

26. INFINITE PRODUCTS

Definition 26.1. Let p_1, p_2, \dots be an infinite sequence of complex numbers.

$$p_1 p_2 p_3 \cdots p_n \cdots = \prod_{i=1}^{\infty} p_i$$

is defined to be the limit of the partial products

$$P_n = p_1 p_2 \cdots p_n.$$

Note that we have to be very careful if any of the terms are zero. We adopt the following convention. We will say that the product converges, if there are only finitely many $p_i = 0$ and the limit of the non-zero terms exists and is not equal to zero.

Note that if the product converges, then in fact $\lim_{n \rightarrow \infty} p_n = 1$. Indeed

$$p_n = \frac{P_n}{P_{n-1}}.$$

Given this, we set $p_n = 1 + a_n$ and rewrite our infinite product in the form,

$$\prod_{n=1}^{\infty} (1 + a_n).$$

If this product converges, then $a_n \rightarrow 0$. Taking logarithms we obtain

$$\sum_{n=1}^{\infty} \log(1 + a_n).$$

Here we take the principal branch of the logarithm.

Theorem 26.2. *The infinite product*

$$\prod_{n=1}^{\infty} (1 + a_n)$$

converges iff

$$\sum_{n=1}^{\infty} \log(1 + a_n)$$

converges.

Proof. Suppose that the series converges. Let the partial sums be S_n and set

$$S = \lim_{n \rightarrow \infty} S_n.$$

Then $P_n = e^{S_n}$ and by continuity

$$P = \lim_{n \rightarrow \infty} P_n = \lim_{n \rightarrow \infty} e^{S_n} = e^S.$$

Thus one direction is clear. The only tricky part of the other direction is to deal with the fact that the logarithm has more than one branch. In general $\log P_n$ does not converge to $\log P$, but to one of its branches. Now as

$$\frac{P_n}{P} \rightarrow 1,$$

it follows that

$$\log\left(\frac{P_n}{P}\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now, for every n , there is an integer h_n such that

$$\log\left(\frac{P_n}{P}\right) = S_n - \log P + h_n(2\pi i).$$

Taking differences, we have

$$(h_{n+1} - h_n)(2\pi i) = \log\left(\frac{P_{n+1}}{P}\right) - \log\left(\frac{P_n}{P}\right) - \log(1 + a_n),$$

so that taking the argument

$$(h_{n+1} - h_n)(2\pi) = \arg\left(\frac{P_{n+1}}{P}\right) - \arg\left(\frac{P_n}{P}\right) - \arg(1 + a_n).$$

Now the first two terms on the left are approaching each other and the absolute value of the last term is at most π . Thus $h_{n+1} = h_n$ for n sufficiently large and so the series converges. \square

Theorem 26.3. *The product*

$$\prod_{n=1}^{\infty} (1 + a_n)$$

converges absolutely iff

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely.

Proof. It suffices to prove that

$$\sum_{n=1}^{\infty} |\log(1 + a_n)|$$

converges iff

$$\sum_{n=1}^{\infty} |a_n|$$

converges. As

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1,$$

and as convergence of either series implies $a_n \rightarrow 0$, it follows that for n sufficiently large, and for any $\epsilon > 0$,

$$(1 - \epsilon)|a_n| < |\log(1 + a_n)| < (1 + \epsilon)|a_n|.$$

The result follows immediately. □