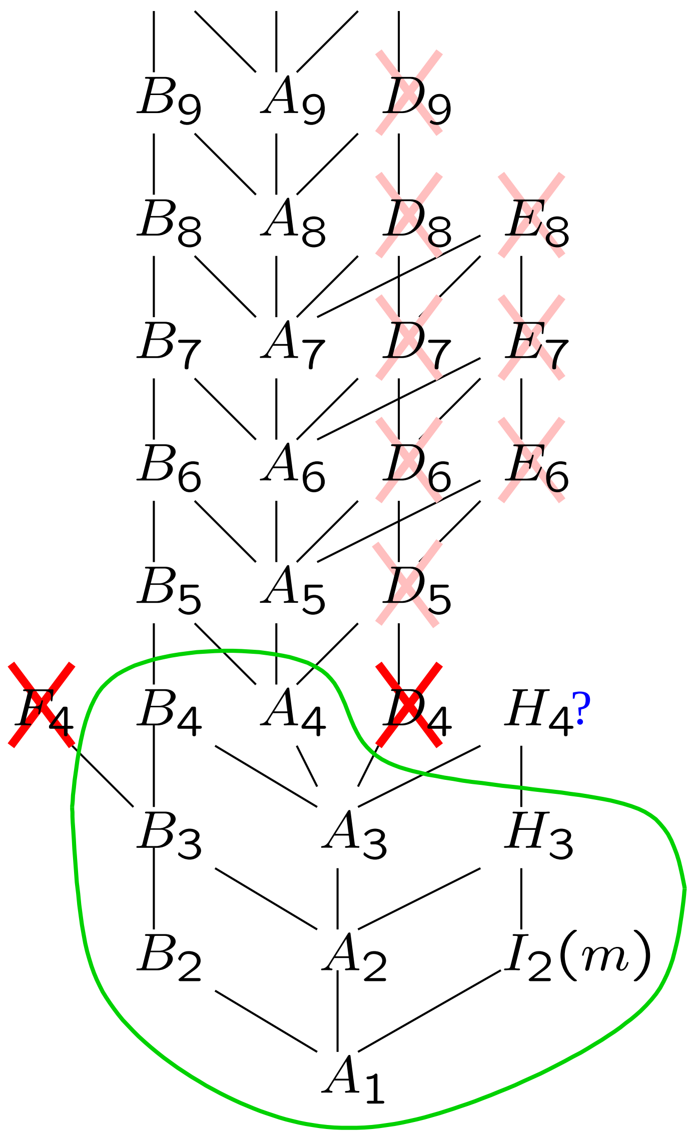
Existence of CAT(0) structures for finite type Artin groups



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Overview

I. Finite-type Artin groups

II. Brady-Krammer complexes

III. Non-positive curvature

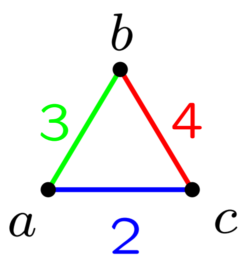
IV. Old and new results

Coxeter and Artin groups

Let Γ be a finite graph with edges labeled by integers greater than 1, and let $(a,b)^n$ be the length n prefix of $(ab)^n$.

Def: The *Artin group* A_Γ is generated by its vertices with a relation $(a,b)^n = (b,a)^n$ whenever a and b are joined by an edge labeled n.

Def: The *Coxeter group* W_Γ is the Artin group A_Γ modulo the relations $a^2 = 1\ \forall a \in \textbf{Vert}(\Gamma)$.

Graph 

Artin presentation

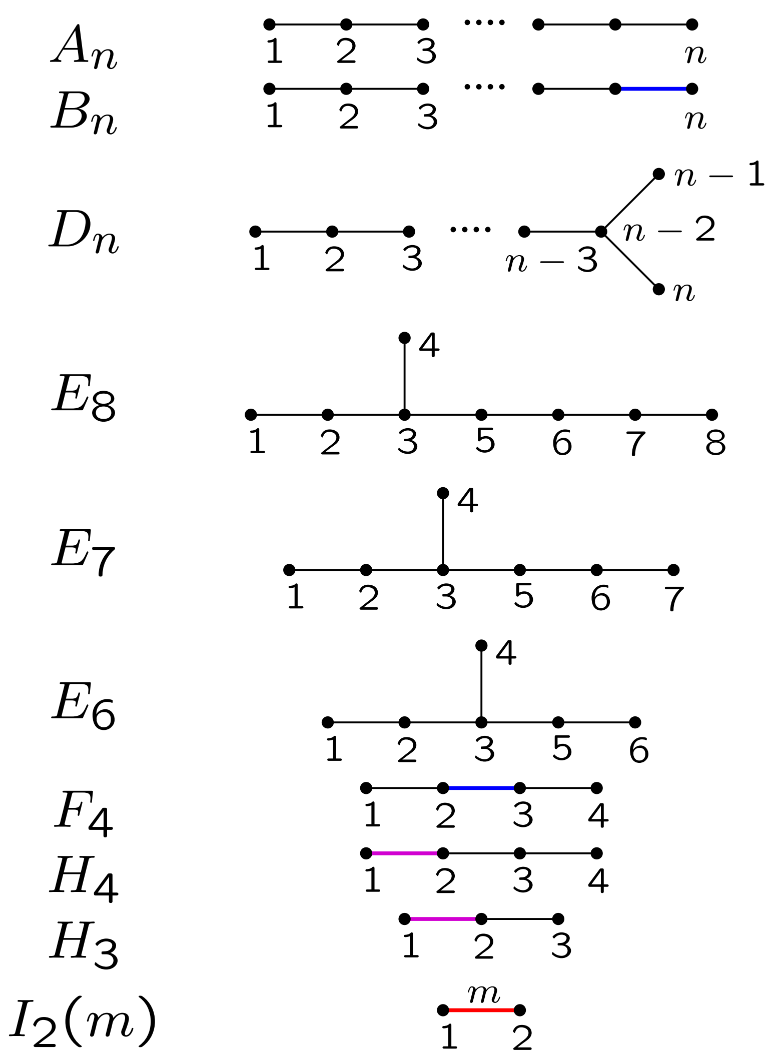
$\langle a,b,c \mid aba = bab, ac = ca, bcbc = cbcb \rangle$

Coxeter presentation

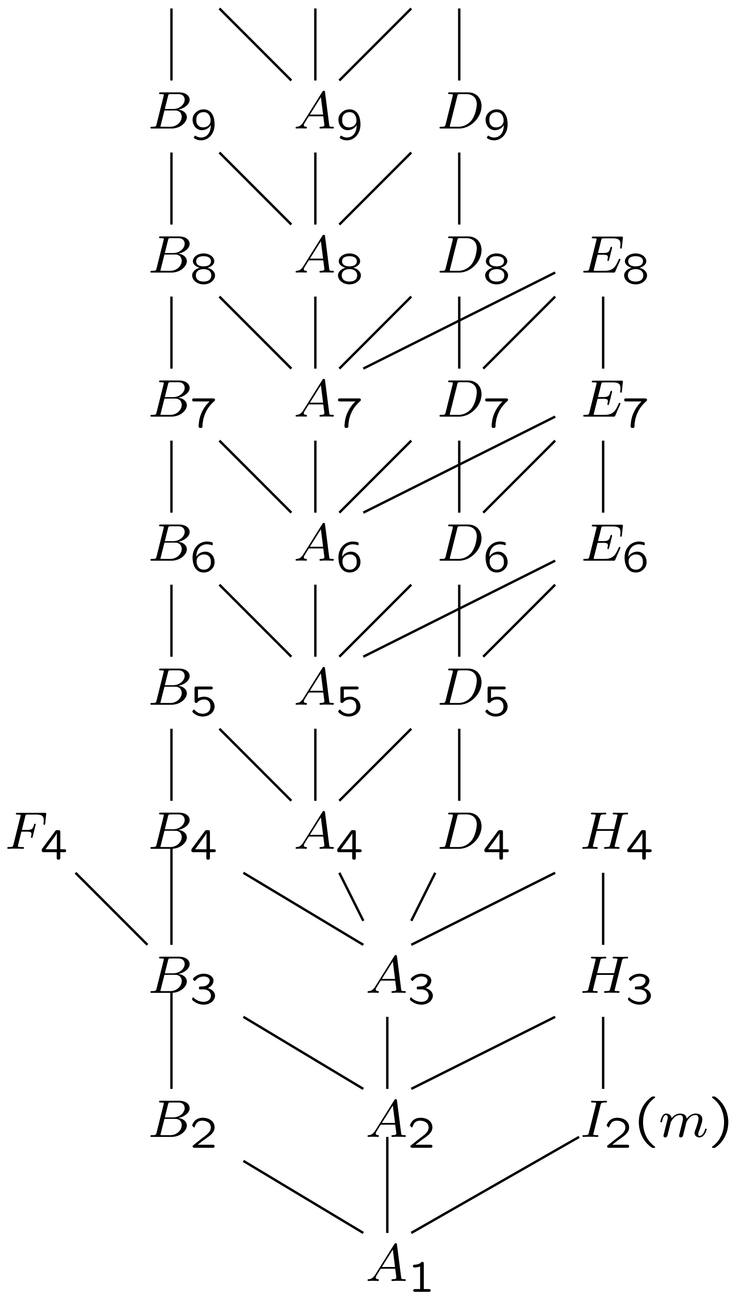
$$\left\langle a,b,c \middle| \begin{array}{c} aba = bab, ac = ca, bcbc = cbcb \\ a^2 = b^2 = c^2 = 1 \end{array} \right\rangle$$

Finite-type Artin groups

The finite Coxeter groups have been classified. An Artin group defined by the same labeled graph as a finite Coxeter is called a *finite-type Artin*.



Irreducible Dynkin diagrams



Eilenberg-MacLane spaces for Artin groups

Finite-type Artin groups are fundamental groups of complexified Coxeter hyperplane arrangements quotiented by the action of the Coxeter group.

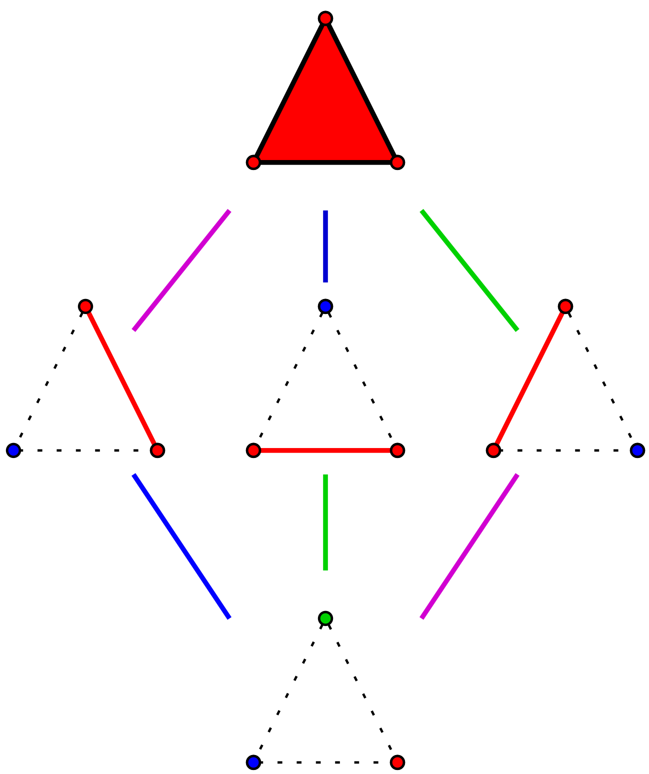
Each finite type Artin group has a
• finite dimensional CAT(0) K(G,1)
• finite dimensional compact K(G,1)
but no known
• finite dimensional compact CAT(0) K(G,1)

Thus they do not yet qualify as CAT(0) groups, but they are good candidates.

Brady-Krammer Complexes

In 1998 Tom Brady and Daan Krammer independently discovered new complexes on which the braid groups and the other Artin groups of finite type act.

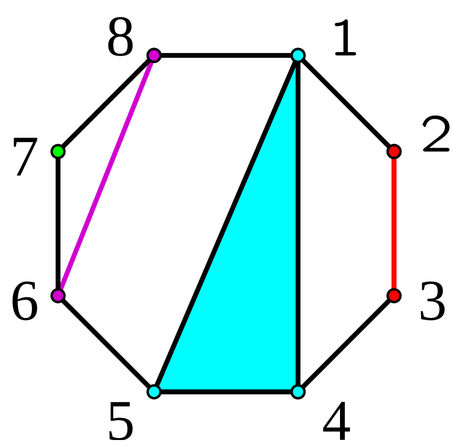
In the case of the braid groups, the link of a vertex in the cross section is the order complex of a well-known combinatorial object known as the noncrossing partition lattice.



Noncrossing Partitions

A noncrossing partition is a partition of the vertices of a regular n-gon so that the convex hulls of the partitions are disjoint.

One noncrossing partition σ is contained in another τ if each block of σ is contained in a block of τ.



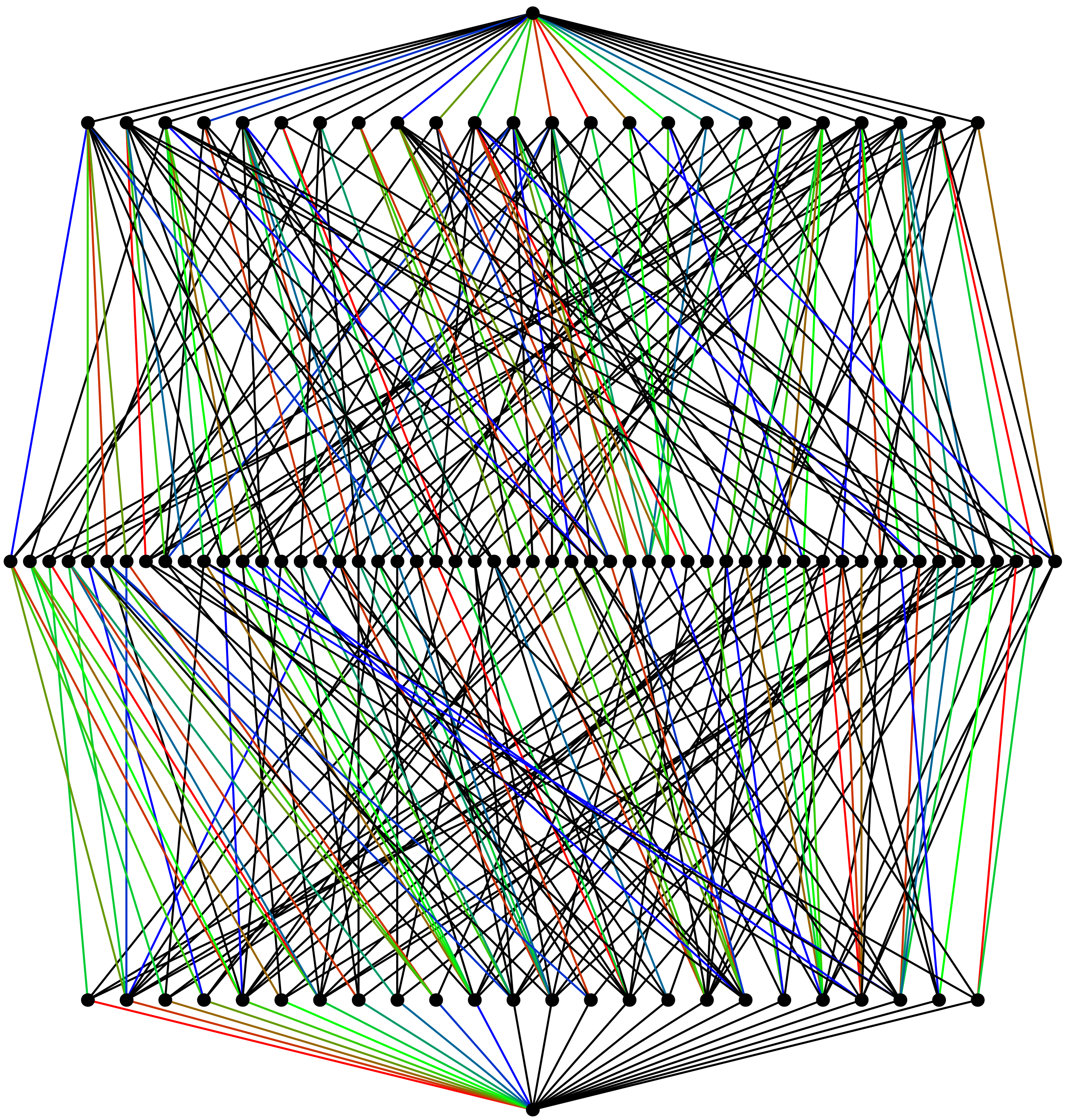
$\{\{1,4,5\},\{2,3\},\{6,8\},\{7\}\}$

Factors of the Coxeter element

A_3	1-6-6-1
B_3	1-9-9-1
H_3	1-15-15-1
A_4	1-10-20-10-1
B_4	1-12-24-12-1
D_4	1-16-36-16-1
F_4	1-24-55-24-1
H_4	1-60-158-60-1
A_5	1-15-50-50-15-1
B_5	1-20-70-70-20-1
D_5	1-25-100-100-25-1

General formulae exist for the A_n, B_n and D_n types as well as explicit calculations for the exceptional ones, but no general formula explains all of these numbers in a coherent framework.

F_4 Poset

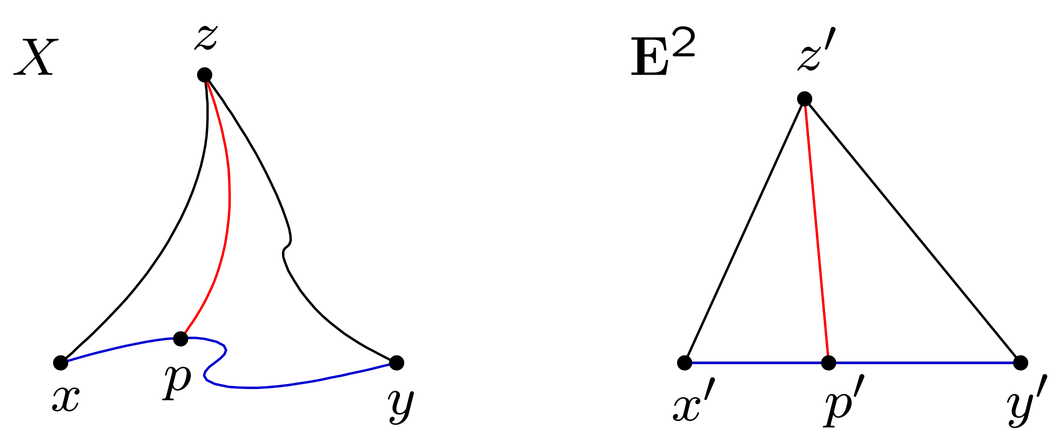


CAT(0)

Def: A geodesic metric space C is called (globally) CAT(0) if
$\forall$ points $x, y, z \in C$
$\forall$ geodesics connecting x, y, and z
$\forall$ points p in the geodesic connecting x to y

$$d(p, z) \leq d(p', z')$$

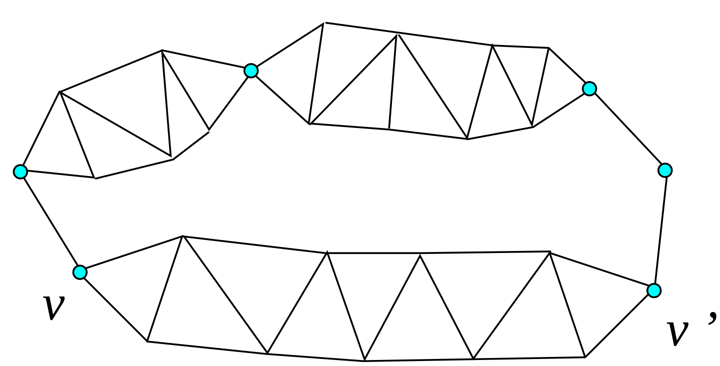
in the corresponding configuration in $\mathbf{E}^2$.



Piecewise Euclidean Complexes

Def: A *piecewise euclidean complex* X is a simplicial complex in which each simplex is given a Euclidean metric and the induced metrics on the intersections always agree.

Thm: A PE complex is CAT(0) iff the link of each cell does not contain a closed geodesic loop of length less than 2π.



CAT(0) **and Artin groups**

Thm(T.Brady-M) The finite-type Artin groups with at most 3 generators are CAT(0)-groups and the Artin groups A_4 and B_4 are CAT(0) groups.

Proof: The link of a vertex in the cross section is the order complex of a fairly small poset. It is then relatively easy to check that using a fairly "natural" metric, each of these links satisfy the link condition.

Conj: The Brady-Krammer complex is CAT(0) for all Artin groups of finite type.

CAT(0) metrics on D_4 and F_4

Thm(Choi): The Brady-Krammer complexes for D_4 and F_4 do not support reasonable PE CAT(0) metrics.

Reasonable means that symmetries of the group should lead to symmetries in the metric.

Proof Idea: First determine what Euclidean metrics on the 3-dimensional cross-section complex have dihedral angles which make the edge links (which are finite graphs) large.

Then check these metrics in the vertex links (which are 2-dimensional PS complexes).

The software

The program `coxeter.g` is a set of GAP routines used to examine Brady-Kramer complexes. Initially developed to test the curvature of the Brady-Krammer complexes using the "natural" metric, the routines were extensively modified by Woonjung Choi so that they

- find the 3-dimensional structure of the cross-section
- find representive vertex and edge links (up to automorphism)
- find the graphs for the edge links
- find the simple cycles in these graph
- find the linear system of inequalities which need to be satisfied by the dihedral angles of the tetrahedra.

Dihedral angles

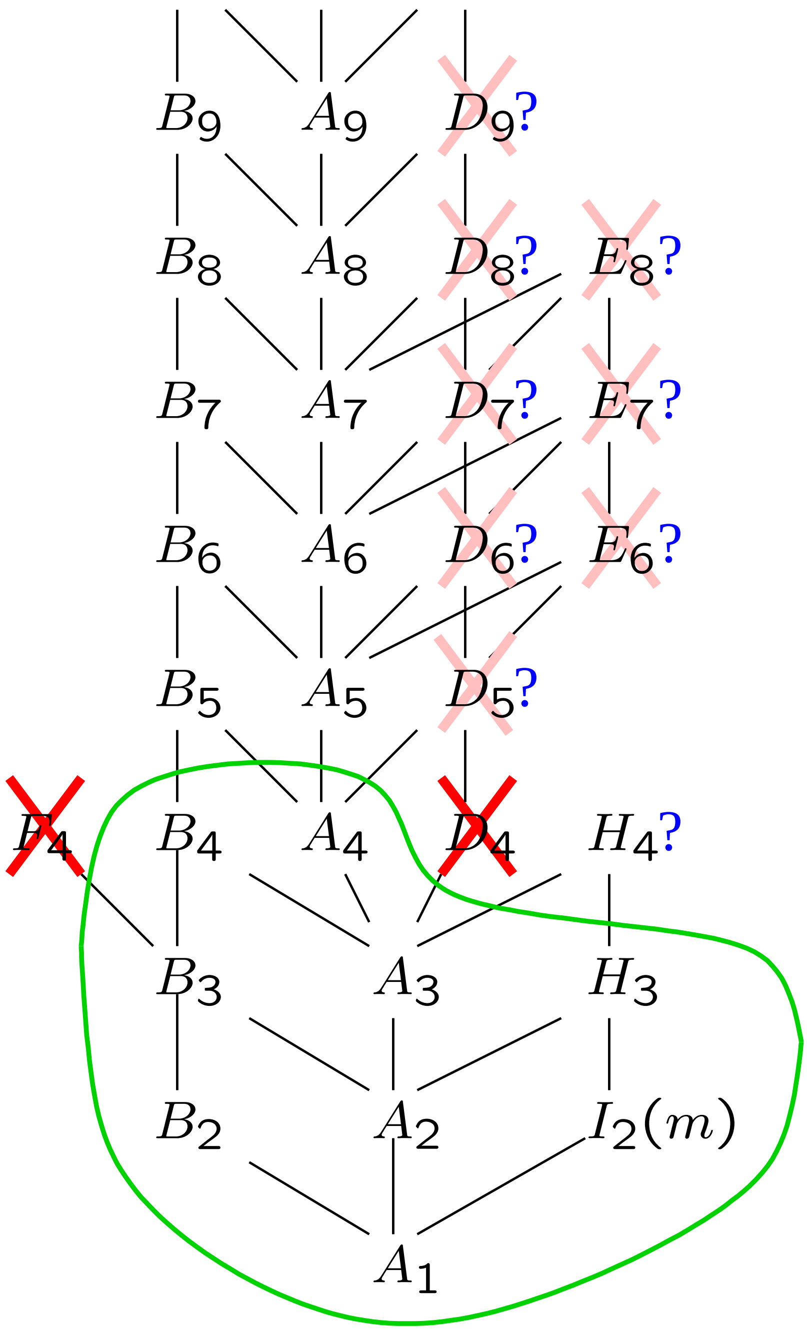
Thm: Let σ and τ be n-simplices and let f be a bijection between their vertices. If the dihedral angle at each codimension 2 face of σ is at least as big as the dihedral angle at the corresponding codimension 2 face of τ, then σ and τ are similar (isometry up to a scale factor).

Proof: $\exists a_i > 0$ s.t. $\sum_i a_i \vec{u_i} = \vec{0}$ (Minkowski).

$$\begin{aligned} 0 = ||\vec{0}||^2 &= \sum_i \sum_j a_i a_j (\vec{u_i} \cdot \vec{u_j}) \\ &\geq \sum_i \sum_j a_i a_j (\vec{v_i} \cdot \vec{v_j}) \\ &= ||\sum_i a_i \vec{v_i}||^2 \geq 0 \end{aligned}$$

This implies $\vec{u_i} \cdot \vec{u_j} = \vec{v_i} \cdot \vec{v_j}$ for all i and j, which shows σ and τ are similar.

CAT(0) and Brady-Krammer complexes



Types D_4 and F_4

D_4 has:

- 162 simplices
- 15 columns
- 3 types of tetrahedra in the cross section
- 4 vertex types to check
- 21 inequalities in 9 variables
- 13 simplified inequalities in 9 variables

F_4 has:

- 432 simplices
- 18 columns
- 4 types of tetrahedra in the cross section
- 7 vertex types to check
- 81 inequalities in 13 variables
- 27 simplified inequalities in 13 variables

Type H_4

The case of H_4 is hard to resolve because the defining diagram has no symmetries which greatly increases the number of equations and variables involved in the computations.

H_4 has:

- 1350 simplices
- 23 columns
- 16 types of tetrahedra in the cross section
- 10 vertex types to check
- 2986 inequalities in 96 variables
- 638 simplified inequalities in 96 variables

The F_4 and D_4 cases produced systems small enough to analyze by hand. This system is not.