Case A. $D_{v^{--}} \neq D_w$.

Then we have the following three cases. In each case, we can slide (a component of) $\bar{\gamma}$ to γ along the disk δ_γ. Moreover, we can isotope Σ to reduce (W_Σ, n_Σ), a contradiction.

Case A-(1). $\bar{\gamma}$ satisfies the condition (1).

Then there are two possibilities: (i) $v^{--} \notin \bar{\gamma}$ and (ii) $v^{--} \in \bar{\gamma}$. In each case, see Figure 71.

Case A-(2). $\bar{\gamma}$ satisfies the condition (2).

See Figure 72.

Case A-(3). $\bar{\gamma}$ satisfies the condition (3).

Then there are two possibilities: (i) $v^{--} \notin \bar{\gamma}$ and (ii) $v^{--} \in \bar{\gamma}$. In each case, see Figure 73.

Case B. $D_{v^{--}} = D_w$.

Since δ_γ is upper, we see that $\bar{\gamma}$ does not satisfy the condition (2). Hence we have the following.

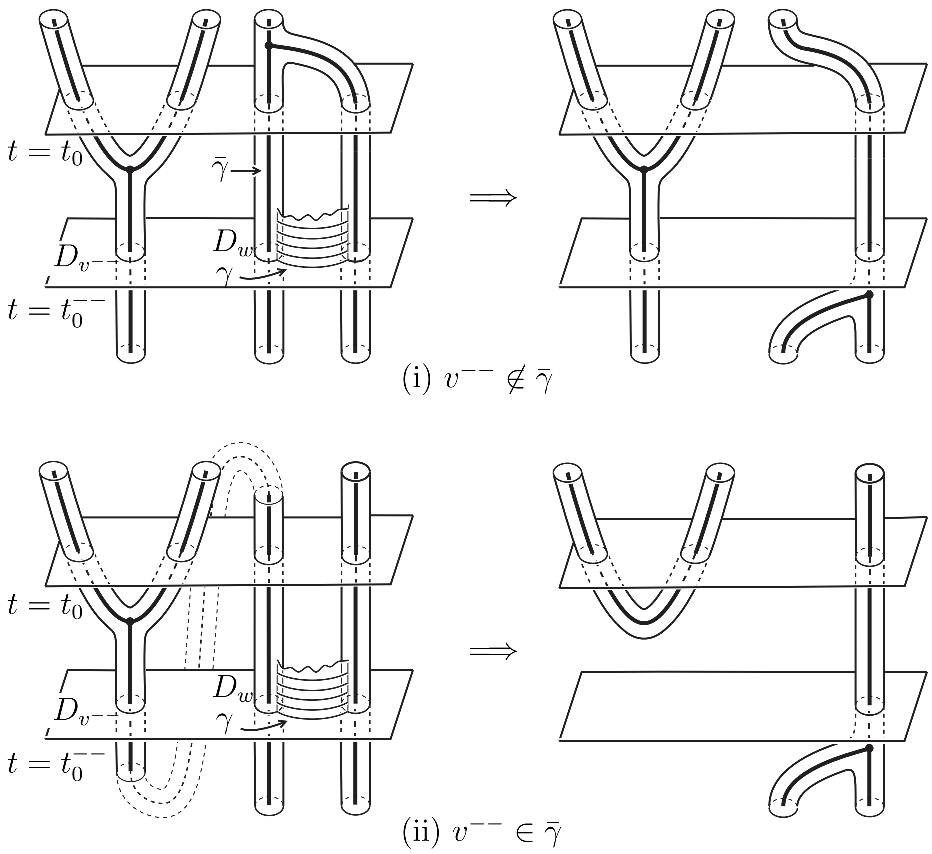

FIGURE 71

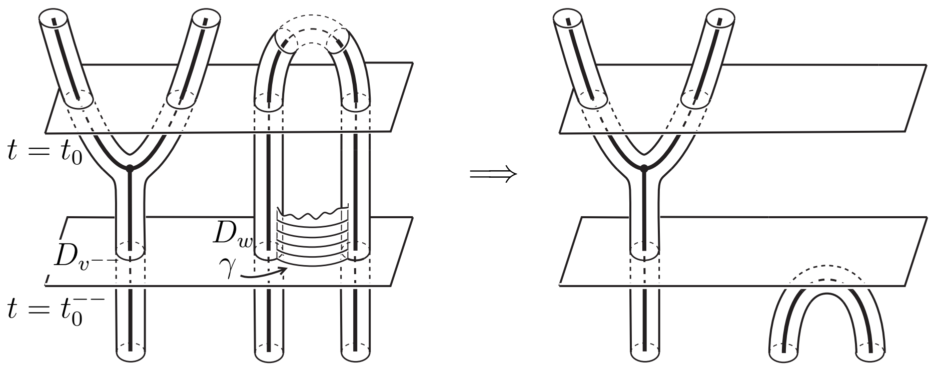

FIGURE 72

Case B-(1). $\bar{\gamma}$ satisfies the condition (1).

Since γ is upper, we see that the y-vertex of Σ at the height t_0^- is an endpoint of $\bar{\gamma}$, i.e., $\bar{\gamma}$ is the short vertical arc joining v^- to v^{--}. Then we can slide $\bar{\gamma}$ to γ along the disk δ_γ to obtain a new graph Σ'. Note that $(W_{\Sigma'}, n_{\Sigma'}) = (W_\Sigma, n_\Sigma)$ (*cf.* Figure 74). However, the critical point for Σ' corresponding to v^- is a minimum. Hence we can apply the arguments in the proof of Lemma 3.3.9 to show that there is an unknotted cycle in Σ'.

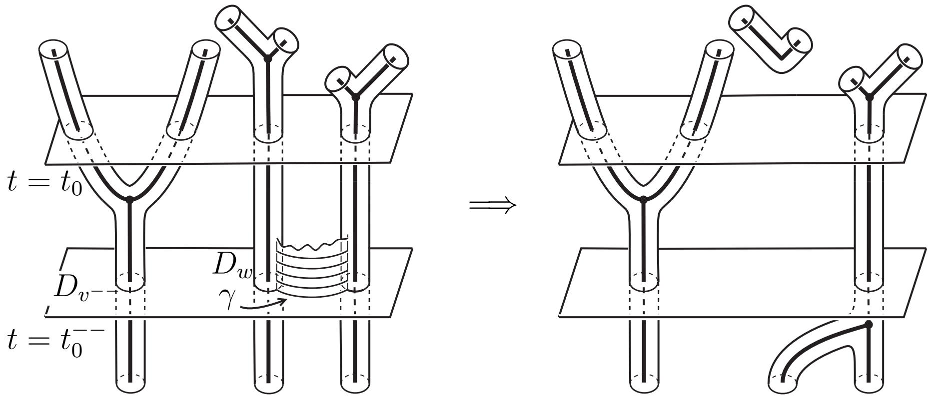

Figure 73

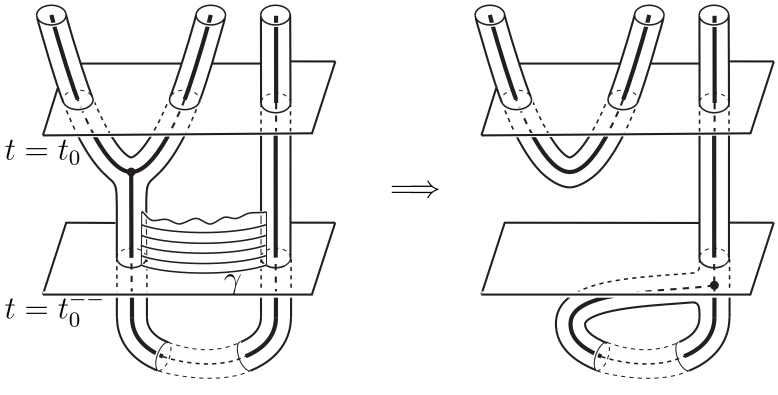

Figure 74

Case B-(3). $\bar{\gamma}$ satisfies the condition (3).

Let $\bar{\gamma}_1$ and $\bar{\gamma}_2$ be the components of $\bar{\gamma}$ with $\partial\bar{\gamma}_1 \ni v^-$. Then we can slide $\bar{\gamma}_2$ into γ along the disk δ_γ. Moreover, we can isotope Σ to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 75).

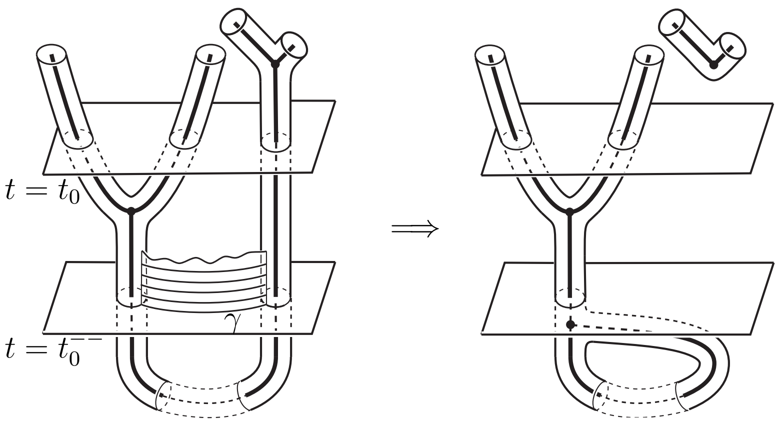

Figure 75

□

Lemma 3.3.11. *Every simple outermost edge for any fat-vertex of* $\Lambda(t_0^{--})$ *is incident to* $D_{v^{--}}$*, or* Σ *is modified so that there is an unknotted cycle.*

Proof. Suppose that $\Lambda(t_0^{--})$ contains a simple outermost edge γ for D_w and is not incident to $D_{v^{--}}$. Then it follows from Lemma 3.3.10 that γ is lower. This means that $\Lambda(t_0)$ contains a lower simple edge, because an edge disjoint from $D_{v^{--}}$ is not affected at all in $[t_0^{--}, t_0]$. This contradicts Lemma 3.3.8.

□

We now prove Proposition 3.3.4.

Proof of Proposition 3.3.4. We first prove the following.

Claim. For any fat-vertex $D_w(\neq D_{v^{--}})$ of $\Lambda(t_0^{--})$, there are no loops of $\Lambda(t_0^{--})$ based on D_w, or Σ is modified by edge slides so that the modified graph contains an unknotted cycle.

Proof. Suppose that there is a fat-vertex $D_w(\neq D_{v^{--}})$ of $\Lambda(t_0^{--})$ such that there is a loop α of $\Lambda(t_0^{--})$ based on D_w. Then α separates $\mathrm{cl}(P(t_0^{--}) \setminus D_w)$ into two disks E_1 and E_2 with $D_{v^{--}} \subset E_2$. By retaking D_w and α, if necessary, we may suppose that there are no loop components of $\Lambda(t_0^{--})$ in $\mathrm{int}(E_1)$. It follows from Lemma 3.3.5 that there is a fat-vertex $D_{w'}$ of $\Lambda(t_0^{--})$ in $\mathrm{int}(E_1)$. Then every outermost edge for $D_{w'}$ of $\Lambda(t_0^{--})$ is simple. Hence it follows from Lemma 3.3.11 that Σ contains an unknotted cycle and therefore we have the claim.

Then we have the following cases.

Case A. The descending edges of Σ from the maximum or λ-vertex v^+ at the height t_0^+ are equal to the ascending edges from v^- (*cf.* Figure 76).

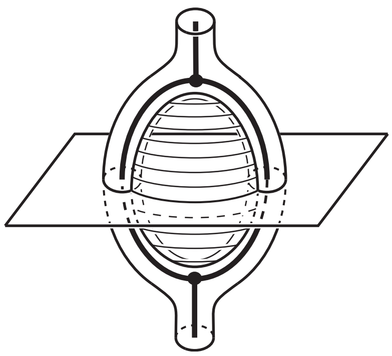

FIGURE 76

Then we can immediately see that there is an unknotted cycle .

Case B. Exactly one of the descending edges from v^+ is equal to one of the ascending edges from v^- (*cf.* Figure 77).

Let σ' be the other edge disjoint from v^-, and let w^{--} be the first intersection point of $P(t_0^{--})$ and the edge σ'. Let γ be an outermost edge for $D_{w^{--}}$ of $\Lambda(t_0^{--})$. By the claim above, we see that γ is simple. It follows from Lemma 3.3.10 that

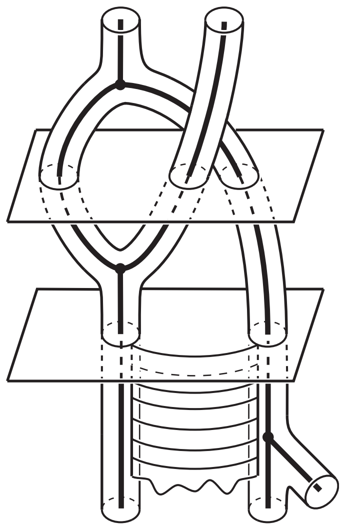

FIGURE 77

we may suppose that γ is lower. It also follows from Lemma 3.3.11 that we may suppose that the endpoints of γ are v^{--} and w^{--}. Let δ_γ be the outermost disk for (D_w, γ). Set $\bar{\gamma} = \sigma' \cap \delta_\gamma$. Since the subarc of σ' whose endpoints are v^{--} and w^{--} is monotonous and γ is lower, we see that $\bar{\gamma}$ cannot satisfy the condition (3) in the proof of Lemma 3.3.8. Hence $\bar{\gamma}$ satisfies the condition (1) or (2). In each case, we can slide $\bar{\gamma}$ to γ along the disk δ_γ to obtain a new graph with an unknotted cycle.

Case C. Any descending edge of Σ from v^+ is disjoint from an ascending edge from v^-.

It follows from 3.3.10, 3.3.11 and the claim that $\Lambda(t^{--})$ contains a lower simple outermost edge γ_i $(i = 1, 2)$ for D_{w_i} which is adjacent to D_{w_i} and $D_{v^{--}}$. Let δ_{γ_i} be the outermost disk for (D_{w_i}, γ_i). Set $\bar{\gamma}_i = \sigma_i \cap \delta_{\gamma_i}$. Since the subarc of σ_i whose endpoints v^+ and w_i are monotonous and δ_{γ_i} is lower, we see that γ_i cannot satisfy the condition (3). Then we have the following.

Case C-(1). Both $\bar{\gamma}_1$ and $\bar{\gamma}_2$ satisfy the condition (1).

If $\sigma_1 = \sigma_2$, then we can slide $\bar{\gamma}_1$ to γ_1 along the disk δ_{γ_1}. We can further isotope Σ to reduce (W_Σ, n_Σ), a contradiction (*cf.* Figure 78). Hence $\sigma_1 \neq \sigma_2$.

Then we can slide $\bar{\gamma}_1 \cup \bar{\gamma}_2$ to $\gamma_1 \cup \gamma_2$ along $\delta_{\gamma_1} \cup \delta_{\gamma_2}$ so that a new graph contains an unknotted cycle (*cf.* Figure 79).

Case C-(2). Either $\bar{\gamma}_1$ or $\bar{\gamma}_2$, say $\bar{\gamma}_1$, satisfies the condition (2).

Since $\bar{\gamma}_1$ satisfies the condition (2), we see that the endpoints of σ_1 are v^+ and v^-. Hence $w_2 \notin \sigma_1$. This implies that $\bar{\gamma}_2$ satisfies the condition (1). Then we first slide $\bar{\gamma}_2$ to γ_2 along δ_{γ_2}. We can further slide $\bar{\gamma}_1$ to γ_1 along δ_{γ_1} so that a new graph contains an unknotted cycle.

This completes the proof of Proposition 3.3.4. □

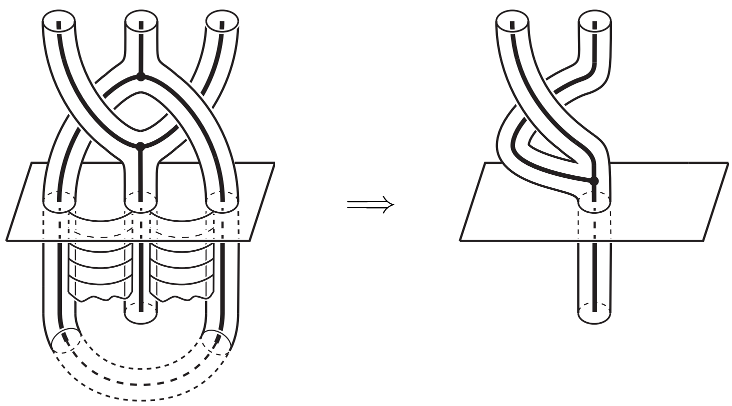

FIGURE 78

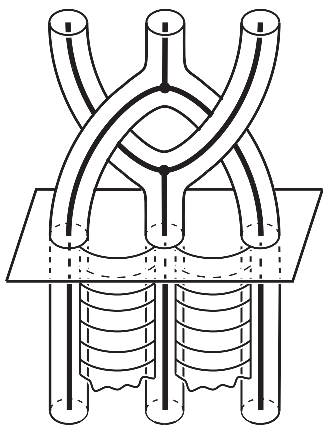

FIGURE 79

3.4. **Applications of Haken's theorem and Waldhausen's theorem.**

Corollary 3.4.1. *Let M be a compact 3-manifold and $(C_1, C_2; S)$ a reducible Heegaard splitting. Then M is reducible or $(C_1, C_2; S)$ is stabilized.*

Proof. Suppose that M is irreducible. Let P be a 2-sphere such that $P \cap S$ is an essential loop. Since M is irreducible, we see that P bounds a 3-ball in M. Hence we can regard M as a connected sum of S^3 and M. By Theorem 3.3.1, the induced Heegaard splitting of S^3 is stabilized. Hence this cancelling pair of disks shows that $(C_1, C_2; S)$ is stabilized. □

Corollary 3.4.2. *Any Heegaard splitting of a handlebody is standard, i.e, is obtained from a trivial splitting by stabilization.*

Exercise 3.4.3. Show Corollary 3.4.2.

Theorem 3.4.4. *Let M be a closed 3-manifold. Let $(C_1, C_2; S)$ and $(C_1', C_2'; S')$ be Heegaard splittings of M. Then there is a Heegaard splitting which is obtained by stabilization of both $(C_1, C_2; S)$ and $(C_1', C_2'; S')$.*

Proof. Let Σ_{C_1} and $\Sigma_{C_1'}$ be spines of C_1 and C_1' respectively. By an isotopy, we may assume that $\Sigma_{C_1} \cap \Sigma_{C_1'} = \emptyset$ and $C_1 \cap C_1' = \emptyset$. Set $M' = \mathrm{cl}(M \setminus (C_1 \cup$

C_1')), $\partial_1 M' = \partial C_1$ and $\partial_2 M' = \partial C_2$. Let $(\bar{C}_1, \bar{C}_2; \bar{S})$ be a Heegaard splitting of $(M'; \partial_1 M', \partial_2 M')$. Set $C_1^* = C_1 \cup \bar{C}_1$ and $C_2^* = C_2 \cup \bar{C}_2$. Then it is easy to see that $(C_1^*, C_2^*; \bar{S})$ is a Heegaard splitting of M. Note that $C_2' = C_1 \cup M' = C_1 \cup (\bar{C}_1 \cup \bar{C}_2) = (C_1 \cup \bar{C}_1) \cup \bar{C}_2$. Here, we note that $(C_1^*, \bar{C}_2; \bar{S})$ is a Heegaard splitting of C_2'. It follows from Corollary 3.4.2 that $(C_1^*, \bar{C}_2; \bar{S})$ is obtained from a trivial splitting of C_2' by stabilization. This implies that $(C_1^*, C_2^*; \bar{S})$ is obtained from $(C_1', C_2'; S')$ by stabilization. On the argument above, by replacing C_1 to C_1', we see that $(C_1^*, C_2^*; \bar{S})$ is also obtained from $(C_1, C_2; S)$ by stabilization. □

Remark 3.4.5. The stabilization problem is one of the most important themes on Heegaard theory. But we do not give any more here. For the detail, for example, see [8], [12], [15], [19] and [20].

4. Generalized Heegaard splittings

4.1. Definitions.

Definition 4.1.1. A 0-*fork* is a connected 1-complex obtained by joining a point p to a point g whose 1-simplices are oriented toward g and away from p. For $n \geq 1$, an n-*fork* is a connected 1-complex obtained by joining a point p to each of distinct n points t_i $(i = 1, ..., n)$ and to a point g whose 1-simplices are oriented toward g and away from t_i. We call p a *root*, t_i a *tine* and g a *grip*.

Remark 4.1.2. An n-fork corresponds to a compression body C such that each of t_i $(i = 1, 2, ..., n)$ corresponds to a component of $\partial_- C$ and g correponds to $\partial_+ C$ (*cf.* Figure 80).

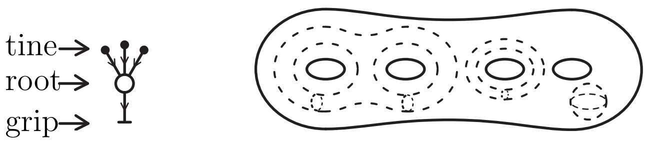

Figure 80

Definition 4.1.3. Let $\mathcal{A}$ ($\mathcal{B}$ resp.) be a collection of finite forks, $T_{\mathcal{A}}$ ($T_{\mathcal{B}}$ resp.) a collection of tines of $\mathcal{A}$ ($\mathcal{B}$ resp.) and $G_{\mathcal{A}}$ ($G_{\mathcal{B}}$ resp.) a collection of grips of $\mathcal{A}$ ($\mathcal{B}$ resp.). We suppose that there are bijections $\mathcal{T} : T_{\mathcal{A}} \to T_{\mathcal{B}}$ and $\mathcal{G} : G_{\mathcal{A}} \to G_{\mathcal{B}}$. A *fork complex* $\mathscr{F}$ is an oriented connected 1-complex $\mathcal{A} \cup (-\mathcal{B})/\{\mathcal{T}, \mathcal{G}\}$, where $-\mathcal{B}$ denotes the 1-complex obtained by taking the opposite orientation of each 1-simplex and the equivalence relation $/\{\mathcal{T}, \mathcal{G}\}$ is given by $t \sim \mathcal{T}(t)$ for any $t \in T_{\mathcal{A}}$ and $g \sim \mathcal{G}(g)$ for any $g \in G_{\mathcal{A}}$. We define:

$$\partial_1 \mathscr{F} = \{(\text{tines of } \mathcal{A}) \setminus T_{\mathcal{A}}\} \cup \{(\text{grips of } \mathcal{B}) \setminus G_{\mathcal{B}}\} \text{ and}$$
$$\partial_2 \mathscr{F} = \{(\text{tines of } \mathcal{B}) \setminus T_{\mathcal{B}}\} \cup \{(\text{grips of } \mathcal{A}) \setminus G_{\mathcal{A}}\}.$$

Definition 4.1.4. A fork complex is *exact* if there exists $e \in \text{Hom}(C_0(\mathscr{F}), \mathbb{R})$ such that

(1) $e(v_1) = 0$ for any $v_1 \in \partial_1 \mathscr{F}$,

(2) $(\delta e)(e_{\mathcal{A}}) > 0$ for any 1-simplex $e_{\mathcal{A}}$ in $\mathcal{A}$ with the standard orientation, $(\delta e)(e_{\mathcal{A}}) < 0$ for any 1-simplex $e_{\mathcal{B}}$ in $\mathcal{B}$ with the standard orientation, where δ denotes the coboundary operator $\mathrm{Hom}(C_0(\mathscr{F}), \mathbb{R}) \to \mathrm{Hom}(C_1(\mathscr{F}), \mathbb{R})$ and
(3) $e(v_2) = 1$ for any $v_2 \in \partial_2 \mathscr{F}$.

Remark 4.1.5. Geometrically speaking, $\mathscr{F}$ is exact if and only if we can put $\mathscr{F}$ in $\mathbb{R}^3$ so that

(1) $\partial_1 \mathscr{F}$ lies in the plane of height 0,
(2) for any path α in $\mathscr{F}$ from a point in $\partial_1 \mathscr{F}$ to a point in $\partial_2 \mathscr{F}$, $h|_\alpha$ is monotonically increasing, where h is the height function of $\mathbb{R}^3$ and
(3) $\partial_2 \mathscr{F}$ lies in the plane of height 1 (*cf.* Figure 81).

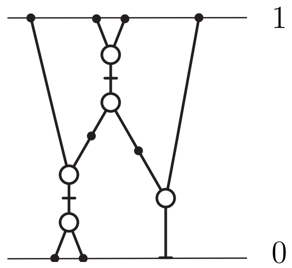

FIGURE 81

In the following, we regard fork complexes as geometric objects, i.e., 1-dimensional polyhedra.

Definition 4.1.6. A *fork of* $\mathscr{F}$ is the image of a fork in $\mathcal{A} \cup \mathcal{B}$ in $\mathscr{F}$. A *grip* (*root* and *tine* resp.) *of* $\mathscr{F}$ is the image of a grip (root and tine resp.) in $\mathcal{A} \cup \mathcal{B}$ in $\mathscr{F}$.

Definition 4.1.7. Let M be a compact orientable 3-manifold, and let $(\partial_1 M, \partial_2 M)$ be a partition of boundary components of M. A *generalized Heegaard splitting* of $(M; \partial_1 M, \partial_2 M)$ is a pair of an exact fork complex $\mathscr{F}$ and a proper map $\rho : (M; \partial_1 M, \partial_2 M) \to (\mathscr{F}; \partial_1 \mathscr{F}, \partial_2 \mathscr{F})$ which satisfies the following.

(1) The map ρ is transverse to $\mathscr{F} - \{\text{the roots of } \mathscr{F}\}$.
(2) For each fork $\mathcal{F} \subset \mathscr{F}$, we have the following (*cf.* Figure 82).
   (a) If $\mathcal{F}$ is a 0-fork, then $\rho^{-1}(\mathcal{F})$ is a handlebody $V_{\mathcal{F}}$ such that (1) $\rho^{-1}(g) = \partial V_{\mathcal{F}}$ and (2) $\rho^{-1}(p)$ is a 1-complex which is a spine of $V_{\mathcal{F}}$, where g is the grip of $\mathcal{F}$.
   (b) If $\mathcal{F}$ is an n-fork with $n \geq 1$, then $\rho^{-1}(\mathcal{F})$ is a connected compression body $V_{\mathcal{F}}$ such that (1) $\rho^{-1}(g) = \partial_+ V_{\mathcal{F}}$, (2) for each tine t_i, $\rho^{-1}(t_i)$ is a connected component of $\partial_- V_{\mathcal{F}}$ and $\rho^{-1}(t_i) \neq \rho^{-1}(t_j)$ for $i \neq j$ and (3) $\rho^{-1}(p)$ is a 1-complex which is a deformation retract of $V_{\mathcal{F}}$, where g is the grip of $\mathcal{F}$, p is the root of $\mathcal{F}$ and $\{t_i\}_{1 \leq i \leq n}$ is the set of the tines of $\mathcal{F}$.

Remark 4.1.8. Let g be a grip of $\mathscr{F}$ which is contained in the interior of $\mathscr{F}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the forks of $\mathscr{F}$ which are adjacent to g. Then $(\rho^{-1}(\mathcal{F}_1), \rho^{-1}(\mathcal{F}_2); \rho^{-1}(g))$ is a Heegaard splitting of $\rho^{-1}(\mathcal{F}_1 \cup \mathcal{F}_2)$.

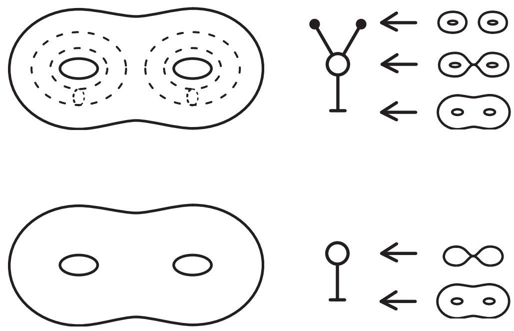

FIGURE 82

Definition 4.1.9. A generalized Heegaard splitting $(\mathscr{F}, \rho)$ is said to be *strongly irreducible* if (1) for each tine t, $\rho^{-1}(t)$ is incompressible, and (2) for each grip g with two forks attached to g, say $\mathcal{F}_1$ and $\mathcal{F}_2$, $(\rho^{-1}(\mathcal{F}_1), \rho^{-1}(\mathcal{F}_2); \rho^{-1}(g))$ is strongly irreducible.

Let $\mathcal{M}$ be the set of finite multisets of $\mathbb{Z}_{\geq 0} = \{0, 1, 2, ...\}$. We define a total order $<$ on $\mathcal{M}$ as follows. For M_1 and $M_2 \in \mathcal{M}$, we first arrange the elements of M_i $(i = 1, 2)$ in non-increasing order respectively. Then we compare the arranged tuples of non-negative integers by lexicographic order.

Example 4.1.10. (1) If $M_1 = \{5, 4, 1, 1\}$ and $M_2 = \{5, 3, 2, 2, 2, 1\}$, then $M_2 < M_1$.
(2) If $M_1 = \{3, 1, 0, 0\}$ and $M_2 = \{3, 1, 0, 0, 0\}$, then $M_1 < M_2$.

Definition 4.1.11. Let $(\mathscr{F}, \rho)$ be a generalized Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$. We define *the width of* $(\mathscr{F}, \rho)$ to be the multiset

$$w(\mathscr{F}, \rho) = \{genus(\rho^{-1}(g_1)), \dots, genus(\rho^{-1}(g_m))\},$$

where $\{g_1, \dots, g_m\}$ is the set of the grips of $\mathscr{F}$. We say that $(\mathscr{F}, \rho)$ is *thin* if $w(\mathscr{F}, \rho)$ is minimal among all generalized Heegaard splittings of $(M; \partial_1 M, \partial_2 M)$.

Example 4.1.12. *The thin generalized Heegaard splittings of the* 3*-ball* B^3 *are two fork complexes illustrated in Figure 83, where* $\rho^{-1}(\mathcal{F}_1)$ *is a* 3*-ball and* $\rho^{-1}(\mathcal{F}_2) \cong S^2 \times [0, 1]$.

FIGURE 83

4.2. **Properties of thin generalized Heegaard splittings.** In this subsection, let $(\mathscr{F}, \rho)$ be a thin generalized Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$.

Observation 4.2.1. *Let* t *be a tine of* $\mathscr{F}$. *Then any* 2*-sphere component of* $\rho^{-1}(t)$ *is essential in* M *unless* M *is a* 3*-ball.*

Proof. Suppose that there is a tine t such that $\rho^{-1}(t)$ is a 2-sphere, say P, which bounds a 3-ball B in M. Let $\mathscr{F}_B$ be the subcomplex of $\mathscr{F}$ with $\rho^{-1}(\mathscr{F}_B) = B$. If $\mathscr{F}_B = \mathscr{F}$, then we see that M is a 3-ball. Otherwise, there is a fork $\mathcal{F}'$ with $t \in \mathcal{F}'$ and $\mathcal{F}' \not\subset \mathscr{F}_B$. Let e_t be the 1-simplex in $\mathcal{F}'$ joining t to the root of $\mathcal{F}'$. Set $\mathscr{F}^* = \mathscr{F} \setminus (\mathscr{F}_B \cup e_t)$. Note that $\rho^{-1}(\mathcal{F}' \cup \mathscr{F}_B)$ $(= \rho^{-1}(\mathcal{F}') \cup B)$ is a compression body V^*. Then it is easy to see that we can modify ρ in V^* to obtain $\rho^* : M \to \mathscr{F}^*$ such that $(\rho^*)^{-1}(\mathcal{F}' \setminus e_t)$ is the compression body V^* (*cf.* Figure 84).

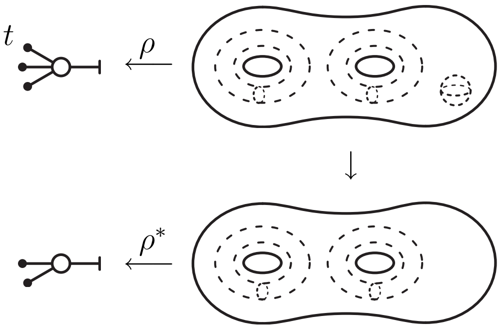

FIGURE 84

Moreover, the generalized Heegaard structure on $(\mathscr{F}, \rho)$ (e.g. $\mathcal{A}, \mathcal{B}$ decomposition etc) is naturally inherited to $(\mathscr{F}^*, \rho^*)$. Then we clearly have $w(\mathscr{F}^*, \rho^*) < w(\mathscr{F}, \rho)$, contradicting the assumption that $(\mathscr{F}, \rho)$ is thin.

□

Lemma 4.2.2. *Suppose that there is a fork $\mathcal{F}$ such that $\rho^{-1}(t)$ is trivial. Let t be the tine of $\mathcal{F}$. Then $\rho^{-1}(t)$ is a component of ∂M and one of the following holds.*

(1) *M is a 3-ball,*
(2) *$M \cong \rho^{-1}(t) \times [0, 1]$ and (or)*
(3) *$\rho^{-1}(t)$ is compressible in M.*

Proof. We first prove that $\rho^{-1}(t)$ is a boundary component of M. Suppose that $\rho^{-1}(t)$ is not a boundary component of M. Let g be the grip of $\mathcal{F}$. If $\rho^{-1}(g)$ is a boundary component of M, then we can reduce the width by removing $\mathcal{F}$, a contradiction (*cf.* Figure 85).

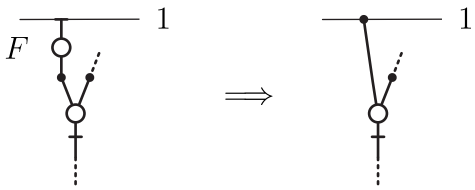

FIGURE 85

Hence $\mathcal{F}$ is contained in the interior of $\mathscr{F}$. Note that $\mathcal{F}$ is a 1-fork. Let $\mathcal{F}_1$ be the fork attached to g and $\mathcal{F}_2$ the fork attached to t. Note that since $\rho^{-1}(\mathcal{F})$ is a trivial compression body, we see that $\rho^{-1}(\mathcal{F}_1 \cup \mathcal{F} \cup \mathcal{F}_2)$ is also a compression body. Hence we can replace $\mathcal{F}_1 \cup \mathcal{F} \cup \mathcal{F}_2$ in $\mathscr{F}$ to a new fork so that we can obtain a new fork complex, say $\mathscr{F}^*$. Moreover, we can modify $\rho : M \to \mathscr{F}$ to obtain $\rho^* : M \to \mathscr{F}^*$ so that $(\mathscr{F}^*, \rho^*)$ is a generalized Heegaard splitting of $(M; \partial_1 M, \partial_2 M)$ with $w(\mathscr{F}^*, \rho^*) < w(\mathscr{F}, \rho)$, a contradiction (*cf.* Figure 86). Hence $\rho^{-1}(t)$ is a boundary component of M.

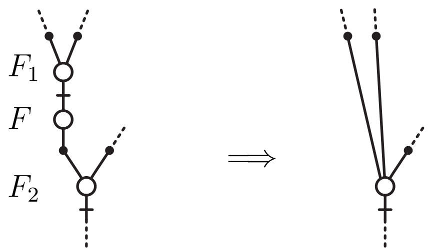

Figure 86

We next show that one of the conclusions (1)-(3) of Lemma 4.2.2 holds. Suppose that both conclusions (1) and (2) of Lemma 4.2.2 do not hold, i.e., M is not a 3-ball and $M \not\cong \rho^{-1}(t) \times [0,1]$. Then there is a fork $\mathcal{F}'(\neq \mathcal{F})$ attached to g. Moreover, since $(\mathscr{F}, \rho)$ is thin and $M \not\cong \rho^{-1}(t) \times [0,1]$, we see that $\rho^{-1}(\mathcal{F}')$ is a non-trivial compression body. Also, since M is not a 3-ball, $\rho^{-1}(t)$ is not a 2-sphere. Hence we see that $\rho^{-1}(t)$ is compressible in $\rho^{-1}(\mathcal{F}) \cup \rho^{-1}(\mathcal{F}')$. This implies that the conclusion (3) of Lemma 4.2.2 holds. □

Proposition 4.2.3. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be forks of $\mathscr{F}$ which have the same grip g of $\mathscr{F}$. Then $(\rho^{-1}(\mathcal{F}_1), \rho^{-1}(\mathcal{F}_2); \rho^{-1}(g))$ is strongly irreducible.*

Proof. Set $A_g = \rho^{-1}(\mathcal{F}_1)$, $B_g = \rho^{-1}(\mathcal{F}_2)$, $S_g = \rho^{-1}(g)$, $M_g = A_g \cup B_g$, $\partial_1 M_g = \partial_- A_g$ and $\partial_2 M_g = \partial_- B_g$. Then $(A_g, B_g; S_g)$ is a Heegaard splitting of $(M_g; \partial_1 M_g, \partial_2 M_g)$. Suppose that $(A_g, B_g; S_g)$ is weakly reducible. Let D_A and D_B be meridian disks of A_g and B_g respectively which satisfy $\partial D_A \cap \partial D_B = \emptyset$. Let Δ_A (Δ_B resp.) be a complete meridian system of A_g (B_g resp.) such that D_A (D_B resp.) is a component of Δ_A (Δ_B resp.) (*cf.* (6) of Remark 3.1.3). Note that A_g is obtained from $\partial_- A_g \times [0,1]$ and 0-handles $\mathcal{H}^0$ by attaching 1-handles $\mathcal{H}^1$ corresponding to Δ_A (*cf.* (3) of Remark 3.1.3) and that B_g is obtained from $S_g \times [0,1]$ by attaching 2-handles $\mathcal{H}^2$ corresponding to Δ_B and 3-handles $\mathcal{H}^3$ (*cf.* Definition 3.1.1). Hence we see that M_g admits the following decomposition (*cf.* Remark 3.1.13):

$$M_g = (\partial_1 M_g \times [0,1]) \cup \mathcal{H}^0 \cup \mathcal{H}^1 \cup \mathcal{H}^2 \cup \mathcal{H}^3.$$

Let h^1 be the component of $\mathcal{H}^1$ corresponding to D_A and h^2 the component of $\mathcal{H}^2$ corresponding to D_B. Then M_g admits the following decomposition:

$$M_g = (\partial_1 M_g \times [0,1]) \cup \mathcal{H}^0 \cup (\mathcal{H}^1 \setminus h^1) \cup h^2 \cup h^1 \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3.$$

Set $A'_g = (\partial_1 M_g \times [0,1]) \cup \mathcal{H}^0 \cup (\mathcal{H}^1 \setminus h^1)$. We divide the proof into the following two cases.

Case 1. ∂D_A or ∂D_B is non-separating in S_g.

Suppose first that ∂D_A is non-separating in S_g. Then A'_g is a compression body (*cf.* (6) of Remark 3.1.3). Since $A'_g = (\partial_1 M_g \times [0,1]) \cup \mathcal{H}^0 \cup (\mathcal{H}^1 \setminus h^1)$, we obtain

$$M_g = A'_g \cup h^2 \cup h^1 \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3.$$

Note that the attaching region of the 2-handle h^2 is contained in $\partial_+ A'_g$. Hence we have:

$$M_g \cong A'_g \cup \big((\partial_+ A'_g \times [0,1]) \cup h^2\big) \cup h^1 \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3.$$

Set $B'_g = (\partial_+ A'_g \times [0,1]) \cup h^2$. Then B'_g is also a compression body and we have:

$$M_g \cong A'_g \cup B'_g \cup h^1 \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3.$$

Note that $\partial_- B'_g$ is homeomorphic to the surface obtained from S_g by performing surgery along $\partial D_A \cup \partial D_B$. Then we have the following subcases.

Case 1.1. ∂D_B is non-separating in S_g and $\partial D_A \cup \partial D_B$ is non-separating in S_g.

Then $\partial_- B'_g$ is connected. Note that

$$\begin{aligned} M_g &\cong A'_g \cup B'_g \cup h^1 \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3 \\ &\cong A'_g \cup B'_g \cup \big((\partial_- B'_g \times [0,1]) \cup h^1\big) \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3. \end{aligned}$$

Set $A''_g = (\partial_- B'_g \times [0,1]) \cup h^1$. Then A''_g is also a compression body and we have:

$$\begin{aligned} M_g &\cong A'_g \cup B'_g \cup A''_g \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3 \\ &\cong A'_g \cup B'_g \cup A''_g \cup \big((\partial_+ A''_g \times [0,1] \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3\big). \end{aligned}$$

Set $B''_g = \partial_+ A''_g \times [0,1] \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3$. Note that $B'_g \cap A''_g = \partial_- B'_g = \partial_- A''_g$. This shows that each handle of $\mathcal{H}^2 \setminus h^2$ and $\mathcal{H}^3$ is adjacent to A''_g along $\partial_+ A''_g$. This implies that B''_g is also a compression body. Hence we have:

$$M_g \cong (A'_g \cup B'_g) \cup (A''_g \cup B''_g).$$

Then we can substitute $\mathcal{F}_1 \cup \mathcal{F}_2$ in $\mathscr{F}$ for $\mathcal{F}'_1 \cup \mathcal{F}'_2 \cup \mathcal{F}''_1 \cup \mathcal{F}''_2$, where $\mathcal{F}'_1$, $\mathcal{F}'_2$, $\mathcal{F}''_1$ and $\mathcal{F}''_2$ are forks corresponding to A'_g, B'_g, A''_g and B''_g respectively. Set $\mathscr{F}^* = (\mathscr{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup (\mathcal{F}'_1 \cup \mathcal{F}'_2 \cup \mathcal{F}''_1 \cup \mathcal{F}''_2)$. Then we can modify $\rho : M \to \mathscr{F}$ in M_g to obtain $\rho^* : M \to \mathscr{F}^*$ such that $(\rho^*)^{-1}(\mathcal{F}'_1) = A'_g$, $(\rho^*)^{-1}(\mathcal{F}'_2) = B'_g$, $(\rho^*)^{-1}(\mathcal{F}''_1) = A''_g$ and $(\rho^*)^{-1}(\mathcal{F}''_2) = B''_g$. It is easy to see that $w(\mathscr{F}^*, \rho^*) < w(\mathscr{F}, \rho)$, a contradiction (*cf.* Figure 87).

Case 1.2. ∂D_B is non-separating in S_g and $\partial D_A \cup \partial D_B$ is separating in S_g.

Then $\partial_- B'_g$ consists of two components, say G_1 and G_2.

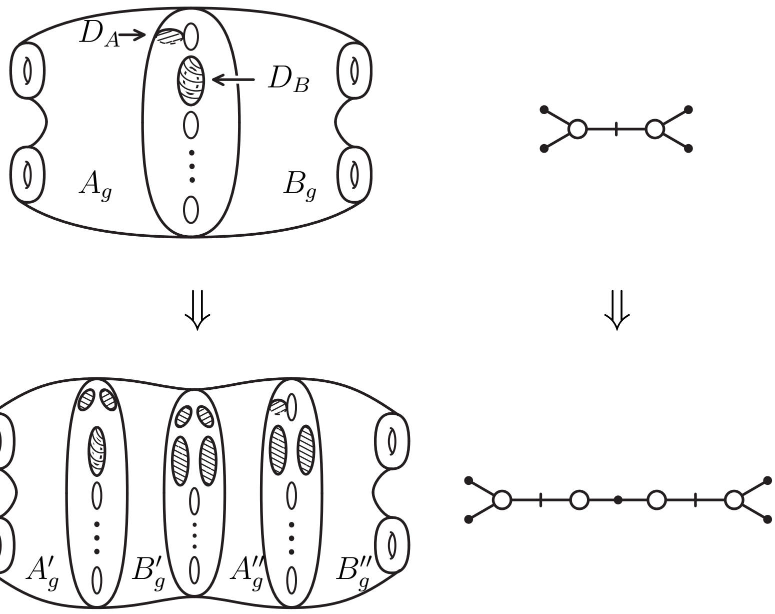

FIGURE 87

$$\begin{aligned} M_g &\cong A'_g \cup B'_g \cup h^1 \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3 \\ &\cong A'_g \cup B'_g \cup (\partial_- B'_g \times [0,1]) \cup h^1 \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3 \\ &\cong A'_g \cup B'_g \cup ((G_1 \cup G_2) \times [0,1]) \cup h^1 \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3. \end{aligned}$$

Set $A''_g = ((G_1 \cup G_2) \times [0,1]) \cup h^1$. Since ∂D_B is non-separating in S_g, we see that h^1 joins G_1 to G_2. Hence A''_g is a compression body and we have:

$$\begin{aligned} M_g &\cong A'_g \cup B'_g \cup A''_g \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3 \\ &\cong A'_g \cup B'_g \cup A''_g \cup (\partial_+ A''_g \times [0,1]) \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3. \end{aligned}$$

Set $B''_g = \partial_+ A''_g \times [0,1] \cup (\mathcal{H}^2 \setminus h^2) \cup \mathcal{H}^3$. Note that $B'_g \cap A''_g = \partial_- B'_g = \partial_- A''_g$. This shows that each handle of $\mathcal{H}^2 \setminus h^2$ and $\mathcal{H}^3$ is adjacent to A''_g along $\partial_+ A''_g$. This implies that B''_g is also a compression body. Hence we have:

$$M_g \cong (A'_g \cup B'_g) \cup (A''_g \cup B''_g).$$

According to this decomposition, we can modify the fork complex $(\mathscr{F}, \rho)$ as in Figure 88 or Figure 89. It is easy to see that for a new complex $(\mathscr{F}^*, \rho^*)$, we have $w(\mathscr{F}^*, \rho^*) < w(\mathscr{F}, \rho)$, a contradiction.

Case 1.3. ∂D_B is separating in S_g (hence $\partial D_A \cup \partial D_B$ is separating in S_g).

Then $\partial_- B'_g$ consists of two components, say $\bar{G}_1$ and $\bar{G}_2$. Since ∂D_B is separating in S_g, we see that h^1 joins $\bar{G}_1$ or $\bar{G}_2$, say $\bar{G}_1$, to itself. Let $\mathcal{H}^2_1$ ($\mathcal{H}^2_2$ resp.) be the components of $\mathcal{H}^2 \setminus h^2$ adjacent to $\bar{G}_1$ ($\bar{G}_2$ resp.). Let $\mathcal{H}^3_1$ ($\mathcal{H}^3_2$ resp.) be the components of $\mathcal{H}^3$ adjacent to $\bar{G}_1$ ($\bar{G}_2$ resp.). Set $\bar{B}'_g = B'_g \cup \mathcal{H}^2_2 \cup \mathcal{H}^3_2$. Then

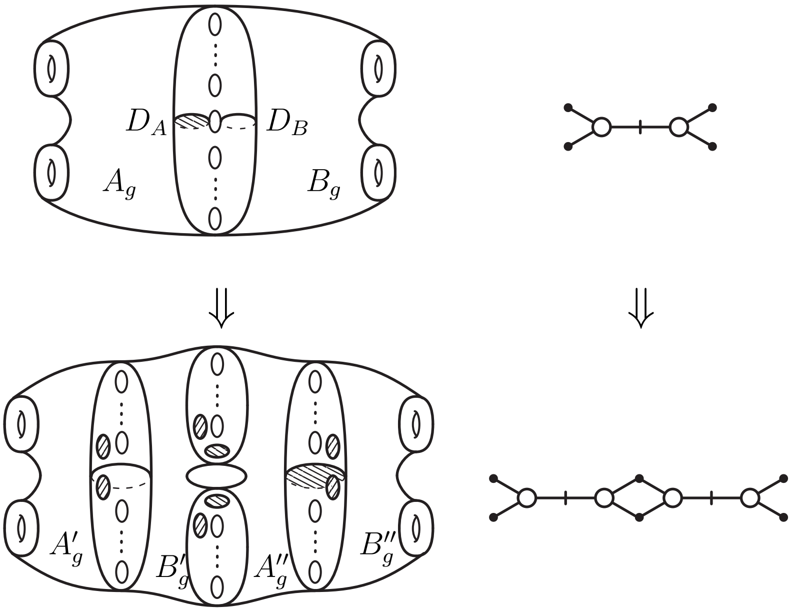

FIGURE 88. *The case of irreducible splittings*

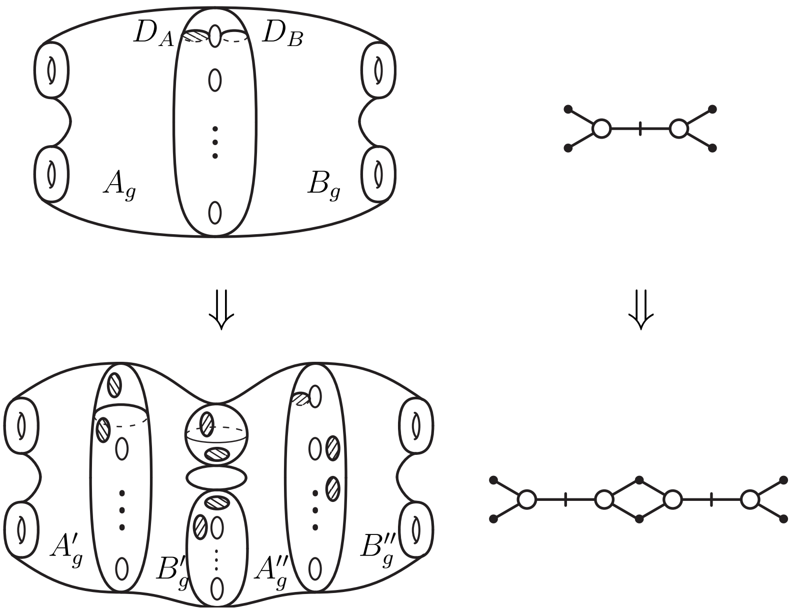

FIGURE 89. *The case of reducible splittings*

$\bar{B}'_g$ is a compression body with $\partial_+\bar{B}'_g = \partial_+ B'_g$. Set $A''_g = (\bar{G}_1 \times [0,1]) \cup h^1$ and $B''_g = (\partial_+ A''_g \times [0,1]) \cup \mathcal{H}_1^2 \cup \mathcal{H}_1^3$. Then each of A''_g and B''_g is a compression body. Hence we have:

$$\begin{aligned} M_g &\cong A'_g \cup B'_g \cup h^1 \cup (\mathcal{H}_1^2 \cup \mathcal{H}_2^2) \cup (\mathcal{H}_1^3 \cup \mathcal{H}_2^3) \\ &\cong A'_g \cup (B'_g \cup \mathcal{H}_2^2 \cup \mathcal{H}_2^3) \cup h^1 \cup \mathcal{H}_1^2 \cup \mathcal{H}_1^3 \\ &\cong A'_g \cup \bar{B}'_g \cup (\bar{G}_1 \times [0,1]) \cup h^1 \cup \mathcal{H}_1^2 \cup \mathcal{H}_1^3 \\ &\cong A'_g \cup \bar{B}'_g \cup A''_g \cup (\partial_+ A''_g \times [0,1]) \cup \mathcal{H}_1^2 \cup \mathcal{H}_1^3 \\ &\cong (A'_g \cup \bar{B}'_g) \cup (A''_g \cup B''_g). \end{aligned}$$

According to this decomposition, we can modify the fork complex $(\mathscr{F}, \rho)$ as in Figure 90. It is easy to see that for a new complex $(\mathscr{F}^*, \rho^*)$, we have $w(\mathscr{F}^*, \rho^*) < w(\mathscr{F}, \rho)$, a contradiction. Therefore if ∂D_A is non-separating, we have the desired conclusion.

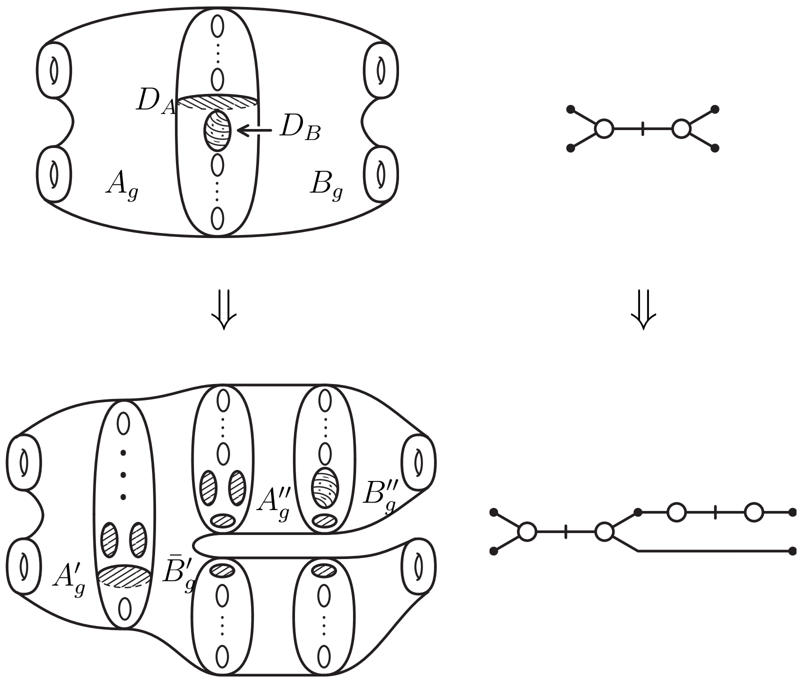

FIGURE 90

Suppose next that ∂D_B is non-separating in S_g. Then we start with the dual handle decomposition

$$M_g = (\partial_2 M_g \times [0,1]) \cup \bar{\mathcal{H}}^0 \cup \bar{\mathcal{H}}^1 \cup \bar{\mathcal{H}}^2 \cup \bar{\mathcal{H}}^3$$

and apply the above arguments which gives a contradiction.

Case 2. Each of ∂D_A and ∂D_B is separating in S.

Then A'_g consists of two compression bodies, say $\bar{A}'_g$ and $\tilde{A}'_g$ (*cf.* (6) of Remark 3.1.3). We may suppose that h^2 is attached to $\partial_+ \bar{A}'_g$. Set $B'_g = (\partial_+ \bar{A}'_g \times [0,1]) \cup h^2$. Since D_B is separating in S, we see that $\partial_- B'_g$ consists of two components, say G_1 and G_2. Note that D_A is also separating in S_g. Hence we may suppose that

$h^1 \cap G_2 \neq \emptyset$ and $h^1 \cap G_1 = \emptyset$. Let $\mathcal{H}_1^2$ be the components of $\mathcal{H}^2 \setminus h^2$ adjacent to G_1 and $\mathcal{H}_1^3$ be the components of $\mathcal{H}^3$ adjacent to G_1. Set $\mathcal{H}_2^2 = \mathcal{H}^2 \setminus (h^2 \cup \mathcal{H}_1^2)$, $\mathcal{H}_2^3 = \mathcal{H}^3 \setminus \mathcal{H}_1^3$ and $\bar{B}'_g = B'_g \cup \mathcal{H}_1^2 \cup \mathcal{H}_1^3$. Then $\bar{B}'_g$ is a compression body. Set $A^*_g = (G_2 \times [0,1]) \cup \tilde{A}'_g \cup h_1$ and $B''_g = (\partial_+ A''_g \times [0,1]) \cup \mathcal{H}_2^2 \cup \mathcal{H}_2^3$. Note that each of A''_g and B''_g is a compression body. Set $A''_g = \tilde{A}'_g \cup A^*_g$. Note also that A''_g is a compression body (*cf.* (5) of Remark 3.1.3). Hence we have:

$$\begin{aligned}
M_g &\cong A'_g \cup h^2 \cup h^1 \cup (\mathcal{H}_1^2 \cup \mathcal{H}_2^2) \cup (\mathcal{H}_1^3 \cup \mathcal{H}_2^3) \\
&\cong (\bar{A}'_g \cup \tilde{A}'_g) \cup h^2 \cup h^1 \cup (\mathcal{H}_1^2 \cup \mathcal{H}_2^2) \cup (\mathcal{H}_1^3 \cup \mathcal{H}_2^3) \\
&\cong \bar{A}'_g \cup \big((\partial_+ \bar{A}'_g \times [0,1]) \cup h^2\big) \cup \tilde{A}'_g \cup h^1 \cup (\mathcal{H}_1^2 \cup \mathcal{H}_2^2) \cup (\mathcal{H}_1^3 \cup \mathcal{H}_2^3) \\
&\cong \bar{A}'_g \cup B'_g \cup ((G_1 \cup G_2) \times [0,1]) \cup \tilde{A}'_g \cup h^1 \cup (\mathcal{H}_1^2 \cup \mathcal{H}_2^2) \cup (\mathcal{H}_1^3 \cup \mathcal{H}_2^3) \\
&\cong \bar{A}'_g \cup B'_g \cup ((G_1 \cup G_2) \times [0,1]) \cup \tilde{A}'_g \cup h^1 \cup (\mathcal{H}_1^2 \cup \mathcal{H}_2^2) \cup (\mathcal{H}_1^3 \cup \mathcal{H}_2^3) \\
&\cong \bar{A}'_g \cup \big(B'_g \cup (G_1 \times [0,1]) \cup \mathcal{H}_1^2 \cup \mathcal{H}_1^3\big) \cup \tilde{A}'_g \\
&\quad \cup \big((G_2 \times [0,1]) \cup \tilde{A}'_g \cup h_1\big) \cup \mathcal{H}_2^2 \cup \mathcal{H}_2^3 \\
&\cong \bar{A}'_g \cup \bar{B}'_g \cup (\tilde{A}'_g \cup A^*_g) \cup \big((\partial_+ A^*_g \times [0,1]) \cup \mathcal{H}_2^2 \cup \mathcal{H}_2^3\big) \\
&\cong (\bar{A}'_g \cup \bar{B}'_g) \cup (A''_g \cup B''_g).
\end{aligned}$$

According to this decomposition, we can modify the fork complex $(\mathscr{F}, \rho)$ as in Figure 91 or Figure 92. It is easy to see that for a new complex $(\mathscr{F}^*, \rho^*)$, we have $w(\mathscr{F}^*, \rho^*) < w(\mathscr{F}, \rho)$, a contradiction.

□

Lemma 4.2.4. *Any component $\rho^{-1}(t)$ is incompressible in M unless M is ∂-compressible, where t is a tine of $\mathscr{F}$.*

Proof. Suppose that $\rho^{-1}(t)$ is compressible in M for a tine t of $\mathscr{F}$. Let D be a compressing disk of $\rho^{-1}(t)$. Let $\mathcal{T}$ be the union of the tines of $\mathscr{F}$. By an innermost disk argument, we may assume that $D \cap \rho^{-1}(\mathcal{T}) = \partial D$. Let $\mathcal{F}_1$ be the fork containing $\rho(\eta(\partial D; D))$. Note that $\rho^{-1}(t)$ is incompressible in $\rho^{-1}(\mathcal{F}_1)$ (*cf.* (4) of Remark 3.1.3). Hence there is a fork $\mathcal{F}_2 (\neq \mathcal{F}_1)$ attached to the grip, say g, of $\mathcal{F}_1$. Since $D \cap \rho^{-1}(\mathcal{T}) = \partial D$, we have $D \subset \rho^{-1}(\mathcal{F}_1 \cup \mathcal{F}_2)$. Hence D is a ∂-compressing disk of $M' = \rho^{-1}(\mathcal{F}_1) \cup \rho^{-1}(\mathcal{F}_2)$. Hence it follows from (2) of Theorem 3.2.1 and Lemma 3.1.20 that the Heegaard splitting $(\rho^{-1}(\mathcal{F}_1), \rho^{-1}(\mathcal{F}_2); \rho^{-1}(g))$ is either weakly reducible or trivial. It also follows from Proposition 4.2.3 that $(\rho^{-1}(\mathcal{F}_1), \rho^{-1}(\mathcal{F}_2); \rho^{-1}(g))$ is strongly irreducible and hence the splitting must be trivial. Since $\rho^{-1}(\mathcal{F}_1)$ contains $\eta(\partial D; D)$, we see that $\rho^{-1}(\mathcal{F}_1)$ is a trivial compression body and that t is the tine of $\mathcal{F}_1$. Hence by Lemma 4.2.2, we have one of the following: (1) M is a 3-ball, (2) $M \cong \rho^{-1}(t) \times [0,1]$ and (3) M is ∂-compressible. We suppose that M does not satisfy the condition (3), i.e., M is ∂-incompressible. If M satisfies the condition (1), i.e., M is a 3-ball, then it follows from Example 4.1.12 that t is the only tine of $\mathscr{F}$ and that $\rho^{-1}(t)$ is incompressible in M. This contradicts that we suppose that $\rho^{-1}(t)$ is compressible in M. If M satisfies the condition (2), i.e., $M \cong \rho^{-1}(t) \times [0,1]$, then $\rho^{-1}(t)$ is incompressible in M, a contradiction. □

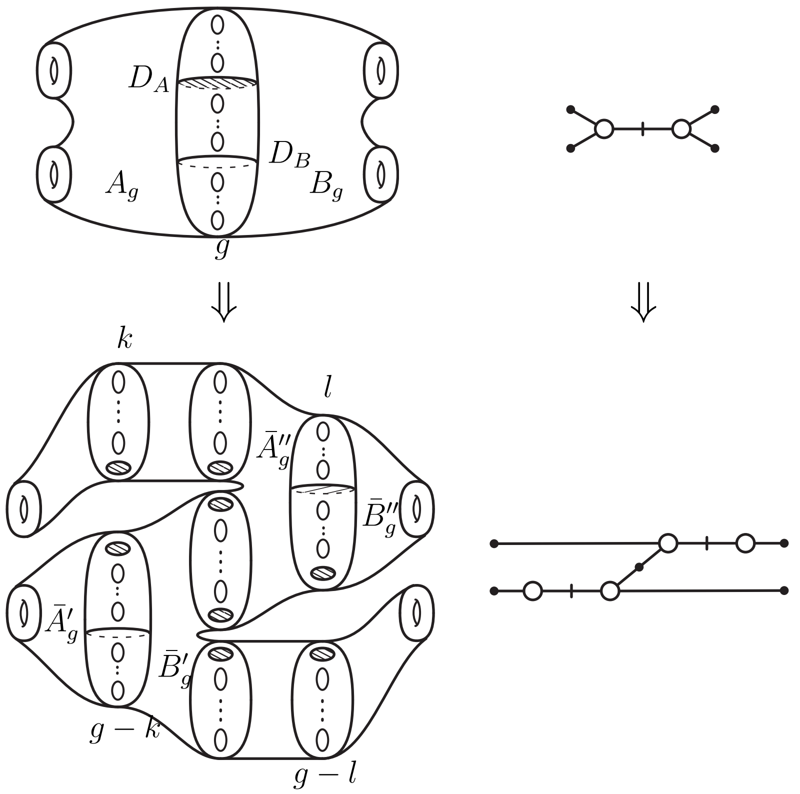

FIGURE 91. *The case of irreducible splittings*

As a direct consequence of Proposition 4.2.3 and Lemma 4.2.4, we have the following.

Corollary 4.2.5. $(\mathscr{F}, \rho)$ *is strongly irreducible unless* M *is* ∂*-compressible.*

Remark 4.2.6. There are strongly irreducible splittings which are not thin. In fact, there are strongly irreducible Heegaard splittings which are not minimal genus (*cf.* [2] and [7]).

Lemma 4.2.7. *Suppose that* $\mathscr{F}$ *contains a tine. There exists a tine* t *of* $\mathscr{F}$ *such that* $\rho^{-1}(t)$ *is a* 2*-sphere if and only if* M *is reducible or is a* 3*-ball.*

Proof. The "only if part" is immediate from Observation 4.2.1. Hence we will give a proof of the "if part".

Suppose that M is reducible or is a 3-ball. If M is a 3-ball, then it follows from Example 4.1.12 that there is exactly one tine, say t, of $\mathscr{F}$ and $\rho^{-1}(t) = \partial M$ is a 2-sphere. Hence in the remainder of the proof, we suppose that M is reducible. Let $\mathcal{T}$ be the union of the tines of $\mathscr{F}$. Let P be a reducing 2-sphere such that $|P \cap \rho^{-1}(\mathcal{T})|$ is minimal among such all reducing 2-spheres. By an innermost disk argument, we see that $P \cap \rho^{-1}(\mathcal{T}) = \emptyset$. Let $\mathcal{F}_1$ be a fork of $\mathscr{F}$ with $\rho^{-1}(\mathcal{F}_1) \cap P \neq \emptyset$.

Suppose first that there are no forks of $\mathscr{F}$ attaching to the grip of $\mathcal{F}_1$. Then this implies that P is an essential 2-sphere in $\rho^{-1}(t)$.

Suppose next that there is a fork of $\mathscr{F}$, say $\mathcal{F}_2$, other than $\mathcal{F}_1$ which attaches to the grip, say g, of $\mathcal{F}_1$. Note that $\rho^{-1}(\mathcal{F}_1) \cup \rho^{-1}(\mathcal{F}_2)$ contains P. It follows from (1) of Theorem 3.2.1 that $\rho^{-1}(\mathcal{F}_1)$ or $\rho^{-1}(\mathcal{F}_2)$ is reducible, or

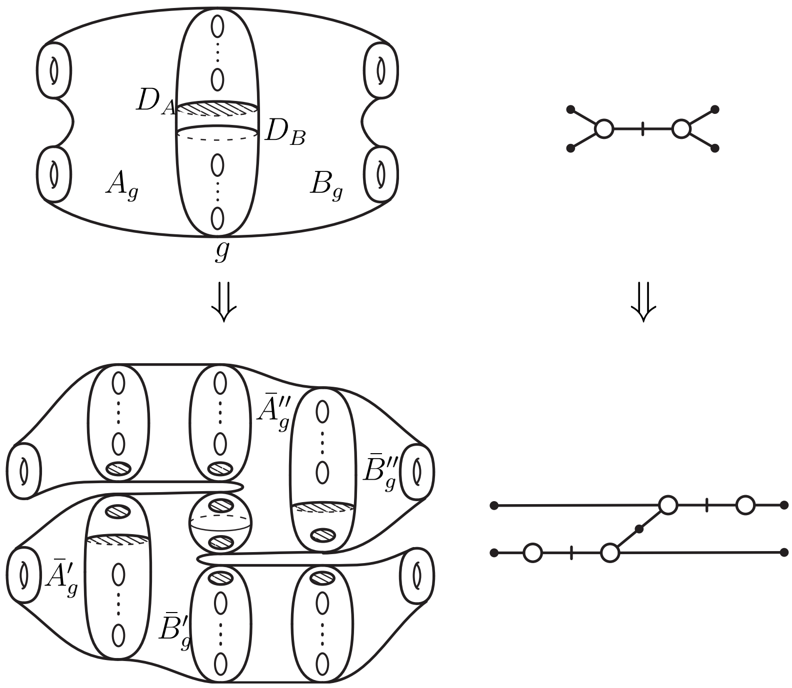

FIGURE 92. *The case of reducible splittings*

$(\rho^{-1}(\mathcal{F}_1), \rho^{-1}(\mathcal{F}_2); \rho^{-1}(g))$ is reducible. The latter condition, however, contradicts Proposition 4.2.3. Hence we may assume that $\rho^{-1}(\mathcal{F}_1)$ is reducible, that is, there is a 2-sphere component P_0 of $\partial_-(\rho^{-1}(\mathcal{F}_1))$ (*cf.* (1) of Remark 3.1.3). This implies that there is a tine t with $\rho^{-1}(t) = P_0$. □

Lemma 4.2.8. *If some $\rho^{-1}(g)$ is a torus, where g is a grip of $\mathscr{F}$, then one of the following holds.*
(1) *M is reducible.*
(2) *M is (a torus) $\times [0,1]$.*
(3) *M is a solid torus.*
(4) *M is a lens space.*

Proof. Suppose that M does not satisfy the conclusion (1) of Lemma 4.2.8, i.e., M is irreducible. Note that $\rho^{-1}(g)$ may be a boundary component of M. Let $\mathcal{F}$ be a fork such that the grip of $\mathcal{F}$ is g. Set $V = \rho^{-1}(\mathcal{F})$.

If V is trivial, then M is either $T^2 \times [0,1]$ or a solid torus by Lemma 4.2.2. Hence conclusion (2) or (3) of Lemma 4.2.8 holds.

If V is non-trivial, then we see that V is a solid torus by Observation 4.2.1 and Example 4.1.12. Suppose further that the conclusion (3) does not hold., i.e., M is not a solid torus. Then there is a fork $\mathcal{F}'(\neq \mathcal{F})$ attached to g. Set $V' = \rho^{-1}(\mathcal{F}')$. If V' is trivial, then it follows from Lemma 4.2.2 that M is a solid torus, a contradiction. If V' is non-trivial, then we see that V' is a solid torus by Observation 4.2.1 and Example 4.1.12. Hence M is a lens space and we have the conclusion (4) of Lemma 4.2.8. □

4.3. **Examples of generalized Heegaard splittings.** In this section, we use some theorems without proofs to obtain generalized Heegaard splittings and associated fork complexes. Let F_g be a connected closed orientable surface of genus g.

- $M = F_g \times [0,1]$.

Set $M = F_g \times [0,1]$, $A = F_g \times [0,1/2]$, $B = F_g \times [1/2,1]$ and $S = F_g \times \{1/2\}$. Clearly, $(A,B;S)$ is a Heegaard splitting of M, and we call this Heegaard splitting the *trivial Heegaard splitting of type I.* Let p be a point in F_g. Set

$$A' = \eta((F_g \times \{0\}) \cup (p \times [0,1]) \cup (F_g \times \{1\}); M),$$

$B' = \mathrm{cl}(M \setminus A')$ and $S' = A' \cap B'$. Then $(A',B';S')$ is also a Heegaard splitting of M, and we call this splitting the *trivial Heegaard splitting of type II.* The proof of the next observation is left to the reader.

Observation 4.3.1. *Both these Heegaard splittings are strongly irreducible.*

In fact, Scharlemann and Thompson proved the following.

Theorem 4.3.2 ([16] 2.11 Main Theorem)**.** *Any irreducible Heegaard splitting of* $F_g \times [0,1]$ *is trivial of type I or II.*

We remark that the fork complexes associated to these Heegaard splittings are illustrated in Figure 93.

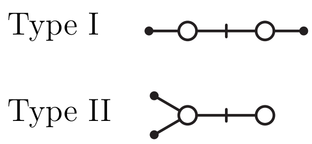

FIGURE 93

- $M = F_g \times S^1$.

Note that S^1 is regarded as $[0,1]/\{0\} \sim \{1\}$. Let p and q be distinct points in F_g. Set

$$A = \mathrm{cl}((F_g \times [0,1/2]) \setminus \eta(p \times [0,1/2]; F_g \times [0,1/2])) \cup \eta(q \times [1/2,1]; F_g \times [1/2,1])$$

and

$$\begin{aligned} B &= \mathrm{cl}(M \setminus A) \\ &= \mathrm{cl}((F_g \times [1/2,1]) \setminus \eta(q \times [1/2,1]; F_g \times [1/2,1])) \cup \eta(p \times [0,1/2]; F_g \times [0,1/2]). \end{aligned}$$

Note that A and B are handlebodies. Set $S = \partial A \cap \partial B$. Then $(A,B;S)$ is a Heegaard splitting of $M = F_g \times S^1$ and is called the *trivial Heegaard splitting* of $M = F_g \times S^1$ (*cf.* Figure 94).

Exercise 4.3.3. *Show that this trivial Heegaard splitting is weakly reducible.*

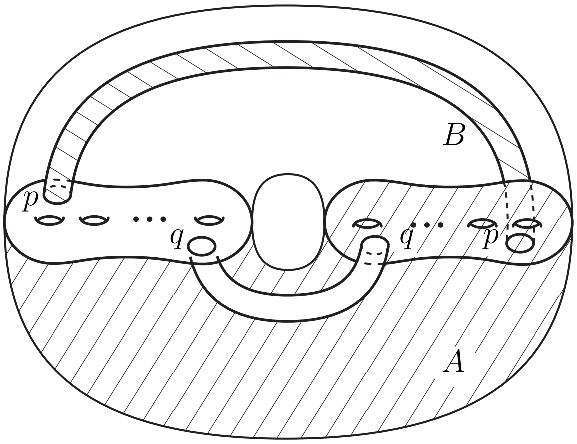

FIGURE 94

Theorem 4.3.4 ([18] Theorem 5.7)**.** *Any irreducible Heegaard splitting of* $F_g \times S^1$ *is the trivial splitting.*

- $T^3 = T^2 \times S^1$.

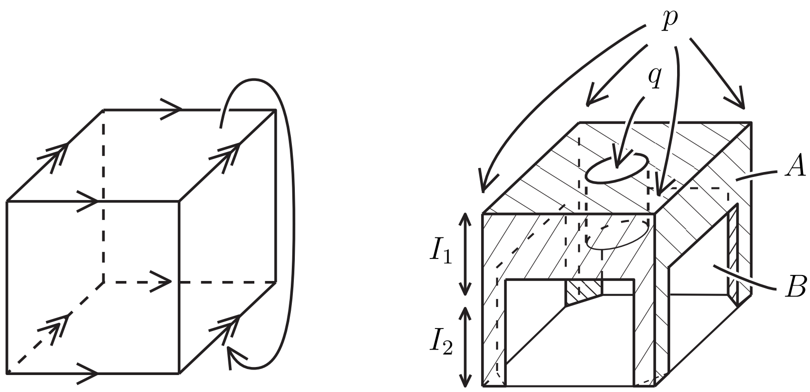

FIGURE 95

It is known that T^3 is obtained from a cube $[0,1] \times [0,1] \times [0,1]$ by attaching corresponding edges and faces as in Figure 95 (a). Set

$$A = \mathrm{cl}((T^2 \times [0,1/2]) \setminus \eta(p \times [0,1/2]; T^2 \times [0,1/2])) \cup \eta(q \times [1/2,1]; T^2 \times [1/2,1])$$

and

$$\begin{aligned} B &= \mathrm{cl}(T^3 \setminus A) \\ &= \mathrm{cl}((T^2 \times [1/2,1]) \setminus \eta(q \times [1/2,1]; T^2 \times [1/2,1])) \cup \eta(p \times [0,1/2]; T^2 \times [0,1/2]). \end{aligned}$$

Then we see that A and B are genus two handlebodies and that it follows from Theorem 4.3.4 that $(A, B; S)$ is the Heegaard splitting of T^3, where $S =$

$\partial A = \partial B$ (*cf.* Figure 95 (b)). Set $h^1 = \eta(q \times [1/2, 1]; T^2 \times [1/2, 1])$ and $h^2 = \eta(p \times [0, 1/2]; T^2 \times [0, 1/2])$. Note that h^1 (h^2 resp.) can be regarded as a 1-handle (2-handle resp.) in a handle decomposition of T^3 obtained from the Heegaard splitting $(A, B; S)$. Since $h^1 \cap h^2 = \emptyset$, we can perform a weak reduction to obtain a generalized Heegaard splitting. We give a concrete description of the generalized Heegaard splitting in the following. First, set $A_1 = \mathrm{cl}(T^3 \times [0, 1/2] \setminus h^1)$ and $B_2 = \mathrm{cl}(T^3 \times [1/2, 1] \setminus h^2)$. That is, A_1 is obtained from A by removing the 1-handle h^1 and B_2 is obtained from B by removing the 2-handle h^2. Then we have:

$$\begin{aligned} T^3 &= A \cup B \\ &= A_1 \cup h^1 \cup h^2 \cup B_2 \\ &\cong A_1 \cup (\partial A_1 \times [0,1]) \cup h^1 \cup h^2 \cup B_2 \\ &= A_1 \cup \big((\partial A_1 \times [0,1]) \cup h^2\big) \cup h^1 \cup B_2. \end{aligned}$$

Set $B_1 = (\partial A_1 \times [0,1]) \cup h^2$. Then B_1 is a compression body such that $\partial_+ B_1 = \partial A_1$ and $\partial_- B$ consists of two tori. Hence we have:

$$\begin{aligned} T^3 &\cong A_1 \cup B_1 \cup h^1 \cup B_2 \\ &\cong A_1 \cup B_1 \cup \big((\partial_- B_1 \times [0,1]) \cup h^1\big) \cup B_2. \end{aligned}$$

Set $A_2 = (\partial_- B_1 \times [0,1]) \cup h^1$. Then A_2 is a compression body such that $\partial_+ A_2 = \partial B_2$ and $\partial_- A_2 = \partial_- B_1$. Hence we have:

$$T^3 = (A_1 \cup B_1) \cup (A_2 \cup B_2).$$

This together with the fork complex as in Figure 96 gives a generalized Heegaard splitting.

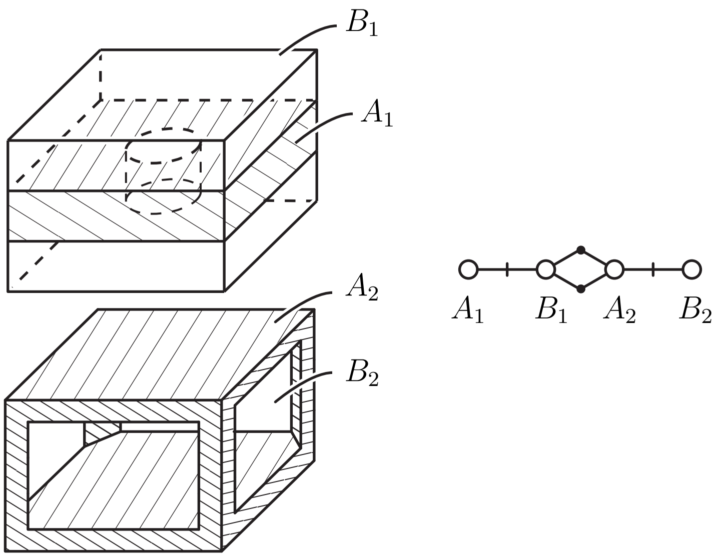

FIGURE 96

Exercise 4.3.5. Show that this is the only fork complex associated with a generalized Heegaard splitting of T^3 via weak reduction.

Remark 4.3.6. The inverse procedure of weak reduction is called *amalgamation*.

Exercise 4.3.7. Show that the above generalized Heegaard splitting of T^3 is strongly irreducible.

- $M = F_2 \times S^1$.

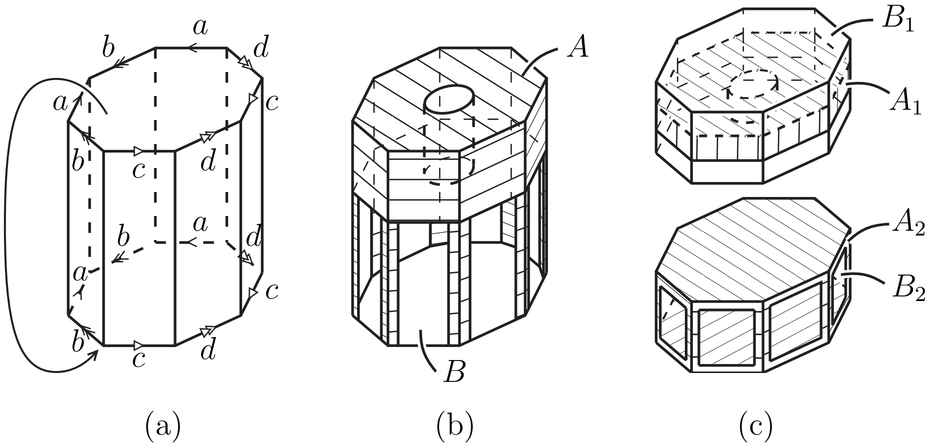

FIGURE 97

M is obtained from a (an octagon) $\times[0,1]$ by attaching corresponding edges and faces as in Figure 97 (a). Set

$$A = \mathrm{cl}((F_g \times [0,1/2]) \setminus \eta(p \times [0,1/2]; F_g \times [0,1/2])) \cup \eta(q \times [1/2,1]; F_g \times [1/2,1])$$

and

$$\begin{aligned} B &= \mathrm{cl}(M \setminus A) \\ &= \mathrm{cl}((F_g \times [1/2,1]) \setminus \eta(q \times [1/2,1]; F_g \times [1/2,1])) \cup \eta(p \times [0,1/2]; F_g \times [0,1/2]). \end{aligned}$$

Then it follows from Theorem 4.3.4 that we obtain the Heegaard splitting $M = A \cup B$ (see Figure 97 (b)). As descibed in case of $M = T^3$, we can perform a weak reduction and we obtain the same fork complex as that illustrated in Figure 96. In this case, each of A_1 and B_2 is a handlebody of genus four and each of A_2 and B_1 is a compression body with $\partial_+ A_2 = \partial B_2$, $\partial A_1 = \partial_+ B_1$ and $\partial_- A_2 = \partial_- B_1$.

For the Heegaard splitting $A \cup B$ of M, we can find another weak reduction as follows. Recall that $M = P_8 \times [0,1]/\sim$, where P_8 is an octagon (*cf.* Figure 97). Then there is a handle decomposition

$$M = h^0 \cup h^1_a \cup h^1_b \cup h^1_c \cup h^1_d \cup h^1_e \cup h^2_a \cup h^2_b \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3,$$

where a 0-handle h^0 corrsponds to a vertex of P_8, a 1-handle h^1_a (h^1_b, h^1_c, h^1_d and h^1_e resp.) corrsponds to a (b, c, d and e resp.) in P_8, a 2-handle h^2_a (h^2_b, h^2_c and h^2_d resp.) corrsponds to the face bounded by $eae^{-1}a^{-1}$ ($ebe^{-1}b^{-1}$, $ece^{-1}c^{-1}$ and $ede^{-1}d^{-1}$ resp.) in $\partial P_8 \times [0,1]$, a 2-handle h^2_e corrsponds to the face bounded by $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ in P_8 and a 3-handle h^3 corrsponds to the vertex in the interior of $P_8 \times [0,1]$. Set $A_1 = h^0 \cup h^1_a \cup h^1_e$. Then A_1 is a genus two handlebody and we have:

$$\begin{aligned} M &= A_1 \cup h^1_b \cup h^1_c \cup h^1_d \cup h^2_a \cup h^2_b \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &\cong A_1 \cup (\partial A_1 \times [0,1]) \cup h^1_b \cup h^1_c \cup h^1_d \cup h^2_a \cup h^2_b \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &= A_1 \cup \big((\partial A_1 \times [0,1]) \cup h^2_a\big) \cup h^1_b \cup h^1_c \cup h^1_d \cup \cup h^2_b \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \end{aligned}$$

Set $B_1 = (\partial A_1 \times [0,1]) \cup h^2_a$. Then B_1 is a compression body such that $\partial_+ B_1 = \partial A_1$ and $\partial_- B_1$ consists of two tori. Then we have:

$$\begin{aligned} M &\cong A_1 \cup B_1 \cup h^1_b \cup h^1_c \cup h^1_d \cup \cup h^2_b \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &\cong A_1 \cup B_1 \cup (\partial_- B_1 \times [0,1]) \cup h^1_b \cup h^1_c \cup h^1_d \cup \cup h^2_b \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &= A_1 \cup B_1 \cup \big((\partial_- B_1 \times [0,1]) \cup h^1_b\big) \cup h^1_c \cup h^1_d \cup \cup h^2_b \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3. \end{aligned}$$

Set $A_2 = (\partial_- B_1 \times [0,1]) \cup h^1_b$. Then A_2 is a compression body such that $\partial_+ A_2$ is a closed surface of genus two and $\partial_- A_2 = \partial_- B_1$. Then we have:

$$\begin{aligned} M &\cong A_1 \cup B_1 \cup A_2 \cup h^1_c \cup h^1_d \cup \cup h^2_b \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &\cong A_1 \cup B_1 \cup A_2 \cup (\partial_+ A_2 \times [0,1]) \cup h^1_c \cup h^1_d \cup \cup h^2_b \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &= A_1 \cup B_1 \cup A_2 \cup \big((\partial_+ A_2 \times [0,1]) \cup h^2_b\big) \cup h^1_c \cup h^1_d \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3. \end{aligned}$$

Set $B_2 = (\partial_+ A_2 \times [0,1]) \cup h^2_b$. Then B_2 is a compression body such that $\partial_+ B_2 = \partial_+ A_2$ and $\partial_- B_2$ consists of a torus. Then we have:

$$\begin{aligned} M &\cong A_1 \cup B_1 \cup A_2 \cup B_2 \cup h^1_c \cup h^1_d \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &\cong A_1 \cup B_1 \cup A_2 \cup B_2 \cup (\partial_- B_2 \times [0,1]) \cup h^1_c \cup h^1_d \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &= A_1 \cup B_1 \cup A_2 \cup B_2 \big((\partial_- B_2 \times [0,1]) \cup h^1_c\big) \cup h^1_d \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3. \end{aligned}$$

Set $A_3 = (\partial_- B_2 \times [0,1]) \cup h^1_c$. Then A_3 is a compression body such that $\partial_+ A_3$ is a closed surface of genus two and $\partial_- A_3 = \partial_- B_2$. Then we have:

$$\begin{aligned} M &\cong A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup h^1_d \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &\cong A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup (\partial_+ A_3 \times [0,1]) \cup h^1_d \cup h^2_c \cup h^2_d \cup h^2_e \cup h^3 \\ &= A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup \big((\partial_+ A_3 \times [0,1]) \cup h^2_c\big) \cup h^1_d \cup h^2_d \cup h^2_e \cup h^3. \end{aligned}$$

Set $B_3 = (\partial_+ A_3 \times [0,1]) \cup h^2_c$. Then B_3 is a compression body such that $\partial_+ B_3 = \partial_+ A_3$ and $\partial_- B_3$ consists of two tori. Then we have:

$$\begin{aligned} M &\cong A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup B_3 \cup h^1_d \cup h^2_d \cup h^2_e \cup h^3 \\ &\cong A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup B_3 \cup (\partial_- B_3 \times [0,1]) \cup h^1_d \cup h^2_d \cup h^2_e \cup h^3 \\ &= A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup B_3 \cup \big((\partial_- B_3 \times [0,1]) \cup h^1_d\big) \cup h^2_d \cup h^2_e \cup h^3. \end{aligned}$$

Set $A_4 = (\partial_- B_4 \times [0,1]) \cup h^1_d$. Then A_4 is a compression body such that $\partial_+ A_4$ is a closed surface of genus two and $\partial_- A_4 = \partial_- B_3$. Then we have:

$$\begin{aligned} M &\cong A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup B_3 \cup A_4 \cup h^2_d \cup h^2_e \cup h^3 \\ &\cong A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup B_3 \cup A_4 \cup (\partial_+ A_4 \times [0,1]) \cup h^2_d \cup h^2_e \cup h^3. \end{aligned}$$

Set $B_4 = (\partial_+ A_4 \times [0,1]) \cup h^2_d \cup h^2_e \cup h^3$. Then B_4 is a genus two handlebody such that $\partial B_4 = \partial_+ A_4$. Therefore we have the following decomposition.

$$T^3 = (A_1 \cup B_1) \cup (A_2 \cup B_2) \cup (A_3 \cup B_3) \cup (A_4 \cup B_4).$$

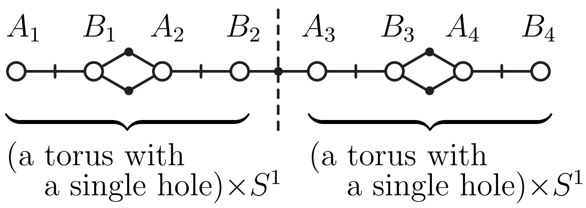

FIGURE 98

This together with the fork complex as in Figure 98 gives a generalized Heegaard splitting. We remark that $(A_1 \cup B_1) \cup (A_2 \cup B_2)$ ($(A_3 \cup B_3) \cup (A_4 \cup B_4)$ resp.) composes a (a torus with a single hole) $\times S^1$.

By changing the attaching order of h^1_b, h^1_c and h^1_d, we can obtain two more strongly irreducible generalized Heegaard splittings via weak reduction (*cf.* Figure 99).

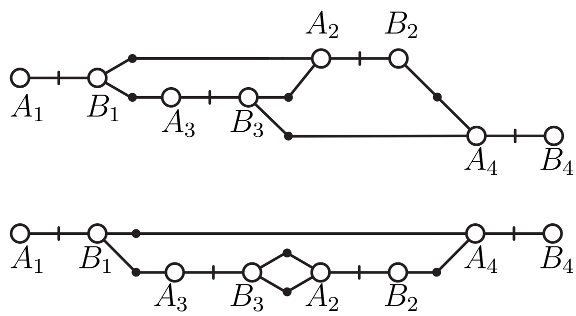

FIGURE 99

Exercise 4.3.8. Show that these are the only fork complexes associated with a generalized Heegaard splitting of $F_2 \times S^1$ via weak reduction.

Remark 4.3.9. The fork complexes associated with distinct weak reduction of a Heegaard splitting need not homotopic.

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