1. Tangent and Cotangent space

Definition 1.1. *A **topological manifold** M is a topological space, which is Hausdorff, has a countable basis for the topology, and which is locally homeomorphic to $\mathbb{R}^n$. (That is $\forall p \in M$, $\exists U$, an open neighbourhood of p, and a homeomorphism $h\colon U \longrightarrow U' \subset \mathbb{R}^n$.)*

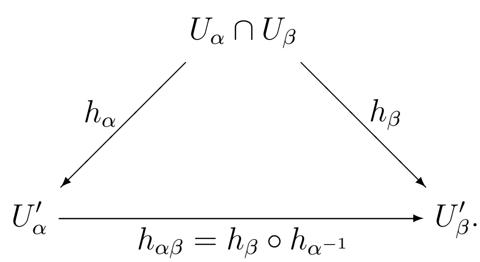
Example 1.2.

Definition 1.3. *A chart for M is any such homeomorphism $h\colon U \longrightarrow U'$. An atlas is a collection is a collection of charts*

$$\{\, h_\alpha \,|\, \alpha \in A \,\},$$

such that $\cup U_\alpha = M$.

Given two charts U_α and U_β, we have a diagram



Definition 1.4. *A **differentiable** (or C^∞) structure on M is an atlas such that $h_{\alpha\beta}$ is differentiable.*

There are three different ways to define and think of the tangent space:

- (A) the algebraist's
- (Ph) the physicists's
- (G) the geometer's.

1.1. **Algebraist's way.** First the algebraic way:

Definition 1.5. *We say that two functions f and $g\colon M \longrightarrow N$ are equivalent, at $p \in M$, if there exists a neighbourhood V of p such that $f|_V = g|_V$. The equivalence classes are called **germs**.*

Equivalently, a germ is a pair (f, U), where f is an function on U an open neighbourhood of p and two such germs (f, U) and (g, V) are considered equivalent, if there is an open subset $W \subset U \cap V$ such that $f|_W = g|_W$.

We denote the set of all C^∞-germs $f\colon (M, p) \longrightarrow \mathbb{R}$ by $\mathcal{C}^\infty_{M,p}$. $C^\infty_{M,p}$ is a real algebra: one can certainly add and multiply germs, just like one adds and multiplies functions. Given a germ $\bar{f} : (M,p) \longrightarrow (N,q)$, we get an algebra homomorphism

$$f^*\colon \mathcal{C}^\infty_{N,q} \longrightarrow \mathcal{C}^\infty_{M,p} \qquad \text{given by} \qquad \bar{\phi} \longrightarrow \bar{\phi} \circ \bar{f}$$

Note that f^* is functorial: the star of the identity is the identity and $(g \circ f)^* = f^* \circ g^*$. Note that if h is a chart which sends p to 0, then

$$h^*\colon \mathcal{C}^\infty_{M,p} \longrightarrow \mathcal{C}^\infty_{\mathbb{R},0},$$

is an isomorphism.

Definition 1.6 (Algebraists)**.** *Let M be a C^∞-manifold and let $p \in M$. A **derivation** of $C^\infty_{M,p}$ is a linear map $X\colon \mathcal{C}^\infty_{M,p} \longrightarrow \mathbb{R}$ such that*

$$X(\bar{\phi} \circ \bar{\psi}) = X(\bar{\phi}) \cdot \bar{\psi}(p) + \bar{\phi}(p) \cdot X(\bar{\psi}).$$

Then the tangent space to M at p, T_pM, is the set of all derivations of $\mathcal{C}^\infty_{M,p}$.

Now if we are given a germ

$$\bar{f}\colon (M,p) \longrightarrow (N,q)$$

then we get a map

$$T_pf\colon T_pM \longrightarrow T_qN$$

as follows. Given a derivation $X\colon \mathcal{C}^\infty_{M,p} \longrightarrow \mathbb{R}$, let $T_p(X) = X \circ f^*$, where f^* is defined as above.

Note also that the set of derivations is indeed a vector space. Also

$$\begin{aligned} X(1) &= X(1 \cdot 1) \\ &= 1 \cdot X(1) + X(1) \cdot 1 \\ &= X(1) + X(1). \end{aligned}$$

So $X(1) = 0$. It follows that $X(c) = 0$, for c any constant. Note that in this way we get a functor, since

$$T_p(\bar{g} \circ \bar{f}) = T_q(\bar{g}) \circ T_p(\bar{f}),$$

where $q = f(p)$. In particular, given a chart $h\colon M \longrightarrow \mathbb{R}^n$, which sends p to 0, we get an isomorphism

$$T_ph\colon T_pM \longrightarrow T_0\mathbb{R}^n \simeq \mathbb{R}^n.$$

Definition 1.7. *Given coordinates $(x_1, x_2, \dots, x_n)$ on $\mathbb{R}^n$, the function*

$$\frac{\partial}{\partial x_i}\colon \mathcal{C}^\infty_{\mathbb{R}^n,p} \longrightarrow \mathbb{R} \qquad \textit{which sends} \qquad \bar{\phi} \longrightarrow \left.\frac{\partial \bar{\phi}}{\partial x_i}\right|_{x=0}$$

is a derivation.

Lemma 1.8. *$\frac{\partial}{\partial x_i}$ are a basis of $T_0\mathbb{R}^n$.*

Lemma 1.9. *Let U be an open ball about the origin of $\mathbb{R}^n$, and let*

$$f\colon U \longrightarrow \mathbb{R}^n,$$

be a differentiable function.

Then $\exists f_1, f_2, \dots, f_n$ such that

$$f(x) = f(0) + \sum_{i=1}^{n} x_i f_i(x).$$

Proof.

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx_1, tx_2, \dots, tx_n)\, dt \\ &= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial}{\partial x_i} f(tx_1, tx_2, \dots, tx_n)\, dt, \end{aligned}$$

and so we are done if we set

$$f_i(x) = \int_0^1 \frac{\partial}{\partial x_i} f(tx_1, tx_2, \dots, tx_n)\, dt.$$

□

Proof of (1.8). Suppose that

$$\sum a_i \frac{\partial}{\partial x_i} = 0.$$

Applying $\bar{x}_j$ to this equation, we get

$$0 = \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \right) (\bar{x}_j) = a_j.$$

Thus these derivations are certainly linearly independent. Now suppose that $X \in T_0(\mathbb{R}^n)$. Set $a_i = X(\bar{x}_i)$, and

$$Y = X - \sum a_i \frac{\partial}{\partial x_i}.$$

Clearly $Y(\bar{x}_j) = 0$, for every j. Suppose that $\bar{f} \in \mathcal{C}^{\infty}_{\mathbb{R}^n,0}$. By (1.9), there are germs $\bar{f}_i$, such that

$$\bar{f} = \bar{f}(0) + \sum_{i=1}^{n} \bar{x}_i \bar{f}_i,$$

and so

$$
\begin{aligned}
Y(\bar{f}) &= Y(\bar{f}(0)) - \sum_{i=1}^{n} Y(\bar{x}_i) \cdot \bar{f}_i(0) \\
&= 0.
\end{aligned}
$$

But then $Y = 0$. $\square$

Suppose that we introduce coordinates $(x_1, x_2, \ldots, x_n)$ about $p \in M$ and $(y_1, y_2, \ldots, y_m)$ about $q \in N$. Then we get a commutative diagram

$$
\begin{array}{ccc}
(N, p) & \xrightarrow{\;f\;} & (M, q) \\
\downarrow & & \downarrow \\
(\mathbb{R}^n, 0) & \xrightarrow{\;\bar{f}\;} & (\mathbb{R}^m, 0).
\end{array}
$$

It is interesting to calculate $T_0 \bar{f}$. Pick $\bar{\phi} \in \mathcal{C}^{\infty}_{\mathbb{R}^m,0}$. Then

$$
\begin{aligned}
T_0 \bar{f} \left(\frac{\partial}{\partial x_i} \right) (\bar{\phi}) &= \frac{\partial}{\partial x_i} (\bar{\phi} \circ \bar{f}) \\
&= \sum_{j=1}^{m} \frac{\partial \phi}{\partial y_j}(0) \cdot \frac{\partial \bar{f}_j}{\partial x_i}(0).
\end{aligned}
$$

So

$$
T_0 \bar{f} \left(\frac{\partial}{\partial x_i} \right) = \sum_{j=1}^{m} \frac{\partial \bar{f}_j}{\partial x_i}(0) \cdot \frac{\partial}{\partial y_j}(0).
$$

The matrix

$$
Df = \left(\frac{\partial f_j}{\partial x_i} \right),
$$

is the Jacobian of f. If

$$
v = \sum a_i \frac{\partial}{\partial x_i} \qquad \text{then} \qquad T_0 f v = \sum_j b_j \frac{\partial}{\partial y_j},
$$

where

$$
b = (Df(0)) \cdot a.
$$

Putting all this together:

Theorem 1.10. *If one introduces coordinates* $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_m)$ *around* $p \in M^n$ *and* $q \in N^n$*, then the derivations* $\frac{\partial}{\partial x_i}$ *and* $\frac{\partial}{\partial y_j}$ *form a basis for* T_pM *and* T_qN*, and if* $\bar{f} \colon (M, p) \longrightarrow (N, q)$ *is a germ, the tangential map* T_pf *is given with respect to these bases as* Df_0*.*

So much for the algebraists definition, which although clean, does not always generalise so well (for example to C^p-manifolds or to infinite dimensional manifolds).

1.2. **Physicist's Way.** Physicists often start with a coordinate version of (1.10). Thus a typical statement might be "A contravariant vector, or a tensor of the first order, is a real n-tuple which transforms according to the Jacobi matrix."

Put differently, first pick one chart $\bar{h} \colon (M, p) \longrightarrow (\mathbb{R}^n, 0)$ and then think of all other charts $\bar{k} \colon (M, p) \longrightarrow (\mathbb{R}^n, 0)$ as being derived from

$$\bar{g} = \bar{k} \circ \bar{h}^{-1} \colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0),$$

so that $\bar{k} = \bar{g} \circ \bar{h}$. Denote the set of all such chart germs $\bar{g}$ by $\mathcal{G}$, which is a group under composition.

Given $\bar{g} \in \mathcal{G}$, assign the Jacobi matrix $Dg(0)$,

$$\mathcal{G} \longrightarrow \mathrm{GL}(n, \mathbb{R}).$$

Definition 1.11 (Physicists)**.** *A tangent vector at* $p \in M^n$ *is a rule which assigns to a chart*

$$\bar{h} \colon (M, p) \longrightarrow (\mathbb{R}^n, 0)$$

a vector $(v_1, v_2, \ldots, v_n)$*, so that the vector* $Dg(0) \cdot v$ *corresponds to the chart germ* $\bar{g} \circ \bar{h}$*.*

Let K_p *be the set of all chart germs*

$$\bar{h} \colon (N, p) \longrightarrow (\mathbb{R}^n, 0)$$

Then $T_p(M)_{Ph}$ *is the set of maps* $v \colon K_p \longrightarrow \mathbb{R}^n$ *such that*

$$v(\bar{g} \circ \bar{h}) = Dg(0)v(\bar{h}) \qquad \forall \bar{g} \in \mathcal{G}.$$

To get an element of K_p, all one needs to do is pick a chart $\bar{h}$ and assign a single vector $v \in \mathbb{R}^n$. Thus $T_p(M)_{\mathrm{Ph}} \simeq \mathbb{R}^n$.

1.3. **Geometer's Way.** Now we turn to the geometers. A tangent vector ought to be the velocity vector of a path based at p.

Definition 1.12. *Let* W_p *be the set of all germs* $\bar{w} \colon (\mathbb{R}, 0) \longrightarrow (M, p)$*. Let* $\sim$ *be the equivalence relation* $\bar{w} \sim \bar{v}$ *iff for every function germ* $\bar{f} \in \mathcal{C}^\infty_{M,p}$*,*

$$\left.\frac{d}{dt} \bar{f} \circ \bar{w}\right|_0 = \left.\frac{d}{dt} \bar{f} \circ \bar{v}\right|_0 .$$

The tangent space T_pM_G is then the set of equivalence classes $W_p/\sim$.

Thus to two path germs define the same tangent vector iff they define the same differentiation of functions in the direction of the curve. Given $[w]$, associate a derivation X_w as follows.

$$X_w(\bar{f}) = \frac{d}{dt}\bar{f}\circ\bar{w}\bigg|_0 .$$

We get an injection

$$T_p(M)_G \longrightarrow T_pM \qquad \text{which sends} \qquad [w] \longrightarrow X_w.$$

The map is also surjective, since if $w(t) = (ta_1, ta_2, \dots, ta_n)$ in local coordinates, then

$$X_w = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}.$$

With this definition, the tangent map is also very clear. A germ

$$\bar{f}\colon (M,p) \longrightarrow (N,q)$$

induces

$$T_pM \longrightarrow T_qN \qquad \text{where} \qquad [w] \longrightarrow [f\circ w].$$

1.4. **Cotangent Space.**

Definition 1.13. *The* ***space of differential forms at*** *p is the dual vector space of $T_pM = T^*_pM$.*

The dual basis to

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_n} \qquad \text{is} \qquad dx_1, dx_2, \dots, dx_n.$$

The total derivative df of a function

$$f\colon (M,p) \longrightarrow (N,q)$$

is the dual of the tangential map T_pf.

Lemma 1.14. *Given $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$,*

$$df = \left(\frac{\partial f}{\partial x_1}\right)\bigg|_0 dx_1 + \left(\frac{\partial f}{\partial x_2}\right)\bigg|_0 dx_2 + \dots \left(\frac{\partial f}{\partial x_n}\right)\bigg|_0 dx_n.$$

Proof. First note that

$$df\colon T^*_0\mathbb{R} \longrightarrow T^*_0\mathbb{R}^n,$$

so that we may identify df with the image of dt. Suppose that we write

$$df = \sum a_i dx_i.$$

Hit both sides with $\frac{\partial}{\partial x_j}$, to get

$$a_j = (df)(\frac{\partial}{\partial x_j}) = \frac{\partial f}{\partial x_j}.$$

□

In general df is given by the transpose of Df.