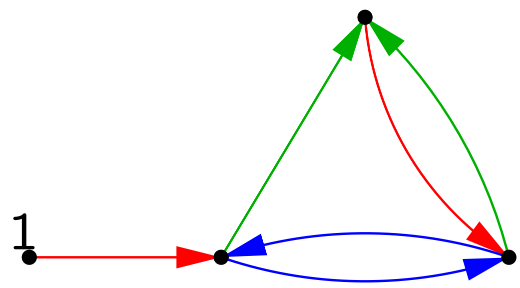
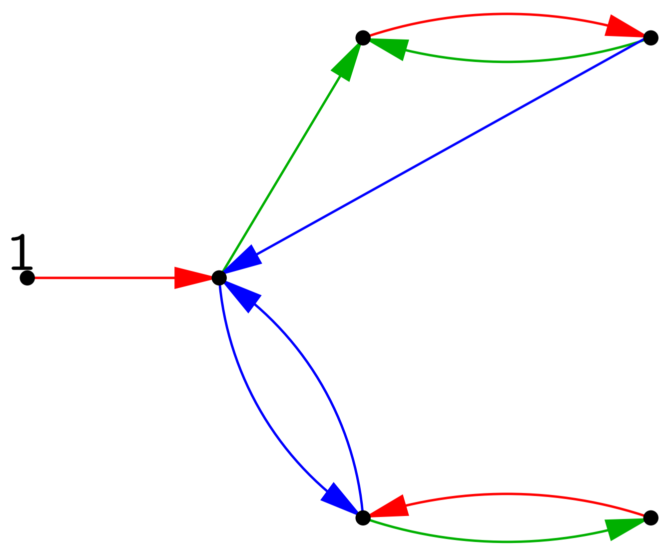
Expansions of Semigroups





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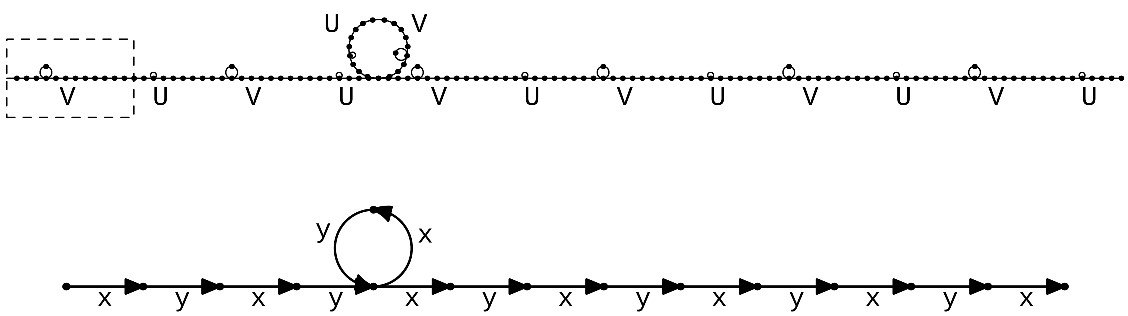
Main theorem

Rough version

Thm(M-Rhodes) If S is a finite A-semigroup then there exists a finite expansion of S such that the right Cayley graph of the expansion has many of the nice geometric properties of the right Cayley graph of the Burnside semigroup $\mathcal{B}(m, n)$, $n \geq 6$.



Burnside semigroups

Def: $\mathcal{B}(m,n) = \langle A \mid a^m = a^{m+n}\ \forall a \in A^+ \rangle$

Why Burnside semigroups?

• Krohn-Rhodes complexity involves aperiodics and groups.

• Free groups are well understood; free aperiodics less so.

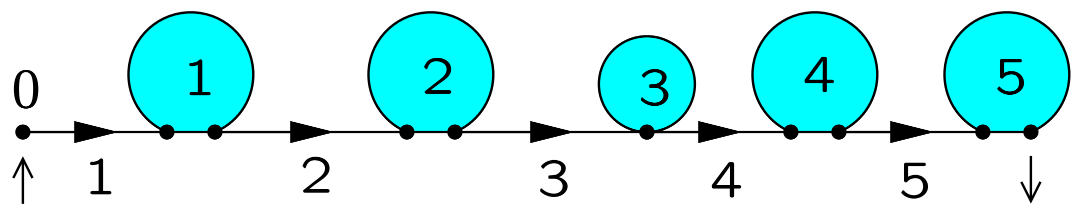
• The structure of the free aperiodic is closely tied to the Burnside semigroups.

Sample "Thm": The term problem for the free aperiodic can be solved by mimicking the solution to the word problem for the Burnside semigroups.

Finite directed graphs

Def: If the strong components of a finite directed graph are totally ordered, we say it is *quasi-linear*.

Def: If a quasi-linear connected graph has a minimal number of edges outside strong components, then it has a *quasibase*.



Lem: If Γ is a finite directed graph with a quasibase and p is a topmost vertex then there exists a directed spanning tree rooted at p.

[transition edges, entry/exit points]

Finitely-generated semigroups

Def: An A-*semigroup* is a semigroup S together with a function $A \to S$ whose image generators.

Def: A morphism $\phi : S \to T$ between A-semigroups such that $A \to T$ factors as $A \to S \to T$ is called an A-*morphism*.

Def: Let **Cayley**(S, A) denote the right Cayley graph of S^1.

Rem: The strong components of **Cayley**(S, A) are the Schützenberger graphs of the $\mathcal{R}$-classes of S^1.

Def: Let **sch**$^S(w)$ be the Schützenberger graph containing the vertex $[w]$.

Finite $\mathcal{J}$-above

Lem: A semigroup S is finite $\mathcal{J}$-above $\Leftrightarrow$ $\exists$ family of co-finite ideals with empty intersection.

Categories

$$\mathbf{FS} \subset \mathbf{FJS} \subset \mathbf{S}$$

$$\mathbf{FS_A} \subset \mathbf{FJS_A} \subset \mathbf{S_A}$$

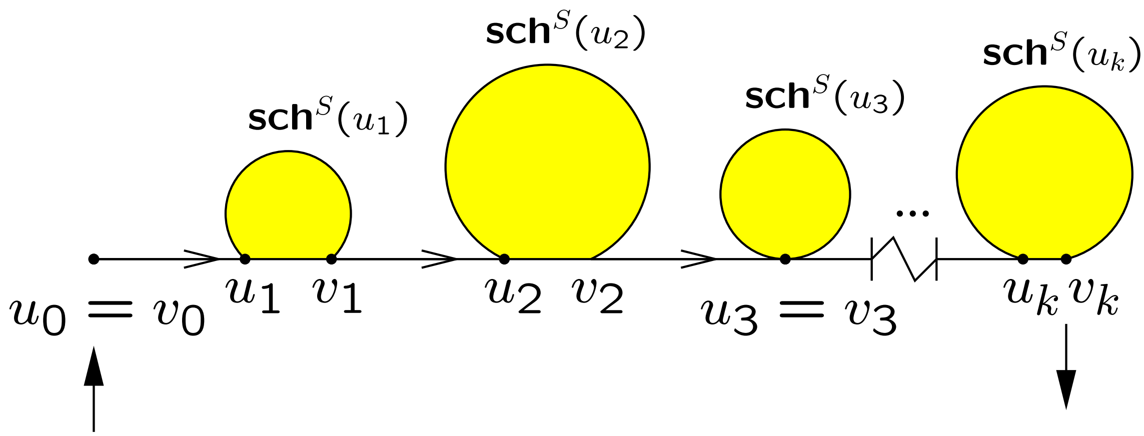
Rem: $\mathbf{S_A}$ is a *poset*; i.e.
• Given S and T there is at most one map $S \to T$.
• Given A-morphisms $f : S \to T$ and $g : T \to S$, $f = g^{-1}$ (canonical).

(actualy a lattice)

Straightline automata

Def: If S is finite $\mathcal{J}$-above A-semigroup and $w \in A^+$ then the *straightline automaton*, $\mathbf{str}^S(w)$, is the path w together with the strong components of its prefixes.

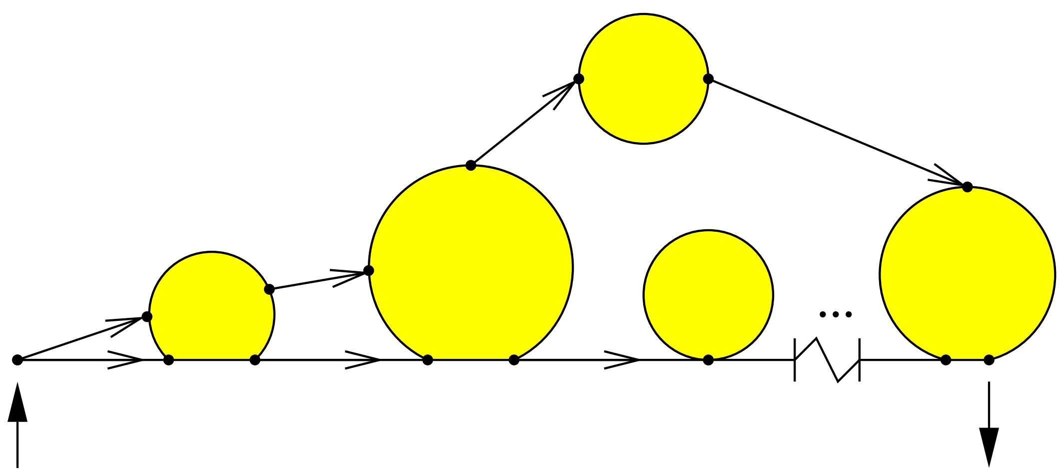
Lem: $\mathbf{str}^S(w)$ is a trim, deterministic FSA which has a quasibase and its strong components are Schützenberger graphs.



Cayley automata

Def: $\mathbf{Cay}^S(w)$ is the full subgraph of $\mathbf{Cayley}(S, A)$ on vertices $\mathcal{R}$-above $[w]$.

Lem: $\mathbf{Cay}^S(w)$ is a trim, deterministic FSA which accepts the language of words equivalent to w in S.



Rem: Want

- $\mathbf{str}^S(w) = \mathbf{Cay}^S(w)$
- to "build" $\mathbf{str}^s(w)$ "geometrically"

Expansions

Def: Let C be a subcategory of $\mathbf{S}$. An *expansion on* C is a functor $F : C \to C$ with a natural transformation to the identity. Explicitly,
$\forall S \in C \ \exists S^{\mathbf{exp}}$ and $\eta : S^{\mathbf{exp}} \to S$.
$\forall S, T \in C \ \exists f^{\mathbf{exp}} : S^{\mathbf{exp}} \to T^{\mathbf{exp}}$
plus consistency conditions.

Rem: For A-semigroups consistency is automatic, and expansions on $\mathbf{S_A}$ are lattice homomorphisms.

Def: **exp** *preserves finiteness* if S is finite implies $S^{\mathbf{exp}}$ finite.

Lem: If **exp** is an expansion on $\mathbf{S_A}$ which preserves finiteness, then S is finite $\mathcal{J}$-above, implies $S^{\mathbf{exp}}$ will be finite $\mathcal{J}$-above.

Digression 1: lattices

Lattices in the combinatorial sense have been around for a long time. Their importance in combinatorics, in semigroup theory, and in group theory is well-established.

One aspect of lattices which has been too little appreciated in geometric group theory is that the lattice property is the key underlying element which drives most combinatorial constructions of Eilenberg-MacLane spaces in the literature (Culler-Vogtmann's outer/auter space, Charney-Davis poset of cosets for Coxeter groups, Charney-Meier-Whittlesey constructions for Garside groups, McCullough-Miller space for free-product decompositions, etc)

One analogy is between a poset construction with a lattice fundamental domain and a spectral sequence which collapses. Few people would be interested (or able) to calculate the resulting topology in the absence of these conditions.

Digression 2: tropical algebra

An idempotent semiring is a set with two commutative monoid operations where “multiplication” distributes over “addition” and “addition” is idempotent ($\forall a, a + a = a$). The naturals (with $+\infty$) under max and plus are an idempotent semiring. Idempotent semirings are equivalent to certain types of lattices via $a + b = a \vee b$.

If the semiring $(\mathbb{N}, \max, +)$ is used instead of the semiring $(\mathbb{N}, +, \times)$ in classical algebraic geometry the result is “tropical” algebraic geometry.

Most semigroup theorists know tropical algebra as a topic closely related to formal language theory and Kleene stars [Simon, Pin]. For geometric group theorists, the most interesting aspect of tropical algebra is the fact that the tropical Grassmannian $G(2, n)$ is precisely the space of metric trees defined by Billera, Holmes and Vogtmann.

Digression 3**: quantales**

Classically a quantale is a lattice with a supremum over every subset, much like the open sets of a topological space. In fact, one theorist has described their study as "pointless topology". (think sheaves)

They have the same advantages as commutative diagrams have over explicit calculations using elements: they force one to think categorically rather than element by element.

Finally, the passage from studying elements to studying operators corresponds to the construction of non-commutative geometry [Connes]

Mal'cev Expansions

Def: Mal'cev kernel of $\phi : S \to T$ is $\{\phi^{-1}(e) \mid e^2 = e \in T\}$.

Thm(Brown) Let $\phi : S \to T$ be a homomorphism. If the Mal'cev kernel of ϕ lies in a locally finite variety $\mathcal{V}$, and T is finite, then S is finite.

Def: The *Mal'cev expansion of S by $\mathcal{V}$* is the largest A-semigroup which maps to S with Mal'cev kernel in $\mathcal{V}$. [intersect congruences] Denote this $S^{\mathcal{V}}$.

Thm: For each $\mathcal{V}$, $S \mapsto S^{\mathcal{V}}$ is an expanion on $\mathbf{S_A}$. If $\mathcal{V}$ is locally finite, it is also an expansion on finite A-semigroups and finite $\mathcal{J}$-above A-semigroups.

Rem: Even $S^{\{1\}}$ is non-trivial! (Ash)

Examples

Name	Notation	Equations
Trivial	$\{1\}$	$x = 1$
Semilattices	$\mathcal{SL}$	$x^2 = x,\ xy = yx$
Right zero	$\mathcal{RZ}$	$xy = y$
Bands	$\mathcal{B}$	$x^2 = x$
Rectangular bands	$\mathcal{RB}$	$x^2 = x,\ xay = xby$

(and many more)

All of these are locally finite

Rectangular bands

Lem: $S^{\mathcal{RB}}$ is defined by
$S^{\mathcal{RB}} = \langle A \mid \alpha\beta = \alpha, \alpha, \beta \in A^*, [\alpha] = [\beta] = e^2 = e$ in $S^{\mathcal{RB}} \rangle$

Notice the circularity!
This is only used to present $S^{\mathcal{RB}}$ once it has been found.

Rem: $S^{\mathcal{RB}} \to S$ is a $\mathcal{J}'$-map and one-to-one on subgroups.

Lem: If $T = S^{\mathcal{RB}}$, then **str**$^T(w)$ is a loop automaton defined by a finite number of loop equations (similar to the Burnside semigroups).

Thm: $S^{\mathcal{RB}}$ is stable under the Rhodes, reverse Rhodes, Birget-Rhodes, Rhodes-Karnofsky, and reverse Rhodes-Karnofsky expansions (similar to the Burnside semigroups).

Cor: **str**$^T(w)$ = **Cay**$^T(w)$ and it only depends on $[w]$ (similar to the Burnside semigroups).

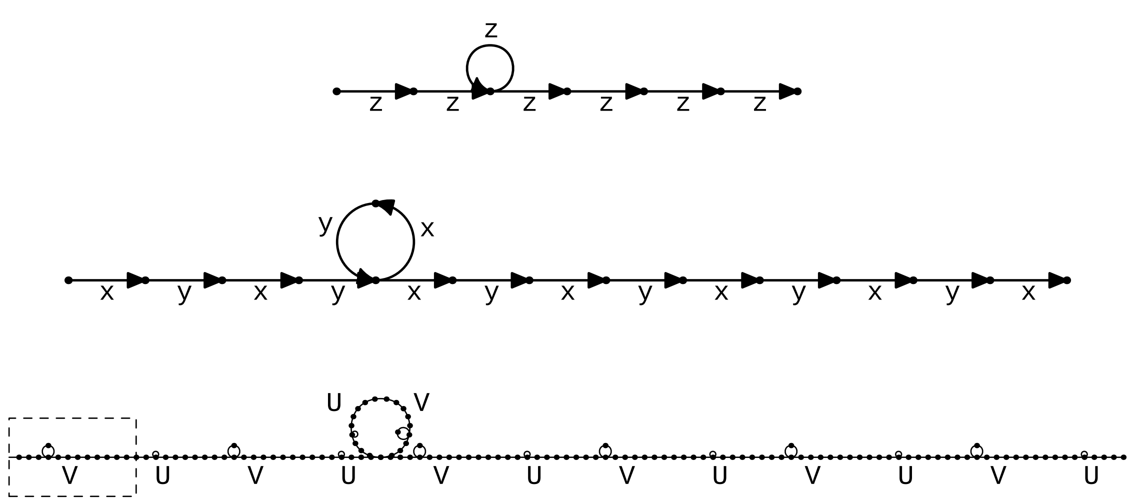
Burnside semigroups

Def: $\mathcal{B}(m,n) = \langle A \mid a^m = a^{m+n} \ \forall a \in A^+ \rangle$

[de Luca-Varricchio, do Lago, M, Guba]

Fact: for m large enough **str**$^{\mathcal{B}(m,n)}(w)$ is a loop automaton which accepts a language described by a unionless Kleene expression.

$V =$ **str**$^{\mathcal{B}(6,1)}(z^6)$
$U =$ **str**$^{\mathcal{B}(6,1)}((xy)^6 x)$
and **str**$^{\mathcal{B}(6,1)}(((xy)^6 xz)^6)$



Why aren't Burnside semigroups completely trivial to work with?

Consider the following sequence of equalities in $\mathcal{B}(6,2)$

$$
\begin{aligned}
&(xy^7)^7 x\underline{y^6} \equiv \\
&(xy^7)^7 x\overline{y^8} = \\
&\underline{(xy^7)^8} y \equiv \\
&\overline{(xy^7)^6} y = \\
&(xy^7)^5 x\underline{y^8} \equiv \\
&(xy^7)^5 x\overline{y^6}
\end{aligned}
$$

So we can replace a not-quite 8-th power with a not-quite 6-th power. If this behavior could propogate, this would be bad.

Knuth-Bendix to the rescue! (along with the $|X| + |Y|$ lemma)

Philosophy

Consider the regular language $\{a + b + c\}^*$.

This has several union-less Kleene expressions. For example, $a^*(ba^*)^*(c(ba^*)^*)^*$

minimum automaton accepting the language	$\Leftrightarrow$	topology of the language
Kleene expressions for the language	$\Leftrightarrow$	geometries imposed on its topology

We think of loops which occur earlier in the Kleene expression as being "shorter" in this "geometry"

Pumping

Once it is noticed that every straightline automaton in $\mathcal{B}(m,n)$ accepts a language described by a union-free Kleene expression, there is a natural way to "pump" this language to a new Burnside semigroup which has more repetition: simply replace each $*$ with a specific number, say k, and then recalcuate the language in a the new group.

Notice that this is dependent on the form of the Kleene expression chosen. For example, $a^*(ba^*)^*(c(ba^*)^*)^*$ becomes $a^{17}(ba^{17})^{17}(c(ba^{17})^{17})^{17}$ when $k = 17$. The behavior of this word is an exaggerated version of the previous behavior. In particular, the "shorter" loops repeat quite often before the next largest loop appears.

Improving stabilizers

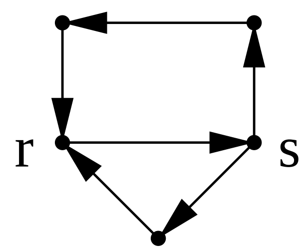
Thm(Le Saec-Pin-Weil) Let S be a finite A-semigroup and p prime. If p is sufficiently large then $\forall t \in T = S^{\mathcal{LZ}.\langle \mathbb{Z}_p \rangle}$ the right stabilizer T_t will be an $\mathcal{R}$-trivial band. In other words, T_t will satisfy $x^2 = x$ and $xyx = xy$.

Cor: The right stabilizer of $S^{\mathcal{RB}.\langle \mathbb{Z}_p \rangle.\mathcal{RB}}$ consists of a finite $\mathcal{L}$-chain of idempotents (within itself) and is $\mathcal{R}$-trivial.

Cor: If $T = S^{\mathcal{RB}.\langle \mathbb{Z}_p \rangle.\mathcal{RB}}$ then the set of loops at state q in $\mathbf{str}^T(w)$ form an $\mathcal{R}$-trivial idempotent subsemigroup which is an $\mathcal{L}$-chain in itself.

Falling back on trees

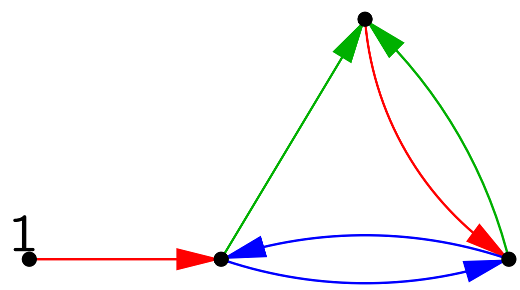
Def: Γ has the *unique simple path property* from q if there does not exist a state p and two distinct simple paths from q to p.

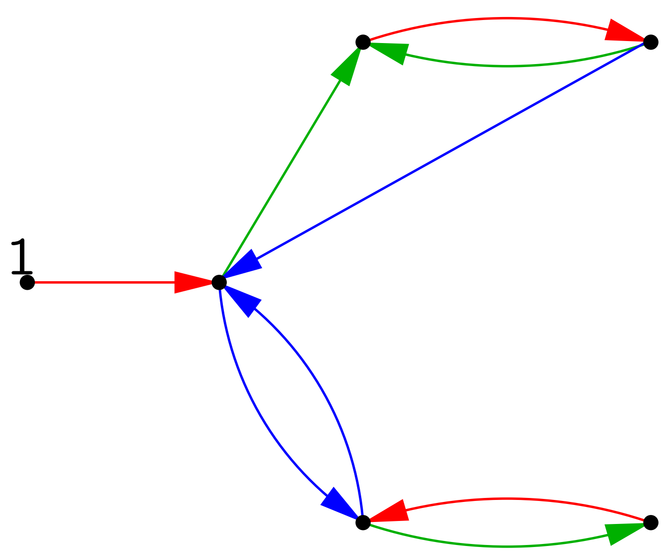


Rem/Def: If Γ is a Cayley graph for a pointed faithful partial transformation semigroup, ∃ an "expanded" graph with u.s.p.p. from 1.

Lem: **Mc** is an expansion on the category of pointed faithful partial A-semigroups. [similar to the Rhodes expansion]

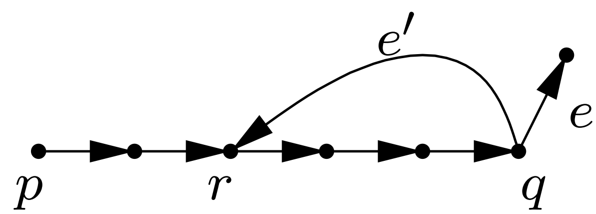
Example 1



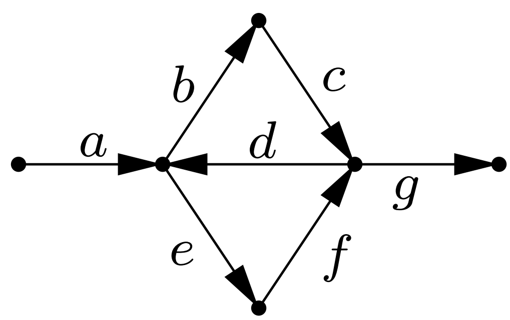


Example 2

Multiplication by an edge:



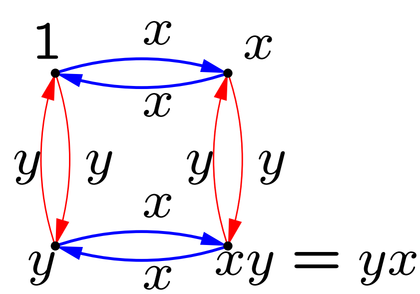
Nonassociativity:

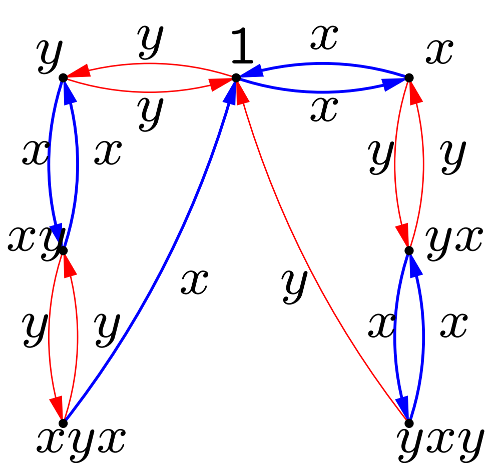


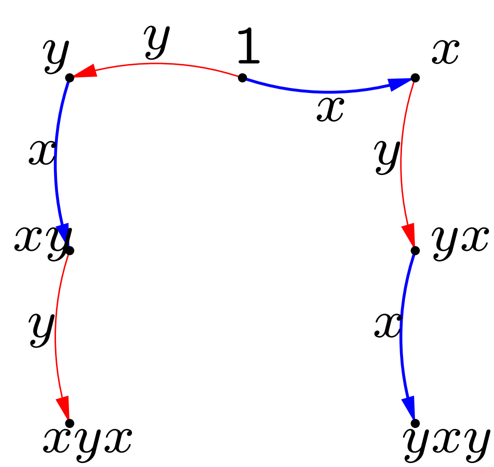
$$aefg = ((abc)d)(efg) \neq (abc)(d(efg)) = abcg$$

Properties

Rem: If $T = S^{\mathcal{RB}.\textbf{Mc}}$ then the labeled graph defining T has the u.s.p.p. from 1 (and $\textbf{sch}^T(w)$ has u.s.p.p. from its entry point). In addition, $\textbf{str}^T(w)$ has a well-defined base, $\textbf{Cayley}(T, A)$ has a well-defined tree and all other edges connect a vertex to a point earlier in the tree. Finally, $\textbf{str}^T(w)$ is an elementary loop automaton.







Adding in delays

Finally, we add in delays so that the loops which occur on any path from 1 occur in a "natural" order.

As an expansion, S^{D_k} is often denoted $S^{\langle k \rangle}$.

Recall that D_k is defined by $x_1 \cdots x_k x = x_1 \cdots x_k$. This ensures that "loops" which occur have paths which have already occurred repeatedly. This provides the ordering of the loops.

Main result: slightly less rough

Let S be a finite (or finite $\mathcal{J}$-above) A-semigroup, and let

$$T = S^{\mathcal{RB}.\langle \mathbb{Z}_p \rangle.\mathcal{RB}.\textbf{Mc}.\langle k \rangle}$$

Then $\textbf{str}^T(w)$ is *very* close to the Burnside automata (and still finite!)

(the Cayley graph is tree-like and smaller rank things have to repeat many many times before they are able to generate a new loop, the automaton is defined by loop equations, and the end result is very fractal-like along each path, etc, etc, etc)

The result is an object which is/should be useful in the study of Krohn-Rhodes complexity.