22. Residues

Let f be a holomorphic function with an isolated singularity at a. Pick a small circle γ centred at a and consider the integral

$$P = \int_\gamma f(z)\, dz.$$

P is called a period of f.

As the function $f(z) = \frac{1}{z-a}$ has period $2\pi i$ the function

$$g(z) = f(z) - \frac{R}{z-a}, \qquad \text{where} \qquad R = \frac{P}{z-a},$$

has period zero, with respect to γ. It follows that g is the derivative of some function.

Definition 22.1. *Let f be a holomorphic function with an isolated singularity at a. The residue of f at a is the unique complex number R, so that the function*

$$g(z) = f(z) - \frac{R}{z-a},$$

for some small $0 < |z-a| < \delta$, is the derivative of another function.

It is useful to employ the following notation for the residue,

$$R = \operatorname{Res}_{z=a} f(z).$$

Theorem 22.2. *(Residue Theorem) Let f be a holomorphic function on a region U, with isolated singularities at $a_1, a_2, \dots$. Let γ be a path in U that is homotopic to zero and does not contain any of the points $a_1, a_2, \dots$.*

Then

$$\frac{1}{2\pi i} \int_\gamma f\, dz = \sum_j n(\gamma; a_j) \operatorname{Res}_{z=a_j} f(z).$$

Proof. Pick small circles γ_j, centred at a_j, contained in U. Consider the path $\gamma' = \gamma - \sum n(\gamma; a_j)\gamma_j$. We want to apply Cauchy's Theorem to γ'. It suffices to check that the winding number of γ' about any complex number $a \in \mathbb{C} - (U - \{a_1, a_2, \dots, a_k\})$ is zero. Note that the regions of $\mathbb{C} - \gamma'$ are equal to the regions of $\mathbb{C} - \gamma$, union the small discs about each a_i. Since γ is homologous to zero, the only non-zero winding numbers for γ are about a_i. By definition of γ' the winding number of γ' about a_i is zero. It follows that γ' has zero winding number about any point.

Thus by Cauchy's Theorem

$$\int_{\gamma'} f(z)\, dz = 0.$$

Rewriting, we get

$$\int_\gamma f(z)\,dz = \sum_j n(\gamma; a_j) P_j,$$

where

$$P_j = \int_{\gamma_j} f(z)\,dz.$$

The result follows by definition of R_j. □

Of course, this Theorem is useless without an effective means of computing the residue.

Lemma 22.3. *Suppose that $f(z)$ has a pole of order one at a. Then*

$$\operatorname{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z).$$

Proof. By assumption

$$f(z) = \frac{b_{-1}}{z-a} + b_0 + b_1(z-a) + b_2(z-a)^2 + \dots = \frac{b_{-1}}{z-a} + g(z).$$

Hence $g(z)$ is a holomorphic function. Thus by definition the residue is b_{-1}. Clearly $b_{-1} = \lim_{z\to a} f(z)$. □

Of course one of the main uses of the residue Theorem is to compute contour integrals.

For example

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Look at the following contour. Let γ be the closed path, that starts at zero, goes along the real axis to R, describes a semi-circle of radius R and then traverses the x-axis from $-R$ to zero. Consider applying the Residue Theorem to $f(z) = \frac{1}{1+z^2}$. Then $f(z)$ has two isolated singularities at $z = \pm i$. The winding number of γ about the first is 1 and about the second is zero. The residue at $z = i$ can be computed in one of two ways. First observe that

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i(z+i)} + \frac{1}{2i(z-i)}.$$

Thus the residue at $z = i$ is by definition $1/2i$. Alternatively take f and multiply it by $(z-i)$, to get

$$\frac{1}{z+i}.$$

At $z = i$ we get $1/2i$. Thus by the residue Theorem

$$\int \frac{1}{1+z^2}\,dz = \pi.$$

On the other hand the integral may by split into two parts. The integral along the real-axis from $-R$ to R and the integral along a semi-circle. Along the semi-circle,

$$|f(z)| \leq \frac{1}{R^2 - 1}$$

so that the integral along the semi-circle is at most

$$\pi \frac{R}{R^2 - 1}$$

which tends to zero as R tends to infinity. The result follows, as the function $\frac{1}{1+x^2}$ is even.

Now consider the integral

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Consider the integral

$$\int_\gamma \frac{e^{iz}}{z}\,dz,$$

where γ is the contour that starts at ρ goes along the x-axis to R, goes around a semi-circle counterclockwise to $-R$, goes back to $-\rho$ and traverses a semi-circle, clockwise around the origin. The only pole of the function

$$f(z) = \frac{e^i z}{z},$$

is at the origin and the winding number of γ about the origin is zero. Thus by the residue Theorem, the integral of $f(z)$ around γ is zero. We split the integral into four pieces.

$$\int_\rho^R \frac{e^i x}{x}\,dx + \int_{\gamma_0} f(z)\,dz + \int_{-R}^{-\rho} \frac{e^i x}{x}\,dx + \int_{\gamma_1} f(z)\,dz.$$

The two integrals along the x-axis, when combined, give

$$\int_\rho^R \frac{e^{ix} - e^{-ix}}{x}\,dx = 2i \int_\rho^R \frac{\sin x}{x}\,dx.$$

Consider the behaviour around the big semi-circle.

$$\left|\int_{\gamma_0} \frac{e^{iz}}{z}\, dz\right| = \left|\int_0^\pi e^{iRe^{i\theta}}\, d\theta\right|$$
$$\leq \left|\int_0^\pi e^{-R\sin\theta}\, d\theta\right|$$
$$\leq \int_0^\delta d\theta + \int_\delta^{\pi-\delta} e^{-R\sin\delta}\, d\theta$$

As R tends to zero, we may let δ approach zero. Thus the integral goes to zero.

Now consider the behaviour around the small semi-circle.

$$\int_\gamma \frac{e^{iz}}{z}\, dz = \int_\gamma \frac{1}{z}\, dz + \int_\gamma \frac{e^{iz}-1}{z}\, dz.$$

Using the Talyor series expansion of e^{iz} it is clear the second integral converges to zero as ρ approaches 1. On the other hand, by direct computation, the first integral comes out as

$$\int_\gamma \frac{1}{z}\, dz = \int_\pi^0 i\, dz = -\pi i.$$

Thus, letting $R \to \infty$, $\rho \to 0$, we get

$$2i\int_0^\infty \frac{\sin x}{x}\, dx - \pi i = 0,$$

whence the result.

Finally consider

$$\int_0^\pi \log\sin\theta\, d\theta.$$

Consider the function

$$1 - e^{2iz} = -2ie^{iz}\sin z.$$

As

$$1 - 2e^{iz} = 1 - e^{-2y}\left(\cos 2x + i\sin 2x\right),$$

we see that this function is only real and negative, if $y < 0$ and $x = n\pi$. So if we delete these half lines, we may assume that log is single-valued and analytic.

We now integrate along the rectangle with corners, 0, π, $\pi + iY$ and iY. At the points 0 and π we choose small arcs of circles, of radius δ, to avoid these points. By periodicity, the integrals along the vertical sides are zero. The integral along the top horizontal line goes to zero, as Y goes to infinity.

I claim that the same is true over the quadrants. The imaginary part of the logarithm is bounded, so we only need worry about the real part. Now

$$\frac{|1 - e^{2iz}|}{|z|} \to 2,$$

for $z \to 0$ so that the logarithm behaves like $\log \delta$. As $\delta \log \delta$ tends to zero, the integral tends to zero around the first quadrant. Similarly for the second quadrant.

Thus

$$\int_0^\pi \log(-2ie^{ix} \sin z)\, dx = 0.$$

Suppose we choose the branch of the logarithm so that

$$\log(e^{ix}) = ix.$$

Then the imaginary part lies between 0 and π. So $\log(-i) = -\pi i/2$. Thus

$$\pi \log 2 - \pi^2 i/2 + \int_0^\pi \log \sin x\, dx + \pi^2/2i = 0.$$

and so

$$\int_0^\pi \log \sin x\, dx = -\pi \log 2.$$