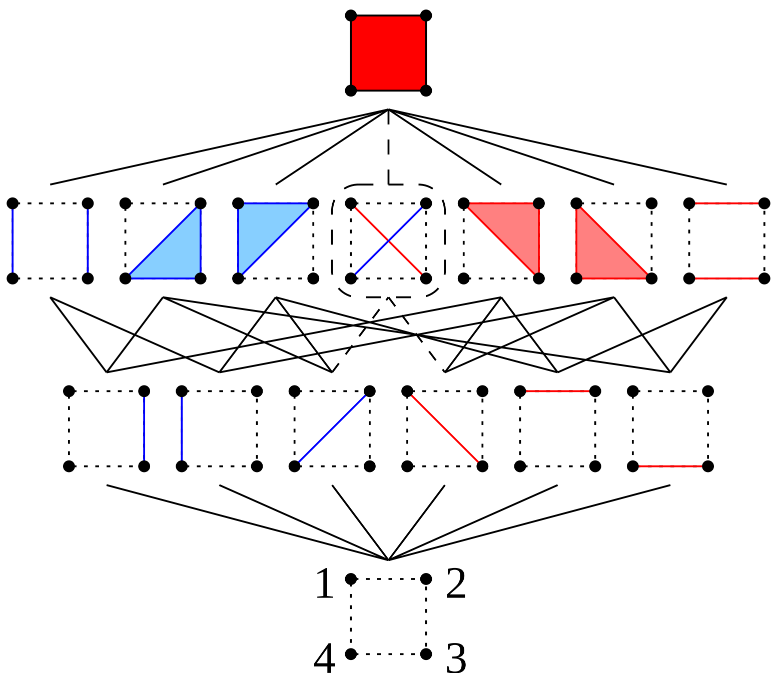
Non-crossing partitions for arbitrary Coxeter groups



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Outline

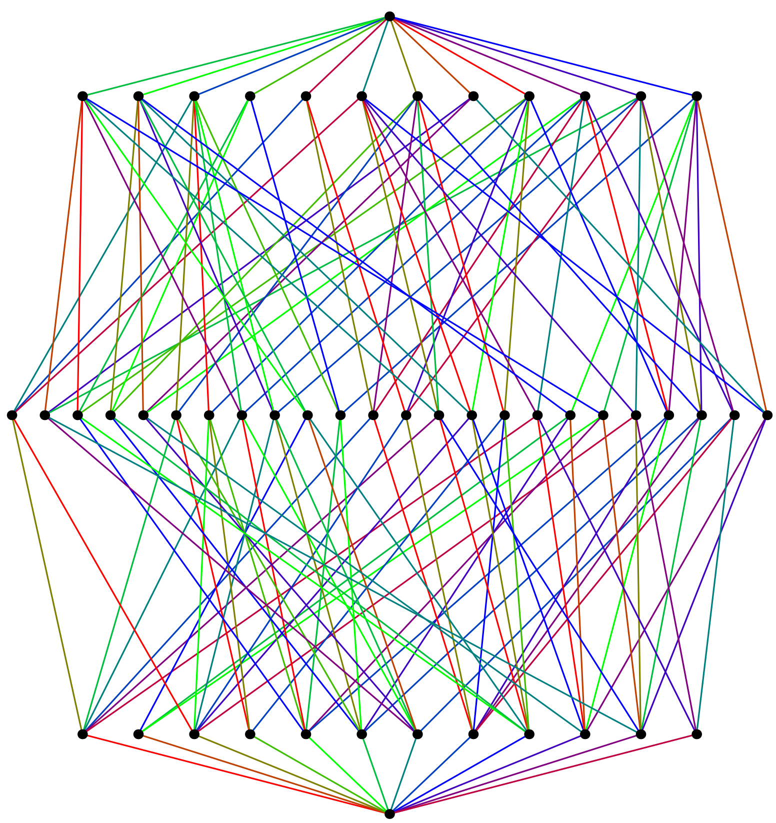
I. Posets and Garside structures

II. Examples and consequences

III. Constructing Garside structures

IV. Coxeter groups and Artin groups

V. General non-crossing partitions



Labeled posets

Main Idea: Let P be a labeled poset. If the labels "act like" group elements, there is often a group in the background which is closely tied to P.

Let P be a bounded graded poset and let $I(P)$ be the set of intervals in P. Bounded and graded imply finite height but P can be arbitrarily "fat" (in Ziegler's terminology). In particular, P need not be finite.

Def: A *labeling of* P by a set S is a map

$$\lambda : I(P) \to S$$

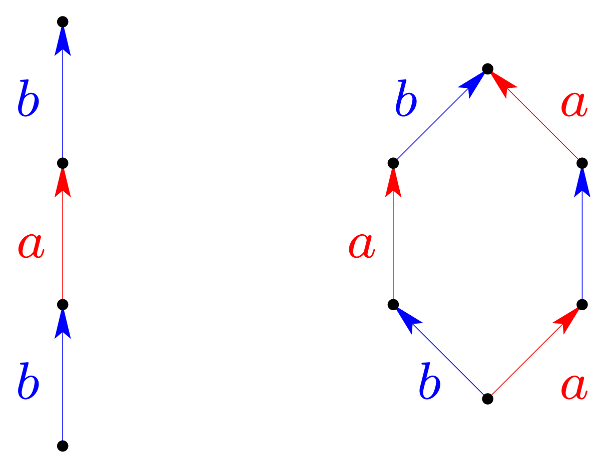
Typically, labels are only given to covering relations, but...

Converting edge labelings

Edge labelings can be converted into interval labelings as follows.

- Label saturated chains with the words.
- Label intervals with languages.

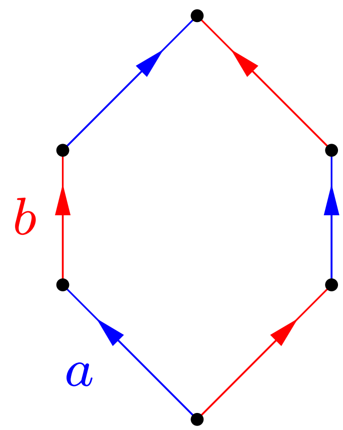
The set S in this case would be a collection of languages.



The "label" on $[\hat{0}, \hat{1}]$ for the righthand poset would be $\{aba, bab\}$.

Posets, Monoids, Groups and Complexes

Define $M(P)$ / $G(P)$ to be the monoid / group generated by the label set and relations equating the the labels on any two chains which start and end at the same elements.



$$M = M(P) = \text{Mon}\langle a, b \mid aba = bab\rangle$$

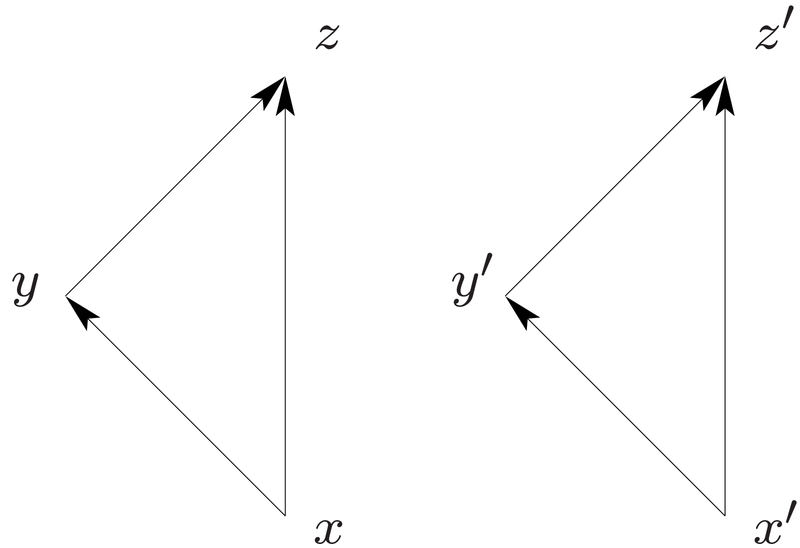
$$G = G(P) = \text{Grp}\langle a, b \mid aba = bab\rangle$$

Now view the labels as elements of the group G and define $K = K(P)$ as the quotient of the order complex of P where simplices with identical labels are identified respecting orientations.

Group-like posets

The main property which helps a labeling lead to an interesting group is being group-like.

Def: A labeled poset P is called *group-like* if whenever chains $x \leq y \leq z$ and $x' \leq y' \leq z'$ have two pairs of corrsponding labels in common, the third pair of labels are also equal.



Thus labels can be multiplied and canceled inside P.

Balanced posets

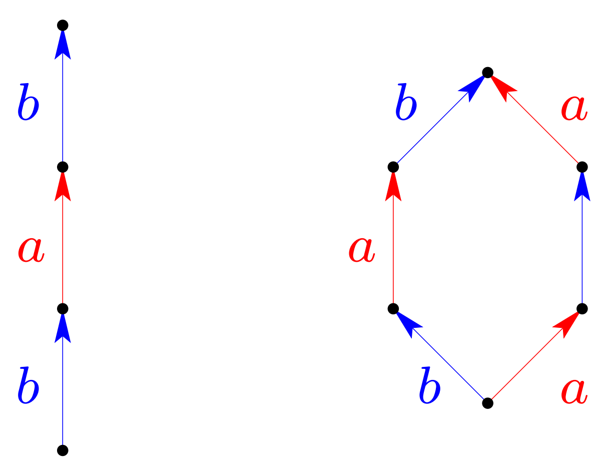
Define

$$L(P) = \{\lambda(\hat{0}, p) : p \in P\}$$

$$C(P) = \{\lambda(p, q) : p, q \in P\}$$

$$R(P) = \{\lambda(p, \hat{1}) : p \in P\}$$

P is *balanced* if $L(P) = R(P)$.

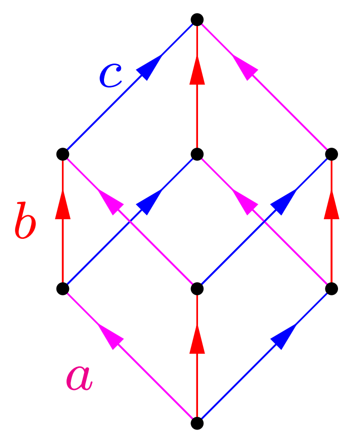


If P is both balanced and group-like, then $L(P) = C(P) = R(P)$. P is also self-dual and locally self-dual.

Garside structures

Let P be a bounded, graded, labeled poset.

If P is balanced and group-like then it is *Garside-like*. If P is also a lattice then it is a *Garside structure*.*



Boolean lattices with the natural labeling are examples of Garside structures.

* There is a slightly more general definition of Garside structures in the group theory context.

What are Garside structures good for?

If P is a Garside structure then

- $P \hookrightarrow M \hookrightarrow G$

- P embeds in the Cayley graph of M (and G)

- G is the group of fractions of M

- The word problem for G is solvable

- $\pi_1(K) = G$ and $\widetilde{K}$ is contractible

As a consequence, the cohomology of G is that of K, the cohomological dimension of G is bounded above by the height of P, and G is torsion-free, etc., etc.

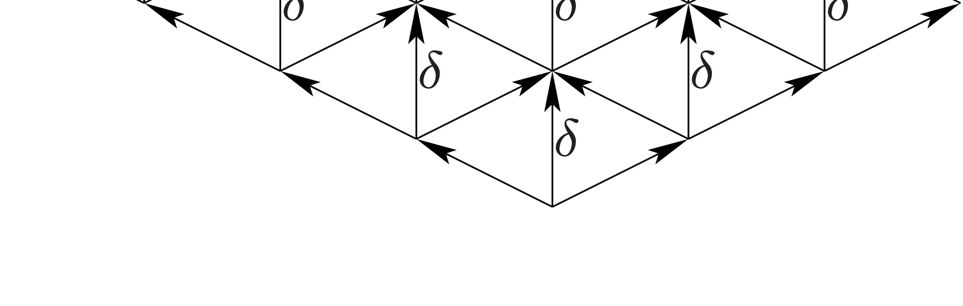
The combinatorics of P (as a labeled poset) dominate and control the structure of G.

Why balanced?

In order for a monoid M to embed in some group, it is necessary for it to be cancellative and it is sufficient for it to be cancellative and have *right common multiples*.

Lem: If P is balanced, then $M(P)$ has right common multiples.

Proof: Let δ be the label in the interval $[\hat{0},\hat{1}]$.



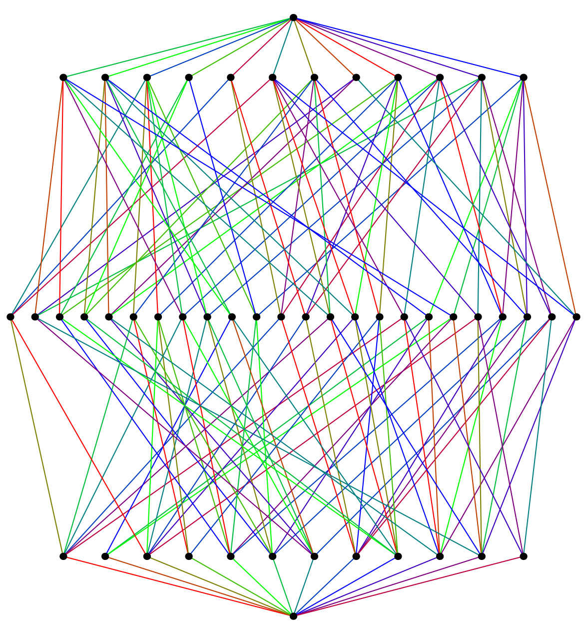
Why lattice?

The lattice property is used twice.

1. It is needed to show that P cancellative implies M is cancellative.

2. It is needed to show that the complex $\widetilde{K}$ is contractible (a Quillen type application).

Unfortunately, our supply of techniques for proving that something is a lattice is rather limited. For example, is this a lattice?



Signed groups

One way to construct Garside-like structures is to start with a group, or better a signed group.

Def: Call a group G generated by a set S *signed* if there is a map $G \to C_2$ such that all the elements of S are sent to the non-identity element of C_2.

Ex: Groups with even presentations are signed. Groups generated by orientation-reversing isometries of a Riemannian manifold are signed.

Orient the edges of the Cayley graph according to the distance from the identity.

Lem: If G generated by S is signed and S' is the closure of S under conjugacy, then every interval in the poset formed by the Cayley graph of G with respect to S' has a Garside-like labeling.

Isom(X^n)

Thm (Brady-Watt) If δ is a isometry of $\mathbb{R}^n$ fixing only the origin, then the interval $[1, \delta]$ in $\text{Isom}_0(\mathbb{R}^n) \cong \text{Isom}(\mathbb{S}^{n-1})$ is isomorphic to the poset of linear subspaces of $\mathbb{R}^n$ under reverse inclusion. The map sends each element to its fixed subspace.

Cor: Every interval in $\text{Isom}(\mathbb{S}^n)$ is a lattice (and hence a Garside structure).

Thm (BCKM) For $n \geq 4$, $\text{Isom}(\mathbb{R}^n)$ is not a lattice, but it can be extended to a complete lattice in a minimal, canonical, and understandable way.

These complicated posets with their continuous set of generators play a crucial role in the understanding of the finite and affine versions of non-crossing partitions.

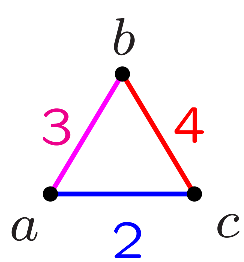
Coxeter groups and Artin groups

Let Γ be a finite graph with edges labeled by integers greater than 1, and let $(a,b)^n$ be the length n prefix of $(ab)^n$.

Def: The *Artin group* A_Γ is generated by its vertices with a relation $(a,b)^n = (b,a)^n$ whenever a and b are joined by an edge labeled n.

Def: The *Coxeter group* W_Γ is the Artin group A_Γ modulo the relations $a^2 = 1$ $\forall a \in \textbf{Vert}(\Gamma)$.

Graph



Artin presentation

$$\langle a, b, c | \; aba = bab, ac = ca, bcbc = cbcb \rangle$$

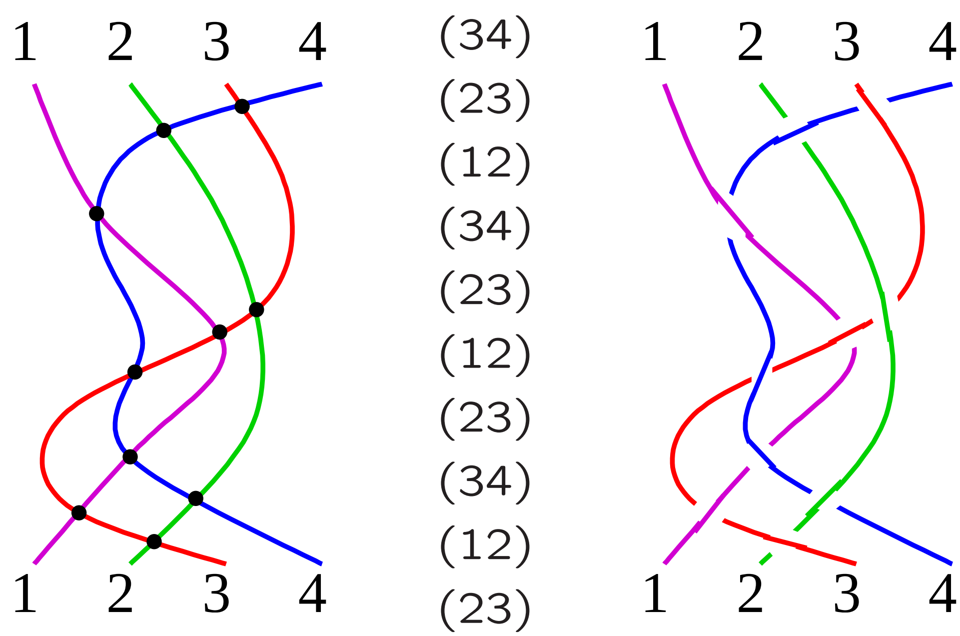
Coxeter presentation

$$\left\langle a, b, c | \begin{array}{c} aba = bab, ac = ca, bcbc = cbcb \\ a^2 = b^2 = c^2 = 1 \end{array} \right\rangle$$

Coxeter groups and arrangements

Each finite Coxeter group corresponds to a highly symmetric reflection arrangement. The fundamental group of the complexified version of the hyperplane arrangement is a (pure) Artin group.

For example, the symmetric group leads to the braid arrangement, whose fundamental group is the (pure) braid group.



Coxeter groups are natural

Coxeter groups are a natural generalization of finite reflection groups and they are amazingly nice to work with.

1. They have a decidable word problem
2. They are virtually torsion-free
3. They act cocompactly on nonpositively curved spaces
4. They are linear
5. They are automatic

Also every Coxeter group acts by reflections on some highly symmetric space, such as $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{H}^n$ or something higher rank.

Artin groups are natural yet mysterious

Artin groups are "natural" in the sense that they are closely tied to the complexified version of the hyperplane arrangements for Coxeter groups.

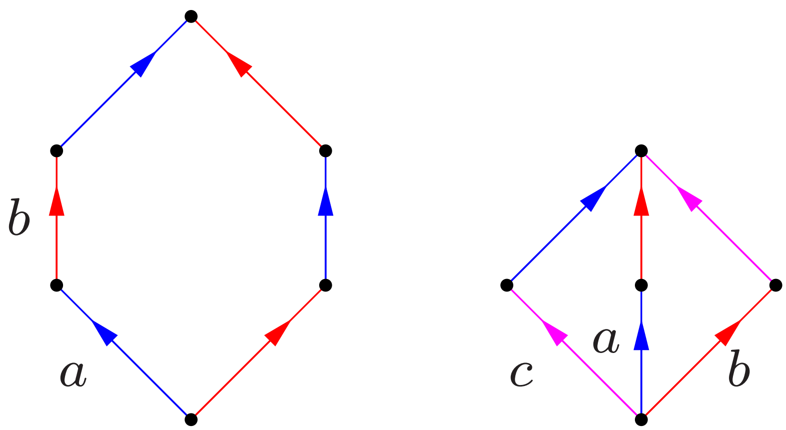
But they are "mysterious" in the sense that it is unknown if

1. They have a decidable word problem
2. They are (virtually) torsion-free
3. They have finite (dimensional) $K(\pi, 1)$s
4. They are linear (i.e. have a faithful matrix representation)

Even the Artin groups corresponding to affine Coxeter groups have been mysterious until this past year.

Examples of Garside structures

Braid groups and other finite-type Artin groups each have two Garside structures. For the 3-string braid group the two posets are shown. The second one is the *dual* of the first.

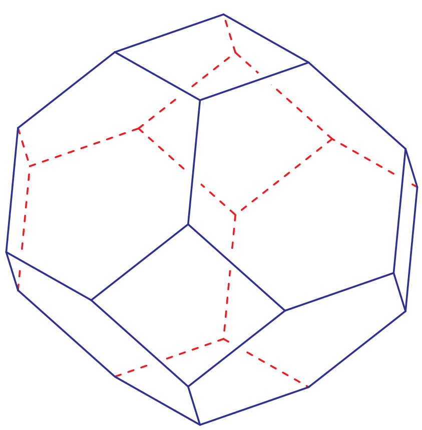


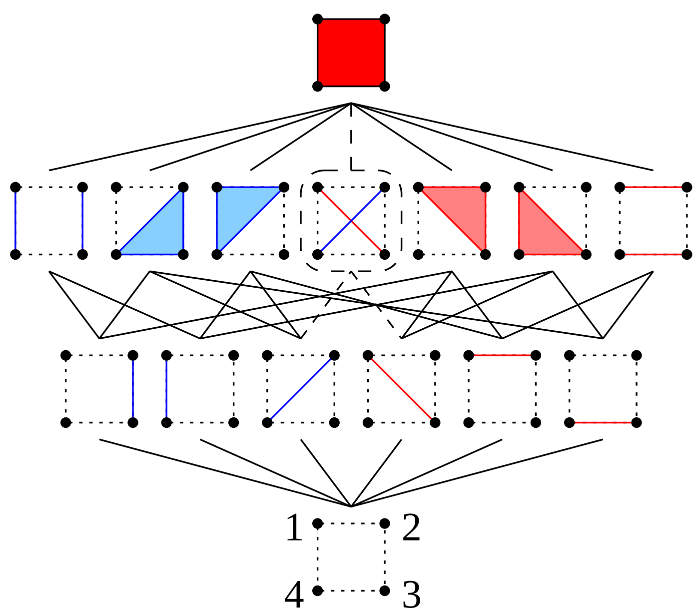
$$\langle a, b | aba = bab \rangle = \langle a, b, c | ab = bc = ca \rangle$$

The A_3 Poset and its dual

The standard Garside structure a braid group is a height function applied to the 1-skeleton of a permutahedron (which is the Cayley graph of S_n with respect to the adjacent transpositions).

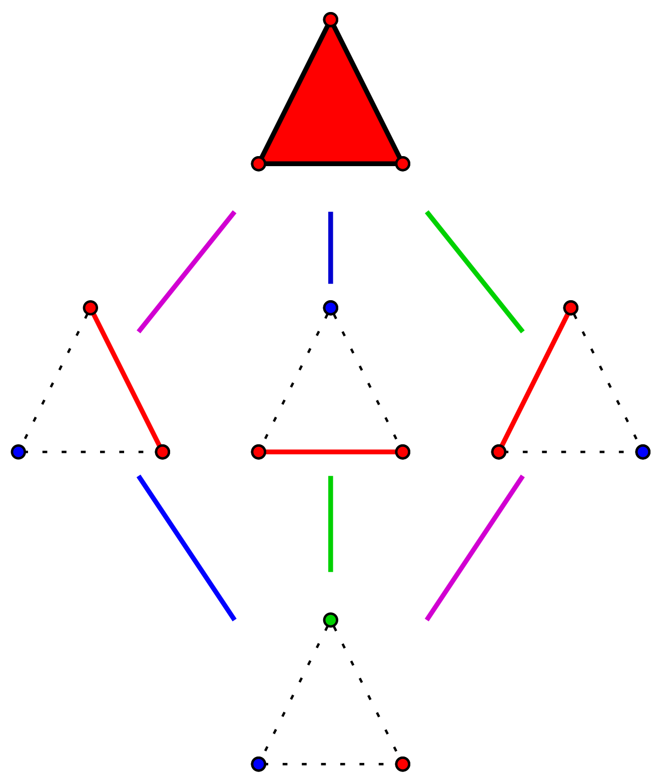
The dual structure is the non-crossing partition lattice for a n-gon.



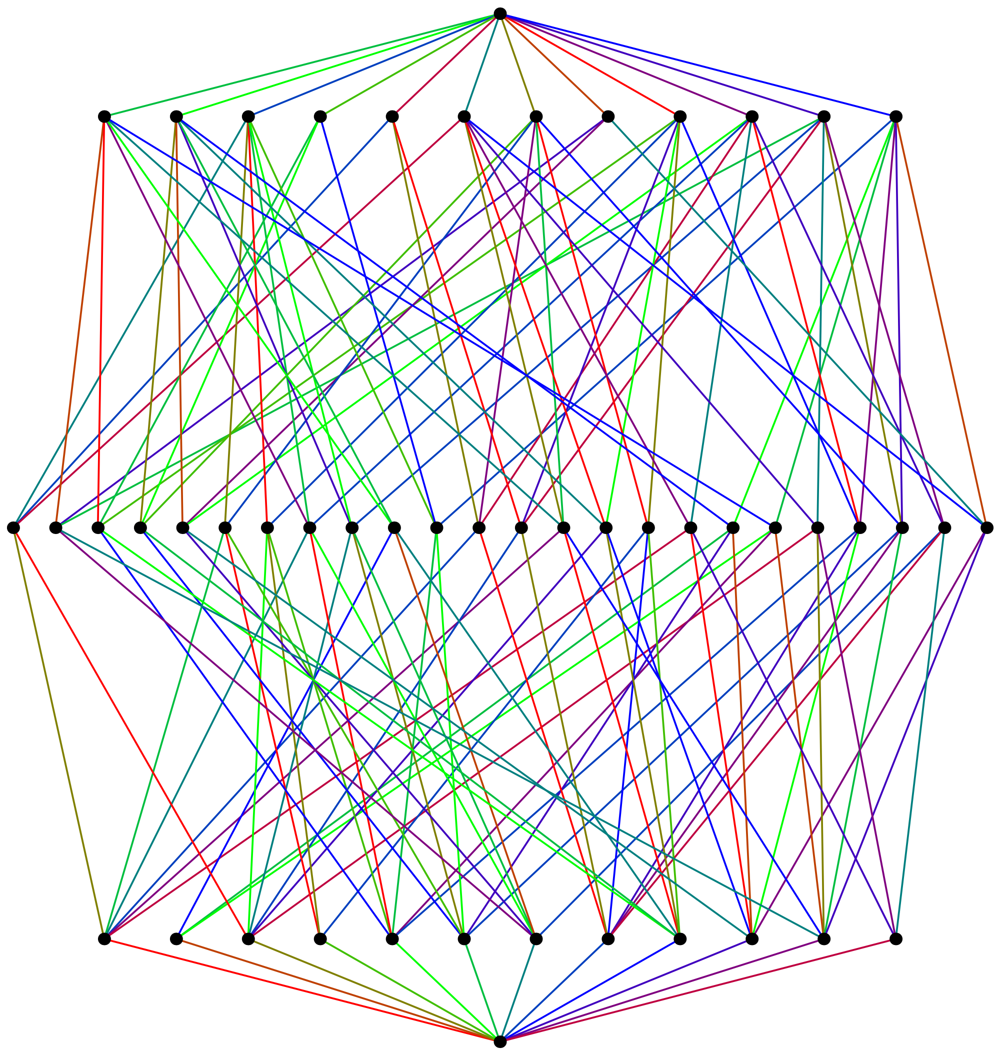


Garside structures for finite Coxeter groups

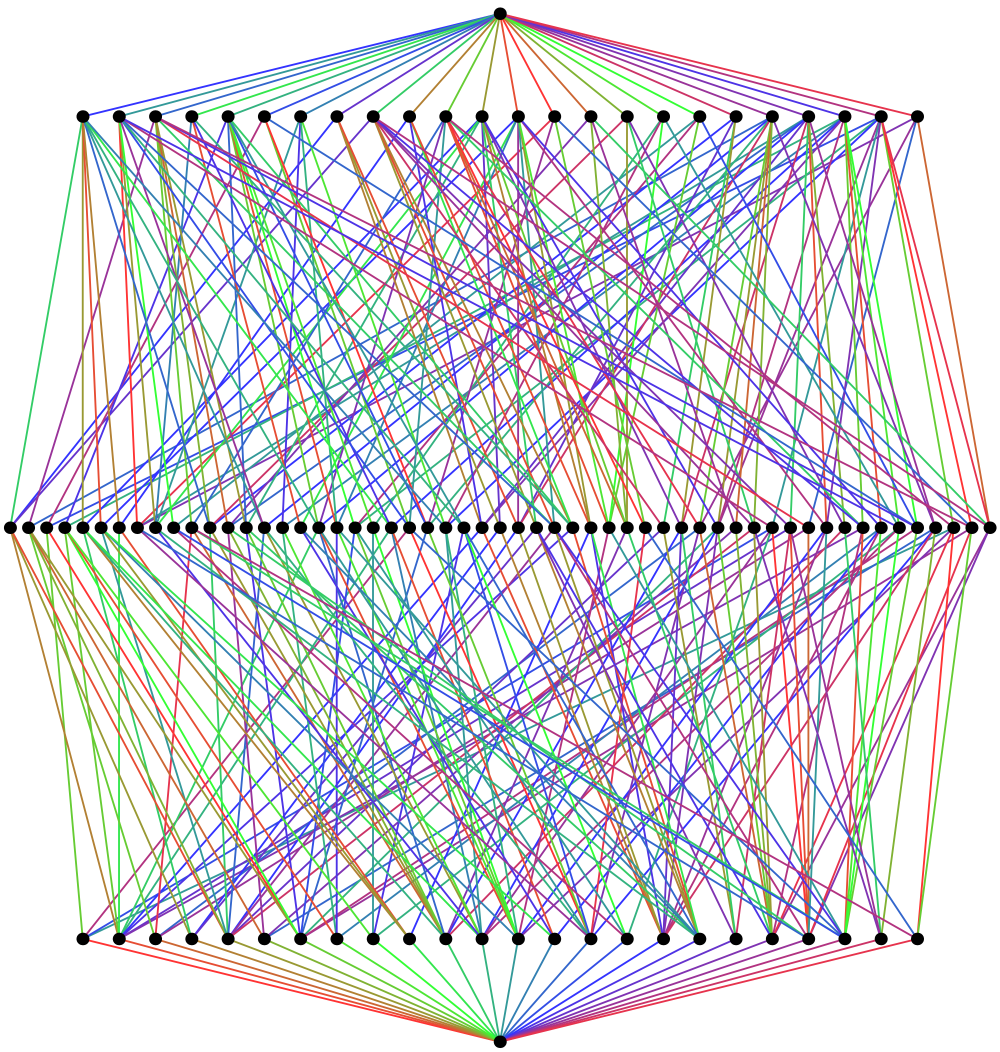
Let W be a finite Coxeter group. The W-permutahedron is one Garside structure. The poset NC^W which records the minimal factorizations of a Coxeter element of W into reflection is a second. The latter is known as either the dual Garside structure of type W or the noncrossing partitions of type W.



The dual D_4 Poset



The dual F_4 Poset



Why “dual”?

[Bessis - “The Dual Braid Monoid”]

S = standard generators
T = set of all “reflections”
c = a Coxeter element = $\prod s$
$\mathbf{w}_0$ = the longest element in W
n = the rank (dimension) of W
N = # reflections = # of positive roots
h = Coxeter number = order of **c**

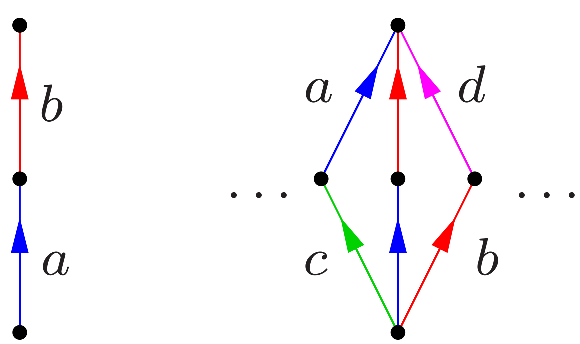
	Classical poset	Dual poset
Other names	Weak order	Noncrossing
Set of atoms	**S**	**T**
Number of atoms	n	N
Product of atoms	**c**	$\mathbf{w}_0$
Regular degree	h	2
Height	N	n
$\lambda(\hat{0}, \hat{1})$	$\mathbf{w}_0$	**c**
Order of $\lambda(\hat{0}, \hat{1})$	2	h

Garside structures for Coxeter groups

The poset of minimal factorizations of a Coxeter element **c** over the set of all reflections in an arbitrary Coxeter group produces a Garside-like poset. All that is missing is a proof of the lattice condition.

Ex: Free Coxeter groups.

$$\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1 \rangle$$



$$G(P) = \langle a_i | a_i a_{i+1} = a_j a_{j+1} \rangle$$

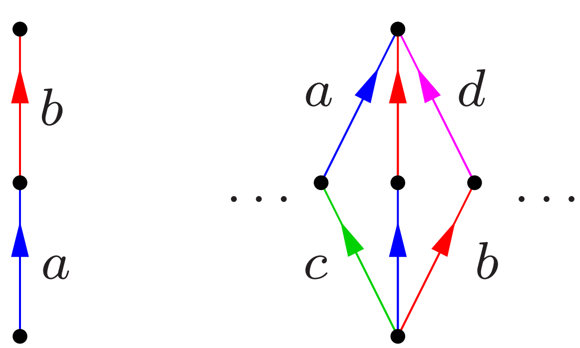
This example is certainly a lattice, but the 3-generated version is less clear. What is this group?

Garside structures for free Coxeter groups

The Artin group defined by this poset is the free group and the construction in this case leads to a universal cover which is an infinitely branching tree cross the reals with a free F_2 action.

Not the usual presentation of F_2 but it is a $K(F_2, 1)$ space with a conjugacy closed generating set.

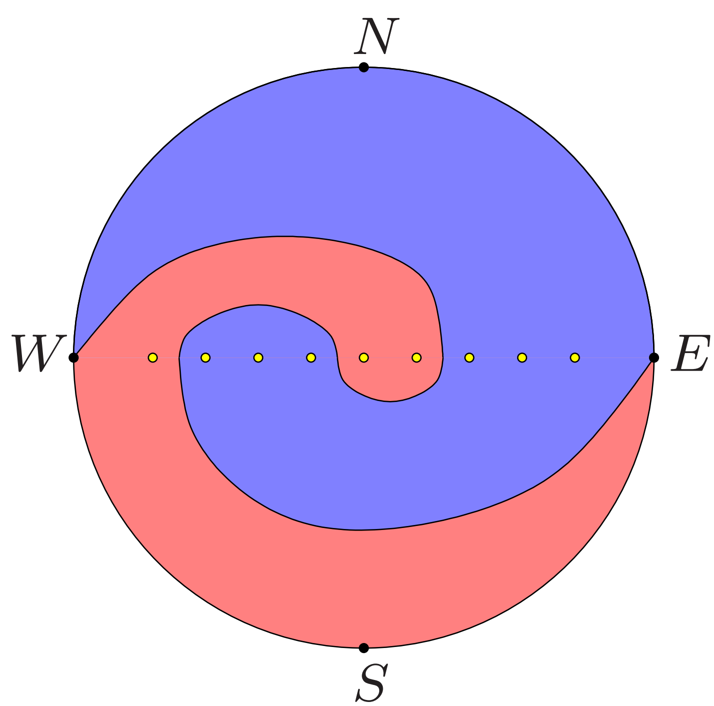
Start with the free Coxeter group generated by $x_1, x_2, \ldots, x_n$ and let $\mathbf{c} = x_1 x_2 \cdots x_n$. We can start building a Garside structure by continuing to add paths (and generators) to create a bounded graded, consistently edge-labeled poset which is balanced.



A more topological definition

Let $\mathbf{D}^*$ denote the unit disc with n puntures and 4 distinguished boundary points, N, S, E and W.

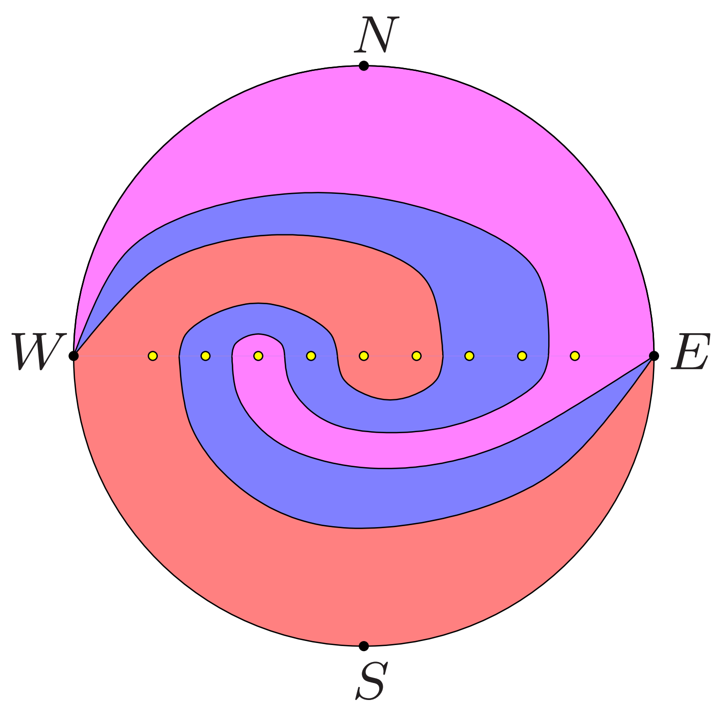
Def: A *cut-curve* is an isotopy class (in $\mathbf{D}^*$) of a path from E to W (rel endpoints, of course).



Notice that cut-curves divide $\mathbf{D}^*$ into two pieces, one containing S and the other containing N. Its *height* is the number of puncture in the lower piece.

Poset of cut-curves

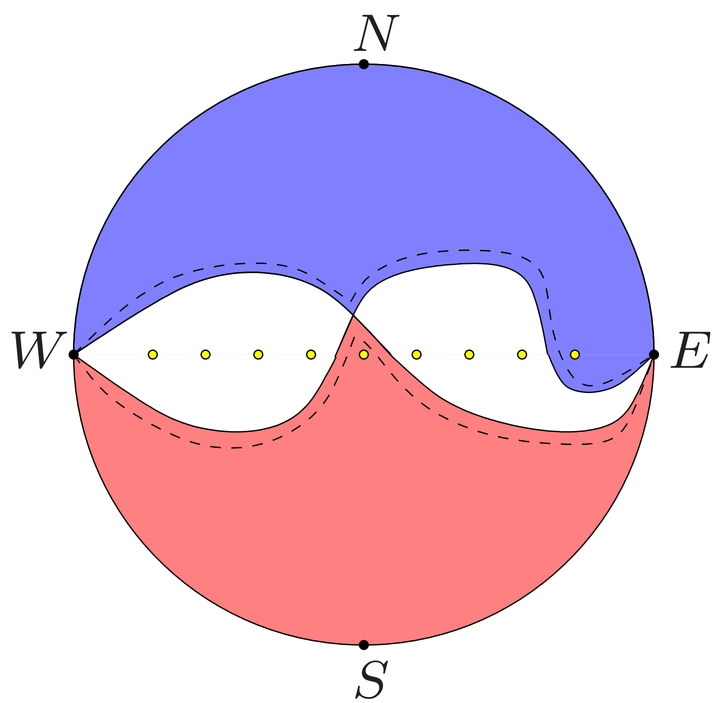
Let $[c]$ and $[c']$ be cut-curves. We write $[c] \leq [c']$ if there are representatives c and c' which are disjoint (except at their endpoints) and c is “below” c'.



Notice that if representative c is given, then we can tell whether $[c] \leq [c']$ by keeping c fixed and isotoping c' into a “minimal position” with respect to c (i.e. no football shaped regions with no punctures).

Proving lattice: the free case

Lemma The poset of cut-curves is a lattice.



Proof: Suppose $[c]$ is above $[c_1]$ and $[c_2]$. Place representatives c_1 and c_2 in minimal position with respect to each other (i.e. no football regions) and then isotope c so that it is disjoint from both. This c is above the dotted line. Thus the dotted line represents a least upper bound for $[c_1]$ and $[c_2]$.

Noncrossing curve diagrams

Using a modification of these noncrossing curve diagrams, we can prove

Thm (BCKM) For every Coxeter group W, the poset NC^W embeds into the Artin group A and the complex K has A as its fundamental group.

Thus, these posets do define the right groups and they are defined using Coxeter groups so they can be worked with concretely, effectively and efficiently.

A modified version of the lattice argument shows the following.

Thm (BCKM) If every relation in W is “long” then the poset NC^W is a lattice and a Garside structure.

Proving lattice: the finite case

As you can see, the trickiest aspect in all of this proving that these well-defined posets are lattices.

Until recently, the only known proof of the lattice property – even in the finite case – was a case by case proof which used a brute force computer check for the exceptional groups.

There is now a uniform proof of the lattice property using the embedding of finite poset NC^W into the continuous poset $\text{Isom}_0(\mathbb{R}^n)$ (and repeatedly applying the results by Tom Brady and Colum Watt).

Thm (M) For every set of reflections closed under conjugacy, the poset of factorizations inside $\text{Isom}_0(\mathbb{R}^n)$ is a lattice.

Proving lattice: the affine case

We have made great progress in understanding the poset NC^W for the affine Coxeter groups using their embedding into the continuous poset Isom($\mathbb{R}^n$). Although they are not always lattices, they are always "close". In particular, they are close enough that we can recover most of the consequences of having a Garside structure.

Thm (BCKM) For every affine Coxeter group W, the poset NC^W is almost a lattice and as a result most of the consequences of a Garside structure also follow for these groups, including that the complex K has a contractible universal cover.