

## 17. THE ALTERNATING GROUPS

Consider the group  $S_3$ . Then this group contains a normal subgroup, generated by a 3-cycle.

Now the elements of  $S_3$  come in three types. The identity, the product of zero transpositions, the transpositions, the product of one transposition, and the three cycles, products of two transpositions. Then the normal subgroup above, consists of all permutations that can be represented as a product of an even number of transpositions.

In general there is no canonical way to represent a permutation as a product of transpositions. But we might hope that the pattern above continues to hold in every permutation group.

**Definition 17.1.** *Let  $\sigma \in S_n$  be a permutation.*

*We say that  $\sigma$  is **even** if it can be represented as a product of an even number of transpositions. We say that  $\sigma$  is **odd** if it can be represented as a product of an odd number of transpositions.*

The following result is much trickier to prove than it looks.

**Lemma 17.2.** *Let  $\sigma \in S_n$  be a permutation.*

*Then  $\sigma$  is not both an even and an odd permutation.*

There is no entirely satisfactory proof of 17.2. Here is perhaps the simplest.

**Definition 17.3.** *Let  $x_1, x_2, \dots, x_n$  be indeterminates and set*

$$f(x_1, x_2, \dots, x_n) = \prod_{i < j} (x_i - x_j).$$

For example, if  $n = 3$ , then

$$f(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

**Definition 17.4.** *Given a permutation  $\sigma \in S_n$ , let*

$$g = \sigma^*(f) = \prod_{i < j} (x_{\sigma(i)} - x_{\sigma(j)}).$$

Suppose that  $\sigma = (1, 2) \in S_3$ . Then

$$g = \sigma^*(f) = (x_2 - x_1)(x_2 - x_3)(x_1 - x_3) = -(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = -f.$$

The following Lemma is the key part of the proof of 17.2.

**Lemma 17.5.** *Let  $\sigma$  and  $\tau$  be two permutations and let  $\rho = \sigma\tau$ . Then*

- (1)  $\sigma^*(f) = \pm f$ .
- (2)  $\rho^*(f) = \sigma^*(\tau^*(f))$ .
- (3)  $\sigma^*(f) = -f$ , whenever  $\sigma$  is a transposition.

*Proof.*  $g$  is clearly a product of terms of the form  $x_i - x_j$  or  $x_j - x_i$ . Thus  $g = \pm f$ . Hence (1).

$$\begin{aligned}\sigma^*(\tau^*(f)) &= \sigma^*(\prod_{i < j} (x_{\tau(i)} - x_{\tau(j)})) \\ &= \prod_{i < j} (x_{\sigma(\tau(i))} - x_{\sigma(\tau(j))}) \\ &= \prod_{i < j} (x_{\rho(i)} - x_{\rho(j)}) \\ &= \rho^*(f).\end{aligned}$$

Hence (2).

Suppose that  $\sigma = (i, j)$ , where  $i < j$ . Then the only terms of  $f$  affected by  $\sigma$  are the ones that involve either  $x_i$  or  $x_j$ . Suppose  $a \neq i$ ,  $a \neq j$ . If  $a < i$ , then  $x_a - x_i$  is sent to  $x_a - x_j$  and there is no change of sign. If  $i < a < j$  then  $x_i - x_a$  is sent to  $x_j - x_a = -(x_a - x_j)$ . Thus there is a change in sign. If  $j < a$  then  $x_i - x_a$  is sent to  $x_j - x_a$  and there is no change in sign. Similarly if we consider  $x_j - x_a$  then we get a similar result and the changes in sign will cancel out. Otherwise we need to consider  $x_i - x_j$  when there is one change of sign. In total then the sign changes. Hence (3).  $\square$

*Proof.* Suppose that  $\sigma$  is a product of an even number of transpositions. Then by (2) and (3) of 17.5,  $\sigma^*(f) = f$ . Similarly if  $\sigma^*(f)$  is a product of an odd number of transpositions, then  $\sigma^*(f) = -f$ . Thus  $\sigma$  cannot be both even and odd.  $\square$

**Definition-Lemma 17.6.** *There is a surjective homomorphism*

$$\phi: S_n \longrightarrow \mathbb{Z}_2$$

*The kernel consists of the even transpositions, and is called the **alternating group**  $A_n$ .*

*Proof.* The map sends an even transposition to 1 and an odd transposition to  $-1$ . (2) of 17.5 implies that this map is a homomorphism.  $\square$

Note that half of the elements of  $S_n$  are even, so that the alternating group  $A_n$  contains  $\frac{n!}{2}$ . One of the most important properties of the alternating group is,

**Theorem 17.7.** *Suppose that  $n \geq 5$ .*

*The only normal subgroup of  $S_n$  is  $A_n$ . Moreover  $A_n$  is simple, that is  $A_n$  has no proper normal subgroups.*