8. PROJECTIVE VARIETIES

Definition 8.1. *Let $X \subset \mathbb{P}^n$ be a projective variety.*

The ***homogeneous coordinate ring*** *of X is equal to the quotient $K[X_0, X_1, \dots, X_n]/I(X)$.*

Note that the homogeneous coordinate ring of X is a graded ring, since the ideal $I(X)$ is homogeneous. Note the following.

Lemma 8.2. *Let I be a homogeneous ideal in a graded ring R.*

Then the radical of I is also a homogeneous ideal.

Proof. Pick $r \in R$, such that $r^n \in I$. Suppose that

$$r = r_0 + r_1 + \dots r_k,$$

is the decomposition of r into its homogeneous pieces. We want to prove that r_i belongs to the radical of I. By induction, it suffices to prove that r_k is in the radical. But if we expand r^n, then r_k^n is the only part of degree nk. Since I is homogeneous, it follows that $r_k^n \in I$. Thus r_k is in the radical of I. □

Theorem 8.3. *Let I be a homogeneous ideal in the polynomial ring and let $X = V(I)$.*

Then $I(X)$ is equal to the radical of I.

Indeed the proof is the same as before, using 8.2. Note that this establishes a correspondence between projective varieties and homogeneous ideals, with the only twist being that the ideal

$$\langle X_0, X_1, \dots, X_n \rangle$$

does not correspond to any projective variety. One subtle point is that the homogeneous coordinate ring remembers the embedding, unlike the coordinate ring of an affine variety. Thus there is no correspondence between projective varieties and finitely generated graded K-algebras without nilpotents.

Note that for a projective variety, unlike for an affine variety there are three different ways in which a collection of homogeneous polynomials can cut out X.

Definition 8.4. *Let X be a projective variety and let $F_1, F_2, \dots, F_k$ be a collection of homogenous polynomials. We say that $F_1, F_2, \dots, F_k$ cuts out X*

1. ***set-theoretically*** *if $V(F_1, F_2, \dots, F_k) = X$.*
2. ***scheme-theoretically*** *if for every i, if $f_1, f_2, \dots, f_k$ denotes the dehomogenisation of $F_1, F_2, \dots, F_k$ in the affine piece $X_i \neq 0$, then $\langle f_1, f_2, \dots, f_k \rangle = I(X \cap U_i)$.*

(3) ***ideal-theoretically*** *if* $\langle F_1, F_2, \dots, F_k\rangle = I(X)$.

Note that $X \cap U_i$ is an affine variety in $\mathbb{A}^n$. Of course we are familiar with the first and last notion. A moments thought will convince the reader that the middle notion is intermediary between the other two. Let us see that this notion is really distinct from the other two.

In fact if $F_1, F_2, \dots, F_k$ generate the ideal of X, note that the products $G_{ij} = X_i F_j$, $0 \leq i \leq n$ and $0 \leq j \leq n$ certainly cut out X set-theoretically (essentially because the common vanishing locus of the X_i is empty). Now the products G_{ij} certainly don't generate the ideal of X (indeed, supposing that the degrees of F_j are increasing, it is clear that F_1 is not a combination of the G_{ij} by reasons of degree). On the other hand, on the affine open piece U_i, the dehomogenisation of G_{ij} is the same as the dehomogenisation of F_j. Thus G_{ij} certainly cut out X scheme-theoretically.

It is interesting to go back to some of the morphisms given before and give intrinsic definitions of these maps, at least in characteristic zero.

Let V be a vector space of dimension two. Then there is a natural map

$$V \longrightarrow \operatorname{Sym}^d(V)$$

obtained by sending a vector v to its dth symmetric power, v^d. This induces a map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^d$$

obtained by sending the line $[v]$ to the line $[v^d]$.

Lemma 8.5. *Suppose that the characteristic is zero (or more generally coprime to d)*

Then the map defined above is precisely the d-uple embedding.

Proof. Pick a basis e and f of V. Then a general vector in V is of the form $v = ae + bf$. Then we expand $(ae + bf)^d$ using the binomial Theorem.

$$(ae + df)^d = a^d e^d + \binom{d}{1} a^{d-1} b e^{d-1} + \dots b^d f^d.$$

Thus the given map is

$$[a, b] \longrightarrow [a^d : \binom{d}{1} a^{d-1} b : \binom{d}{2} a^{d-2} b^2 : \dots : b^d]$$

Replacing a by S and b by T, note that the entries gives us a basis for the polynomials of degree d. Thus changing coordinates we get the d-uple embedding. □

Note that this interpretation sheds new light on the fact that the rational normal curve is a determinental variety. Indeed we have identified the rational normal curve as being the locus of rank one symmetric tensors in $\mathrm{Sym}^d(V)$.

Similarly, in characteristic zero, the general d-uple embedding, has the same description. In particular the Veronese surface in $\mathbb{P}^5$, may be identified as the locus of rank one symmetric tensors inside $\mathbb{P}(\mathrm{Sym}^2(V))$, where V is a three dimensional vector space.

Proposition 8.6. *Let $X \subset \mathbb{P}^n$ be a projective variety.*

Then we can embed X into projective space so that X is cut out scheme-theoretically by quadratic equations.

Proof. Suppose that $F_1, F_2, \dots, F_k$ generate the ideal of X. Multiplying each F_i by all monomials of a given degree, we may assume that the degree of each F_i is the same (as above, the new polynomials still cut out X scheme-theoretically). Now consider the d-uple embedding of $\mathbb{P}^n$ into $\mathbb{P}^N$. Let Y be the image of X. Then the polynomials $F_1, F_2, \dots, F_k$ correspond to linear polynomials in $\mathbb{P}^N$. Since the image of $\mathbb{P}^N$ is cut out ideal theoretically by quadrics and the restriction of a quadric to a hyperplane is a quadric, it follows that Y is cut out by quadrics in some linear space contained in $\mathbb{P}^N$. $\square$

Note that we in fact get a little more than is suggested by 8.6. We can take the ranks of the quadrics to be small.

Definition 8.7. *Let*

$$\mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{mn+m+n}$$

denote the map given by

$$([X_0, X_1, \dots, X_m], [Y_0, Y_1, \dots, Y_n]) \longrightarrow [X_iY_j].$$

This map is easily seen to be a bijection and the image is a closed subset, defined by the quadratic polynomials

$$Z_{ij}Z_{kl} = Z_{il}Z_{kj},$$

(where of course Z_{ij} corresponds to X_iX_j). The image is called the **Segre variety**, and we define the product using this map, that is we are aiming for:

Proposition 8.8. *Let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$. Then the image of $X \times Y$ under the Segre map is the product (in the sense of category theory) of X and Y.*

Lemma 8.9. *The Segre Variety V is the product, in the sense of category theory, of $\mathbb{P}^m \times \mathbb{P}^n$.*

Proof. We have to exhibit two morphisms $p\colon V \longrightarrow \mathbb{P}^m$ and $q\colon V \longrightarrow \mathbb{P}^n$ and show that they satisfy the given universal property. Fix l and let $U_l \subset V$ be the open subset where at least one of Z_{il} is non-zero. Define a map

$$U_l \longrightarrow \mathbb{P}^m$$

by sending $[Z_{ij}]$ to $[Z_{il}]$. This is clearly a morphism, and these maps agree on overlaps. Moreover, varying l, the U_l cover $\mathbb{P}^{mn+m+n}$ so that we get a morphism on the whole of V.

By symmetry, this gives us two morphisms p and q. Moreover, under the identification of V with $\mathbb{P}^m \times \mathbb{P}^n$, it is clear that p and q are the ordinary projection maps. Since $\mathbb{P}^m \times \mathbb{P}^n$ is a product in the category of sets, given any morphisms $p'\colon Z \longrightarrow \mathbb{P}^m$ and $q'\colon Z \longrightarrow \mathbb{P}^n$, there is an induced unique function

$$f\colon Z \longrightarrow V$$

It suffices to check that f is a morphism. We check this locally. Let $U_{ij} \subset V$ be the locus where $Z_{ij} \neq 0$. Then U_{ij} corresponds to $U_i \times U_j$. We first check that U_{ij} is isomorphic to $\mathbb{A}^{m+n}$, which we have already seen is the product of $\mathbb{A}^m$ and $\mathbb{A}^n$. By symmetry, we may assume that $i = j = 0$. In this case, dehomogenising, the equations of U_{ij} become

$$z_{ij} = z_{i0}z_{0j} \qquad \text{and} \qquad z_{ij}z_{kl} = z_{il}z_{kj}.$$

Note that therefore the coordinate algebra of Uij is freely generated by z_{i0} and z_{0j} as expected. Thus $U_{ij} \simeq \mathbb{A}^{n+m}$.

Thus, by the universal property of the product, f_{ij}, the restriction of f to the inverse image of U_{ij}, is a morphism. $\square$

The general case, follows by the same argument, provided we can prove that the image of $X \times Y$ is a closed subset. In other words we have to say something about which subsets of V are closed.

Definition 8.10. *Let $F(X, Y)$ be a polynomial in $X_0, X_1, \dots, X_m$ and $Y_0, Y_1, \dots, Y_n$. We say that $F(X, Y)$ is bihomogeneous of bi-degree (d, e) if it is homogeneous of degree d in the variables $X_0, X_1, \dots, X_m$ and of degree e in the variables $Y_0, Y_1, \dots, Y_n$.*

For example, $X_0Y_1^2 + X_1Y_0Y_1$ is bihomogeneous of bi-degree $(1, 2)$.

Note that the zero locus of a bihomogeneous polynomial is a well-defined subset of the product.

Lemma 8.11. *Let $Z \subset V$ be a subset defined by bihomogeneous polynomials.*

Then Z is a closed subset.

Proof. Topping up the degree, we may as well assume that X is defined by bihomogeneous polynomials F of bi-degree (d, d). It suffices then to prove that there is a polynomial G on $\mathbb{P}^{mn+m+n}$ which pullsback to F. By linearity, it suffices to prove this for monomials. But Z_{ij} pullsback to X_iY_j, and we can clearly build any monomial X^IY^J, as product of such monomials, provided that X^I and Y^J have the same degree. □

Proof of 8.8. By 8.11 the image of $X \times Y$ is a closed subset of the Segre variety, under the Segre map, and the rest of the the proof goes through as before. □

It is interesting to see what happens in a specific example. Suppose we take the twisted cubic in $\mathbb{P}^3$. This lies in the quadric $XW = YZ$, that is it lies in the Segre variety. Now it also lies in the quadric Y^2-XZ. Pulling back to $\mathbb{P}^1\times\mathbb{P}^1$ we get the bihomogeneous polynomial

$$(X_0Y_1)^2 - (X_0Y_0)(X_1Y_0) = X_0(X_0Y_1^2 - X_1Y_0^2).$$

Now the equation $X_0 = 0$ corresponds to a line in the quadric (see below), and what is left defines the twisted cubic. Thus the twisted cubic is defined by a bi-homogeneous polynomial of type $(1, 2)$.

It is also interesting to see what happens to $\mathbb{P}^1 \times \{p\}$ and $\{q\} \times \mathbb{P}^1$. Indeed $[\lambda_0 : \lambda_1] \times [Y_0 : Y_1]$ is sent to $[\lambda_0Y_0 : \lambda_0Y_1 : \lambda_1Y_0 : \lambda_1Y_1]$, which is the parametric form of a line. In fact the equations of the line are $\lambda_1X = \lambda_0Z$ and $\lambda_1Y = \lambda_0W$.

Similarly $[X_0 : X_1]\times[\mu_0 : \mu_1]$ is sent to the line $[\mu_0X_0 : \mu_1X_0 : \mu_0X_1 : \mu_1X_1]$. This line has equations $\mu_1X = \mu_0Y$ and $\mu_1Z = \mu_0W$.

It follows that the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is covered by two 1-parameter families of lines.

There is in fact another way to look at all of this. Let V and W be two vector spaces of dimension two. Consider the natural map

$$V \times W \longrightarrow V \otimes W$$

This induces a map

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

Let us calculate what this map is in terms of coordinates. A general vector $v \in V$ has the form $v = ae + bf$, where $\{e, f\}$ is a basis of V. Similarly a general vector $w \in W$, is of the form $ce + df$. Thus the pair (v, w) is sent to

$$\begin{aligned} v \otimes w &= (ae + bf) \otimes (ce + df) \\ &= ac(e \otimes e) + ad(e \otimes f) + bc(f \otimes e) + bd(f \otimes f). \end{aligned}$$

The induced map is then

$$\begin{aligned}([v],[w]) &\longrightarrow [v\otimes w] = [ac:ad:bc:bd] \\ ([a:b],[c:d]) &\longrightarrow [ac(e\otimes e)+ad(e\otimes f)+bc(f\otimes e)+bd(f\otimes f)]\end{aligned}$$

which is clearly the Segre map. Thus the Segre variety consists of all tensors of rank one. The two families of lines, are given as $[v]\times\mathbb{P}^1$ and $\mathbb{P}^1\times[w]$.

Cearly this generalises in an obvious way to the general Segre variety, which is covered by two families of linear spaces. A family of linear spaces of dimension m, parametrised by $\mathbb{P}^n$ and a family of linear spaces of dimension n, parametrised by $\mathbb{P}^m$.

This also sheds some light on the fact that the twisted cubic is not the intersection of two quadrics. If one of the quadrics has maximal rank 4 (or better one of the quadrics in the pencil, which is in fact always true), then it projectively equivalent to the Segre variety. In this case the other quadric cuts out a curve of bi-degree $(2,2)$ on $\mathbb{P}^1\times\mathbb{P}^1$. As the twisted cubic has bi-degree $(1,2)$, it follows that we get not only the twisted cubic, but a line (something of bi-degree $(1,0)$), so that the union has bi-degree $(2,2)$. Now the line is a fibre of one of the rulings, and a general fibre meets the cubic in two points (since a quadratic polynomial has two roots in general).

In fact projecting a curve C of bi-degree (d,e) to either factor defines a morphism $C\longrightarrow\mathbb{P}^1$ which has degree d (respectively e), that is the typical fibre contains d points (at least in characteristic zero).