

A COMPLEX EUCLIDEAN REFLECTION GROUP WITH AN ELEGANT COMPLEMENT COMPLEX

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ABSTRACT. The complement of a hyperplane arrangement in \mathbb{C}^n deformation retracts onto an n -dimensional cell complex, but the known procedures only apply to complexifications of real arrangements (Salvetti) or the cell complex produced depends on an initial choice of coordinates (Björner-Ziegler). In this article we consider the unique complex euclidean reflection group acting cocompactly by isometries on \mathbb{C}^2 whose linear part is the finite complex reflection group known as G_4 in the Shephard-Todd classification and we construct a choice-free deformation retraction from its hyperplane complement onto an elegant 2-dimensional complex K where every 2-cell is a euclidean equilateral triangle and every vertex link is a Möbius-Kantor graph. Since K is non-positively curved, the corresponding braid group is a CAT(0) group, despite the fact that there are non-regular points in the hyperplane complement, the action of the reflection group on K is not free, and the braid group is not torsion-free.

INTRODUCTION

The complement of a hyperplane arrangement in \mathbb{C}^n is obtained by removing the union of its hyperplanes. When the arrangement under consideration is a complexified version of a real arrangement, there is a classical construction due to Salvetti that provides a deformation retraction onto an n -dimensional cell complex now known as the *Salvetti complex* of the arrangement [Sal87]. Björner and Ziegler extended Salvetti's construction so that it works for an arbitrary complex hyperplane arrangement, but their construction depends on an initial choice of a coordinate system [BZ92]. In this article we deformation retract the complement of a specific infinite affine hyperplane arrangement in \mathbb{C}^2 onto an elegant 2-dimensional piecewise euclidean complex that involves no choices along the way. The arrangement we consider is the

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set of hyperplanes for the reflections in a complex euclidean reflection group that we denote $\text{REFL}(\tilde{G}_4)$. This is the unique complex euclidean reflection group acting cocompactly by isometries on \mathbb{C}^2 whose linear part is the finite complex reflection group known as G_4 in the Shephard-Todd classification.

Theorem A (Complement complex). *The hyperplane complement of $\text{REFL}(\tilde{G}_4)$ deformation retracts onto a non-positively curved piecewise euclidean 2-complex K in which every 2-cell is an equilateral triangle and every vertex link is a Möbius-Kantor graph.*

The essence of our construction is easy to describe. We use the set of 0-dimensional hyperplane intersections to form Voronoi cells and then construct a deformation retraction from the hyperplane complement onto the portion of the Voronoi cell structure contained in the complement. For the group $\text{REFL}(\tilde{G}_4)$ all of the Voronoi cells are isometric and their shape is that of the regular 4-dimensional polytope known as the 24-cell. The 0-dimensional intersection at the center of each Voronoi cell means that as a first step one can remove its interior by radially retracting onto its 3-dimensional polytopal boundary built out of regular octahedra. This procedure works for this particular complex euclidean reflection group but it appears that this is one of the few cases where it can be carried out without significant modifications. See Remark 9.3.

Next we use the complement complex K to study the structure of the braid group of $\text{REFL}(\tilde{G}_4)$ acting on \mathbb{C}^2 . Recall that for any group G acting on a space X a point $x \in X$ is said to be *regular* when its G -stabilizer is trivial, the *space of regular orbits* is the quotient of the subset of regular points by the free G -action and the *braid group of G acting on X* is the fundamental group of the space of regular orbits. The name “braid group” alludes to the fact that when the symmetric group SYM_n acts on \mathbb{C}^n by permuting coordinates, the braid group of this action is Artin’s classical braid group BRAID_n . For complex spherical reflection groups, one consequence of Steinberg’s theorem is that the hyperplane complement is exactly the set of regular points [Ste64, Leh04, LT09]. For complex euclidean reflection groups the two spaces can be distinct and they are distinct in this case.

Theorem B (Isolated fixed points). *The space of regular points for the complex euclidean reflection group $\text{REFL}(\tilde{G}_4)$ acting on \mathbb{C}^2 is properly contained in its hyperplane complement because of the existence of isolated fixed points.*

Concretely, for every vertex v in the 2-complex K located inside the hyperplane complement there is a non-trivial group element that fixes v and acts as the antipodal map in the coordinate system with v as its origin. Moreover, the set of isolated fixed points that form the vertices of K are the only non-regular points contained in the hyperplane complement. Let $\text{BRAID}(\tilde{G}_4)$ denote the braid group of $\text{REFL}(\tilde{G}_4)$ acting on \mathbb{C}^2 . The well-behaved geometry of K and the isolated fixed points in the hyperplane complement lead to an unusual mix of properties for a braid group of a reflection group.

Theorem C (Braid group). *The group $\text{BRAID}(\tilde{G}_4)$ is a $\text{CAT}(0)$ group and it contains elements of order 2.*

The group $\text{BRAID}(\tilde{G}_4)$ is a $\text{CAT}(0)$ group because it acts properly discontinuously and cocompactly by isometries on the $\text{CAT}(0)$ universal cover of K and it has elements of order 2 that are caused by the stabilizers of the isolated fixed points in the hyperplane complement. Since every finitely generated Coxeter group is a $\text{CAT}(0)$ group that contains 2-torsion, this combination is not unusual in the broader world of $\text{CAT}(0)$ groups. However, torsion is unusual in the braid group of a reflection group. The braid groups of finite complex reflection groups are torsion-free [Bes15], as are the braid groups of complexified euclidean Coxeter groups, also known as euclidean Artin groups or affine Artin groups [MS]. In fact, it is conjectured that the braid groups of all complexified Coxeter groups, i.e. all Artin groups, are torsion-free [GP12]. Thus, this example is a departure from the norm.

The article is structured from general to specific. We begin with basic definitions and results about general complex spherical and complex euclidean reflection groups. Then we restrict attention to complex dimension at most two and describes how quaternions can be used to give efficient linear-like descriptions of arbitrary isometries of the complex euclidean plane. Next, we describe the 4-dimensional regular polytope known as the 24-cell, and investigate the natural action of $\text{REFL}(G_4)$ on this polytope. The main tool is a novel visualization technique that makes it easy to understand the isometries of the 4-dimensional regular polytopes. Finally, the last part of the article describes the complex euclidean reflection group $\text{REFL}(\tilde{G}_4)$ in detail and proves our three main results.

1. COMPLEX SPHERICAL REFLECTION GROUPS

This section reviews the definition and classification of the complex spherical reflection groups. Recall that in geometric group theory one

seeks to understand groups via their actions on metric spaces and that the connection between the two is particularly close when the action is geometric in the following sense.

Definition 1.1 (Geometries and geometric actions). A metric space X is called a *proper metric space* or a *geometry* when for every point $x \in X$ and for every positive real r , the closed metric ball of radius r around x is a compact subspace of X and a group G acting on a geometry X is said to act *geometrically* when the action of G on X is properly discontinuous and cocompact by isometries.

The first geometry we wish to consider is that of the unit sphere in a complex vector space with a positive definite inner product.

Definition 1.2 (Complex spherical geometry). Let $V = \mathbb{C}^n$ be an n -dimensional complex vector space. When V comes equipped with a positive definite hermitian inner product that is linear in the second coordinate and conjugate linear in the first, we say that V is a *complex spherical geometry*. For an appropriate choice of basis, the inner product of vectors v and w in V can be written as $\langle v, w \rangle \stackrel{\text{def}}{=} v^*w = \sum_{i=1}^n \bar{v}_i w_i$ where v and w are viewed as column vectors or as n by 1 matrices and for any matrix A , A^* denotes its *adjoint* or conjugate transpose. The *length* of a vector v is $|v| = \sqrt{\langle v, v \rangle}$ and *unit vectors* are those of length 1. The linear transformations of V that preserve the inner product are the *unitary transformations*, they form the *unitary group* $U(V)$ or $U(n)$ and they are precisely those linear transformations that preserve the sphere of unit vectors in V and its complex structure, sending complex lines in V to complex lines and the corresponding oriented circles in the unit sphere \mathbb{S}^{2n-1} to oriented circles.

A complex reflection is an elementary isometry of such a geometry.

Definition 1.3 (Complex reflections). Let V be a complex spherical geometry. Vectors in V are *orthogonal* or *perpendicular* when their inner product is 0 and the *orthogonal complement* of a vector v is the set of all vectors perpendicular to v . A *complex reflection* r is a unitary transformation of V that multiplies some unit vector v by a unit complex number $z \in \mathbb{C}$ and pointwise fixes the vectors in the orthogonal complement of v . The formula for the reflection $r = r_{v,z}$ is $r(w) = w - (1 - z)\langle v, w \rangle v$. The reflection r has finite order if and only if $z = e^{ai}$ where a is a rational multiple of π and when this occurs we say that r is a *proper* complex reflection. The name refers to the fact that the action of the cyclic subgroup generated by r on the unit sphere is properly discontinuous if and only if r is a proper reflection. Since

properly discontinuous actions require proper complex reflections, only proper reflections are considered and we drop the adjective. When the complex number z is of the form $z = e^{\frac{2\pi}{m}i}$ for some positive integer m , the complex reflection $r_{v,z}$ is said to be *primitive*, and note that every finite cyclic subgroup generated by a single proper complex reflection contains a unique primitive generator.

We are interested in groups generated by complex reflections.

Definition 1.4 (Complex spherical reflection groups). A group G is called a *complex spherical reflection group* if it is generated by complex reflections acting on a complex spherical geometry V so that the action restricted to the unit sphere in V is geometric in the sense of Definition 1.1. Such groups are also known as *finite complex reflection groups*. If there is an orthogonal decomposition $V = V_1 \oplus V_2$ preserved by all of the elements of G , then G is *reducible* and it is *irreducible* when such a decomposition does not exist. In 1954 Shephard and Todd completely classified the irreducible complex spherical reflection groups. There is a single triply-indexed infinite family $G(d, e, r)$ where d , e and r are positive integers that they split into 3 subcases G_1 , G_2 and G_3 based on some additional properties and 34 exceptional cases that they label G_4 through G_{37} [ST54, Coh76]. Since this article discusses both reflection groups and the corresponding braid groups, we use the symbol G_k with k between 4 and 37 to indicate a *Shephard-Todd type* analogous to the Cartan-Killing types that index so many objects in Lie theory and we write $\text{REFL}(G_k)$ to denote the exceptional complex spherical reflection group of type G_k identified by Shephard and Todd.

The main group of interest here is a euclidean extension of the smallest exceptional complex spherical reflection group $\text{REFL}(G_4)$.

2. COMPLEX EUCLIDEAN REFLECTION GROUPS

The transition from complex spherical to complex euclidean geometry involves replacing the underlying vector space and its distinguished origin with the corresponding affine space where all points are on an equal footing.

Definition 2.1 (Affine space). For any vector space V , the abstract definition of the corresponding *affine space* is a set E together with a simply transitive V action on E . The elements of E are *points*, the elements of V are *vectors* and we write $x+v$ for the image of point $x \in E$ under the action of $v \in V$. For each linear subspace $U \subset V$ and point $x \in E$ there is an *affine subspace* $x+U = \{x+v \mid v \in U\} \subset E$ that collects the images of x under the action of the vectors in U and the functions

$f: E \rightarrow E$ that send affine subspaces to affine subspaces are *affine maps*. For each vector $v \in V$ there is a *translation map* $t_v: E \rightarrow E$ that sends each point x to $x + v$ and this is an affine map. The collection of all translation maps is an abelian group isomorphic to the vector space V under addition and it is a normal subgroup of the group $\text{AFF}(E)$ of all affine transformations. If we pick a point $x \in E$ as our *basepoint* then every point y in E can be labeled by the unique vector $v \in V$ that sends x to y so that E based at x is naturally identified with V and the group of all affine maps can be identified with the semidirect product of the translation group and the invertible linear transformations of E based at x now identified with V . In other words, for each point $x \in E$ there is a natural isomorphism between the group $\text{AFF}(E)$ and the semidirect product $V \rtimes GL(V)$.

When the vector space V is a complex spherical geometry, it makes sense to restrict attention to those affine transformations that preserve the hermitian inner product.

Definition 2.2 (Complex euclidean space). Let E be an affine space for a complex vector space V . When V is a complex spherical geometry, then E is a *complex euclidean geometry*. Since an ordered pair (x, x') of points in E determines a vector $v_{x,x'} \in V$ that sends x to x' , an ordered quadruple (x, x', y, y') of points in E determines an ordered pair of vectors $(v_{x,x'}, v_{y,y'})$ in V to which the hermitian inner product can be applied. An affine map $f: E \rightarrow E$ is called a *complex euclidean isometry* when f preserves the hermitian inner product of the ordered pair of vectors derived from an ordered quadruple of points in E . In other words $\langle v_{x,x'}, v_{y,y'} \rangle = \langle v_{f(x),f(x')}, v_{f(y),f(y')} \rangle$ for all $x, x', y, y' \in E$. The group of all complex euclidean isometries is denoted $\text{ISOM}(E)$. All translations are complex euclidean isometries and an affine map fixing a point x is a complex euclidean isometry if and only if the corresponding linear transformation of V is a unitary transformation. Therefore, for each point $x \in E$ there is a natural isomorphism between the group $\text{ISOM}(E)$ and the semidirect product $V \rtimes U(V)$ or $\mathbb{C}^n \rtimes U(n)$ once an orthonormal coordinate system has been introduced.

The spherical notion of a complex reflection is extended to complex euclidean space as follows.

Definition 2.3 (Complex euclidean reflection groups). An isometry of a complex euclidean space E is called a *complex reflection* if it becomes a complex reflection in the sense of Definition 1.3 for an appropriate choice of origin and identification of the space E with the vector space V . For us, a *complex euclidean reflection group* is any group generated

by complex reflections that acts geometrically on a complex euclidean space. In the literature, the complex euclidean reflection groups that act geometrically on E are called *crystallographic*. The image of a complex euclidean reflection group G under the projection map from $\text{ISOM}(E) \rightarrow U(V)$ is called its *linear part* and the kernel is its *translation part*. In many but not all examples the group G has the structure of a semidirect product of its linear and translation parts. The group G is called *reducible* or *irreducible* depending on the corresponding property of its linear part. Two complex euclidean reflection groups G and G' acting on complex euclidean spaces E and E' are called *equivalent* when there is an invertible affine map from E to E' (that need not preserve the complex euclidean metric) so that the action of G on E corresponds to the action of G' on E' under this identification.

Remark 2.4 (Known examples). The collection of known inequivalent irreducible complex euclidean reflection groups includes 30 infinite families and 22 isolated examples. Some of the infinite families have a discrete parameter that indicates the dimension of the space on which it acts, some of the infinite families have a continuous complex parameter which, when varied, produces inequivalent reflection groups that all act on the same space, and some have both a discrete and a continuous parameter. The 17 infinite families with a continuous complex parameter correspond exactly to those whose linear part is an irreducible finite real reflection group. There is one such family for each simply-laced Cartan-Killing type (A_n , D_n , E_6 , E_7 , E_8) and multiple families for each of the others (G_2 has 4, F_4 has 3 and $B_n = C_n$ has 5 – except in dimension $n = 2$ where the identification $\text{COX}(\tilde{B}_2) \cong \text{COX}(\tilde{C}_2)$ reduces the number of parameterized families from 5 to 3). The 7 families of type A , $B = C$ and D have both a continuous parameter and a discrete parameter, the 10 families of type E , F and G have a continuous parameter only. Next there are 13 infinite families with primitive linear part indexed by a discrete parameter but with only one instance in each dimension. And finally, there are 7 isolated examples with primitive linear part that only occur in low dimensions (3 in dimension 1 and 4 in dimension 2) and 15 isolated examples whose linear part is one of the 34 exceptional complex spherical reflection groups (5 in dimension 2, 7 in dimension 3 and one each in dimensions 4, 5 and 6).

Remark 2.5 (Classification). The inequivalent irreducible complex euclidean reflection groups were essentially classified by Popov in [Pop82]. He established many structural results about these groups and gave algorithms in each of the various subcases that together could be used to produce a complete list. Some of the details of the computations that

connect the algorithms with the explicit tables of examples, however, were not included and in 2006 Goryunov and Man found an isolated example in dimension 2 that was not among those listed by Popov, thus calling the completeness of the tables into question [GM06].

We write $\text{REFL}(\tilde{G}_4)$ to denote the unique complex euclidean reflection group whose linear part is $\text{REFL}(G_4)$. Popov denotes it [K_4].

3. ISOMETRIES OF THE COMPLEX EUCLIDEAN LINE

The inequivalent complex euclidean reflection groups that act geometrically on the complex euclidean line are easily classified. In this section we review their classification and preview the Voronoi cell argument in this easy-to-visualize context.

Definition 3.1 (Isometries and reflections). Every isometry of the complex euclidean line is a function of the form $f(x) = e^{ai}x + z$ where a is real and z is an arbitrary complex number. And since complex euclidean reflections acting on \mathbb{C} must fix some point z_0 (i.e. some affine copy of \mathbb{C}^0), they are precisely those isometries of the form $e^{ai}(x - z_0) + z_0$.

The fact that we are only interested in discrete actions places a strong restriction on the orders of the complex euclidean reflections that can be used.

Lemma 3.2 (Crystallographic). *If r and s are primitive complex reflections of order m with distinct fixed points acting on \mathbb{C} , then the action of the group they generate is indiscrete unless $m \in \{2, 3, 4, 6\}$.*

Proof. The product $t = rs^{-1}$ is a translation and the composition of the translations rtr^{-1} and $r^{-1}tr$ is another translation in the same direction as t but its translation distance is $2 \cos \frac{\pi}{m}$ times that of t . In particular, the group of translations in this direction act indiscretely unless $\cos \frac{\pi}{m}$ is rational, and this is true exactly for $m \in \{2, 3, 4, 6\}$. \square

The crystallographic restriction makes it easy to classify the complex euclidean reflection groups that act on the complex euclidean line.

Theorem 3.3 (Classification). *If G is a complex euclidean reflection group that acts geometrically on the complex euclidean line, then every reflection in G has order 2, 3, 4 or 6 and its reflections of maximal order generate G . When the maximal order is 2 there is a 1-parameter family of such groups, but when it is 3, 4 or 6 there is a unique such group up to affine equivalence.*

We include a brief description of each case.

Example 3.4 (Order 2). When all reflections have order 2, their fixed points form a lattice in \mathbb{C} , i.e. a discrete \mathbb{Z}^2 subgroup in \mathbb{C} once one of these fixed points has been chosen as the origin. After rescaling so that there are fixed points at 0 and 1 and no pair of fixed points less than 1 unit apart, the various inequivalent cases are described by a third generator fixing a point z with $|z| \geq 1$ and the real part of z in the interval $[-\frac{1}{2}, \frac{1}{2}]$ with some identifications along the boundary.

Example 3.5 (Orders 3, 4 and 6). For $m = 3$, $m = 4$ and $m = 6$ we start with an equilateral triangle, an isosceles right triangle and a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{6}$, respectively. There is a triangular tiling of \mathbb{C} generated by the real reflections in the sides of this triangle. The real reflection groups generated are the euclidean Coxeter groups $\text{COX}(\tilde{A}_2)$, $\text{COX}(\tilde{B}_2)$ and $\text{COX}(\tilde{G}_2)$, respectively. In each case, the index 2 subgroup of orientation-preserving isometries is generated by those complex reflections rotating through an angle of $\frac{2\pi}{m}$ fixing a point where $2m$ triangles met. These groups are denoted $[K_3(m)]$ in Popov's notation and $\text{REFL}(\tilde{G}_3(m))$ in ours.

Remark 3.6 (Fixed points and translations). Let G be a complex euclidean reflection group acting on \mathbb{C} , let T be the subgroup of translations, let T_0 be the images of the origin under the translations in T and let FP_m be the fixed points of the primitive reflections of order m (assuming they exist) and assume that the origin is fixed by a primitive reflection r of order m . The computation $t_v \circ r \circ t_v^{-1}(x) = z(x - v) + v = zx + (1 - z)v = t_{(1-z)v} \circ r(x)$ with $z = e^{\frac{2\pi}{m}i}$ shows that $(1 - z) \cdot FP_m = T_0$. In the group $\text{REFL}(\tilde{G}_3(6))$, for example, $2 \cdot FP_2 = (1 - \omega) \cdot FP_3 = FP_6 = T_0$, where $\omega = e^{2\pi i/3}$ is a primitive cube-root of unity.

Definition 3.7 (Voronoi cells). Let S be a discrete set of points in some euclidean space E . The *Voronoi cell* around s is the set of points in E that are as close to s as they are to any point in S . These regions are delineated by the hyperplanes that are equidistant between two points in S . Thus, the Voronoi cells are euclidean polytopes so long as these regions are bounded (as they are in our context). The union of these euclidean polytopes gives the entire euclidean space E a piecewise euclidean cell structure that we call the *Voronoi cell structure*. The Voronoi cell structure of a complex euclidean reflection group G is the cell structure obtained when S is the set of 0-dimensional intersections of the fixed hyperplanes of the complex reflections in G .

As should be clear from its definition, the Voronoi cell structure is preserved by the complex euclidean group used to create it.

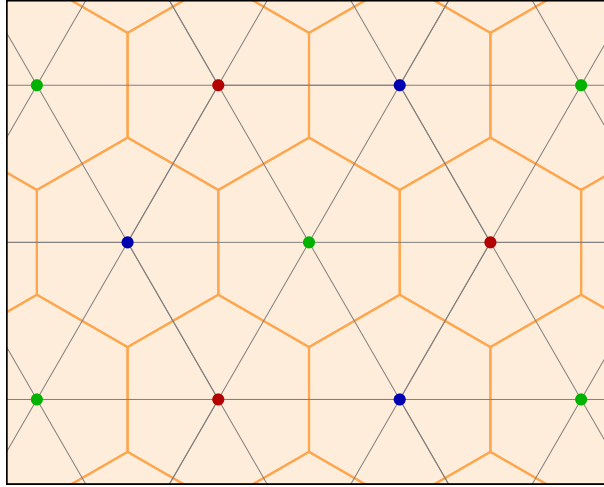


FIGURE 1. The Voronoi cell structure for the complex euclidean reflection group $\text{REFL}(\tilde{G}_3(3))$ is a hexagonal tiling of \mathbb{C} and the hyperplane complement deformation retracts to its 1-skeleton.

Example 3.8 (Voronoi cells). Let G be one of the complex euclidean reflection groups $\text{REFL}(\tilde{G}_3(m))$ with $m \in \{3, 4, 6\}$ and let S be the set of fixed points for the reflections in G . The vertices of the Voronoi cells in this case are the centers of the inscribed circles of the triangles in the corresponding triangular tiling, the edges are built out of the altitudes from these centers to the sides of the triangles and the Voronoi cells themselves are regular polygons, hexagons for $m = 3$, squares and octagons for $m = 4$ and squares, hexagons and dodecagons for $m = 6$. The case $m = 3$ is illustrated in Figure 1.

The Voronoi cells can be used to understand the braid groups.

Theorem 3.9 (Braid groups). *For $m = 3, 4$ and 6 , the braid group $\text{BRAID}(\tilde{G}_3(m))$ is isomorphic to the free group of rank 2.*

Proof. In all three cases, once the fixed points of the reflections are removed, the remainder deformation retracts to the 1-skeleton of the Voronoi cell structure. The group acts freely on the 1-skeleton but it does not act transitively on the vertices. The quotient graph has 2 vertices with 3 edges connecting them, a graph whose fundamental group is the free group of rank 2. \square

4. QUATERNIONS AND THEIR COMPLEX STRUCTURES

In this section we recall basic properties of the quaternions and their subalgebras isomorphic to the complex numbers. The goal is to establish notation for the quaternions with a specified complex structure.

Definition 4.1 (Quaternions). Let \mathbb{H} denote the *quaternions*, the skew field and normed division algebra of dimension 4 over the reals with standard basis $\{1, i, j, k\}$ where $i^2 = j^2 = k^2 = ijk = -1$ and i, j and k pairwise anticommute. The reals \mathbb{R} are identified with the \mathbb{R} -span of 1 inside \mathbb{H} and they form its center: every real is central and every central element is real. If $q = a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$ then $\text{REAL}(q) = a$ is its *real part* and $\text{IMAG}(q) = bi + cj + dk$ is its *imaginary part*. A quaternion is *purely imaginary* if its real part is 0 and *real* if its imaginary part is 0. The *conjugate* of q is $\bar{q} = a - (bi + cj + dk)$, its *norm* $\text{NORM}(q) = q\bar{q} = \bar{q}q = a^2 + b^2 + c^2 + d^2$ and its *length* $|q|$ is the square root of its norm. The *distance* between q and q' is the length of $q - q'$. This distance function makes \mathbb{H} into a 4-dimensional euclidean space with $\{1, i, j, k\}$ as an orthonormal basis. The quaternions in the unit 3-sphere in \mathbb{R}^4 have norm 1 and are the set of *unit quaternions*. Every nonzero quaternion can be *normalized* by dividing by its length.

The unit quaternions show that the 3-sphere has a Lie group structure. It can be identified with the compact symplectic Lie group $\text{SP}(1)$, the spin group $\text{SPIN}(3)$ (the double cover of $\text{SO}(3)$) or special unitary group $\text{SU}(2)$ once a complex structure has been chosen. The quaternions have a canonical copy of the reals and thus a canonical euclidean structure, but they contain a continuum of subalgebras isomorphic to \mathbb{C} and a corresponding continuum of ways to specify a complex spherical structure.

Definition 4.2 (Complex subalgebras). For each purely imaginary unit quaternion u , $u^2 = -1$ and the \mathbb{R} -span of 1 and u is a subalgebra of \mathbb{H} isomorphic to the complex numbers with u playing the role of $\sqrt{-1}$. More generally, note that every nonreal quaternion q_0 determines a complex subalgebra of \mathbb{H} in which q_0 has positive imaginary part. Concretely, the \mathbb{R} -span of 1 and q_0 is a complex subalgebra and the isomorphism with \mathbb{C} identifies $\sqrt{-1}$ with the normalized imaginary part of q_0 . We call this the *complex subalgebra determined by q_0* .

The choice of a complex subalgebra determines a complex structure.

Definition 4.3 (Complex structures). Let q_0 be a nonreal quaternion and identify \mathbb{C} with the complex subalgebra of \mathbb{H} determined by q_0 . The right cosets $q\mathbb{C}$ of \mathbb{C} inside \mathbb{H} partition the nonzero quaternions

into right *complex lines*. Vector addition and this type of right scalar multiplication turn \mathbb{H} into a 2-dimensional right vector space over this subalgebra \mathbb{C} . In addition, there is a unique positive definite hermitian inner product on this 2-dimensional complex vector space so that the unit quaternions have length 1 with respect to this inner product. We call this the *right complex structure on \mathbb{H} determined by q_0* and we write \mathbb{H}_{q_0} to denote the quaternions with this choice of complex structure. Note that when $q_1 = a + bq_0$ with a real and b positive real, \mathbb{H}_{q_0} and \mathbb{H}_{q_1} define the same complex structure.

The complex structure used in our computations is \mathbb{H}_ω where $\omega = \frac{-1+i+j+k}{2}$ is a cube root of unity. The pure unit quaternion that plays the role of $\sqrt{-1}$ in the chosen complex subalgebra is $\frac{i+j+k}{\sqrt{3}}$.

Definition 4.4 (Unit complex numbers). Because the complex subalgebra we use in our computations does not contain the quaternion i , we do not use i as a notation for $\sqrt{-1}$ in the distinguished copy of \mathbb{C} , but we make an exception for the unit complex numbers. Specifically, we write $z = e^{ai}$ with a real for the numbers on the unit circle in \mathbb{C} even though the chosen copy of \mathbb{C} does not contain the quaternion i . Since this misuse of the letter i only occurs as an exponent and only in this particular formulation, the improvement in clarity, in our opinion, outweighs any potential confusion.

Those who prefer computations over \mathbb{C} can select an ordered basis and work with coordinates. Note that we use the letter z rather than q when we wish to emphasize that a particular quaternion lives in the distinguished copy of \mathbb{C} .

Definition 4.5 (Bases and Coordinates). Let \mathbb{H}_{q_0} be the quaternions with a complex structure. Every ordered pair of nonzero quaternions q_1 and q_2 that belong to distinct complex lines form an *ordered basis* of \mathbb{H}_{q_0} viewed as a 2-dimensional right complex vector space. In particular, their right \mathbb{C} -linear combinations $q_1\mathbb{C} + q_2\mathbb{C}$ span all of \mathbb{H}_{q_0} and for every $q \in \mathbb{H}_{q_0}$ there are unique *coordinates* $z_1, z_2 \in \mathbb{C}$ such that $q = q_1z_1 + q_2z_2$. When the basis $\mathcal{B} = \{q_1, q_2\}$ is ordered we view the coordinates of q as a column vector. When the complex structure is determined by j and the ordered basis $\mathcal{B} = \{1, i\}$, for example, the quaternion $q = a + bi + cj + dk$ has coordinates $z_1 = a + cj$ and $z_2 = b + dj$ because $q = 1(a + cj) + i(b + dj)$. In other words, inside \mathbb{H}_j

$$q = a + bi + cj + dk = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} a + cj \\ b + dj \end{bmatrix}_{\mathcal{B}}.$$

5. ISOMETRIES OF THE COMPLEX EUCLIDEAN PLANE

This section concisely describes each isometry of the complex euclidean plane using an elementary quaternionic map. We begin with the left and right multiplication maps.

Definition 5.1 (Spherical maps). For each quaternion q there is a left multiplication map $L_q(x) = qx$ and a right multiplication map $R_q(x) = xq$ from \mathbb{H} to itself and these maps are isometries of the canonical euclidean structure of \mathbb{H} if and only if q has length 1. When q is not a unit, they are euclidean similarities but not isometries since they change lengths. When q is a unit quaternion, both L_q and R_q are orientation preserving euclidean isometries that fix the origin, send the unit 3-sphere to itself and move every point in \mathbb{S}^3 the same distance. For each pair of unit quaternions q and q' , there is a function defined by the composition $f = L_q \circ R_{q'} = R_{q'} \circ L_q$ or explicitly by the equation $f(x) = qxq'$ that we call a *spherical map*. Every spherical map induces an orientation preserving isometry of \mathbb{S}^3 and every orientation preserving isometry of \mathbb{S}^3 can be represented as a spherical map in precisely two ways. The second representation is obtained from the first by negating both q and q' . This correspondence essentially identifies the topological space $\mathbb{S}^3 \times \mathbb{S}^3$ of pairs of unit quaternions with the Lie group $\text{SPIN}(4)$, the double cover of $\text{SO}(4)$. For details see [CS03].

The spherical maps that preserve a complex structure are special.

Definition 5.2 (Complex spherical maps). Once a complex structure is added to the quaternions, only some spherical maps preserve this structure and we call those that do *complex spherical maps*. For every unit quaternion q the left multiplication map L_q sends the complex lines in \mathbb{H}_{q_0} to complex lines and it is a complex spherical isometry. Right multiplication is different because of the noncommutativity of quaternionic multiplication. The only right multiplication maps that sent complex lines to complex lines are those of the form R_z where z is number in the chosen complex subalgebra and the only isometries among them are those where z is a unit. When $z = e^{ai}$ is unit complex number (i.e. a unit quaternion in the complex subalgebra generated by 1 and q_0), the map R_z is a complex spherical isometry that stabilizes each individual complex line $q\mathbb{C}$ setwise and rotates it by through an angle of a radians. As z varies through the unit complex numbers, this motion is called the *Hopf flow*.

The next proposition records the fact that left multiplication and the Hopf flow are sufficient to generate all complex spherical isometries.

Proposition 5.3 (Complex spherical isometries). *The spherical maps that preserve the complex structure of \mathbb{H}_{q_0} are precisely those of the form $x \mapsto qxz$ where q is a unit quaternion and z is a unit complex number in the chosen complex subalgebra.*

As with general spherical maps, each complex spherical map can be represented in two ways because of the equality $qxz = (-q)x(-z)$. This gives a map from $\mathbb{S}^3 \times \mathbb{S}^1 \rightarrow \mathrm{U}(2)$ with kernel $\{\pm 1\}$, which corresponds to the short exact sequence $\mathrm{O}(1) \hookrightarrow \mathrm{SP}(1) \times \mathrm{U}(1) \rightarrow \mathrm{U}(2)$. For later use we concretely describe the action of L_z and R_z for any unit complex number z in some detail.

Remark 5.4 (Left and Right). Let \mathbb{H}_{q_0} be the quaternions with a complex structure, let q_1 be any unit quaternion orthogonal to both 1 and q_0 , and let $z = e^{ai}$ with a real be a unit complex number. Both maps L_z and R_z stabilize the complex lines $1\mathbb{C}$ and $q_1\mathbb{C}$ setwise, but their actions on these lines are slightly different. The map R_z rotates both lines through an angle of a radians while the map L_z rotates $1\mathbb{C}$ through an angle of a radians and the line $q_1\mathbb{C}$ through an angle of $-a$ radians. The minus occurs because q_1 , being orthogonal to 1 and q_0 , is a pure imaginary quaternion that commutes with 1 and anticommutes with the pure imaginary part of q_0 . Thus $zq_1 = q_1\bar{z}$ and $\bar{z} = e^{-ai}$.

In the ordered basis $\mathcal{B} = \{1, q_1\}$ the hermitian inner product is the standard one, $q_1\mathbb{C}$ is the unique complex line that is orthogonal to $1\mathbb{C}$, and the maps R_z and L_z can be represented as left multiplication by 2×2 matrices over the complex numbers on the column vector of coordinates with respect to \mathcal{B} . Let x be the quaternion with coordinates x_1 and x_2 with respect to \mathcal{B} so that $x = 1x_1 + q_1x_2$ with $x_1, x_2 \in \mathbb{C}$. The element $xz = x_1z + q_1x_2z = zx_1 + q_1zx_2$ because elements in \mathbb{C} commute. Thus:

$$R_z(x) = xz = \begin{bmatrix} e^{ai} & 0 \\ 0 & e^{ai} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathcal{B}}$$

On the other hand, the element $zx = zx_1 + zq_1x_2 = zx_1 + q_1\bar{z}x_2$ as discussed above. Thus:

$$L_z(x) = zx = \begin{bmatrix} e^{ai} & 0 \\ 0 & e^{-ai} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathcal{B}}$$

Note that the matrix for R_z lies in the center of $\mathrm{U}(2)$ and the matrix for L_z lies in $\mathrm{SU}(2)$.

Definition 5.5 (Complex reflections). Let \mathbb{H}_{q_0} be the quaternions with a complex structure, let q_1 be any unit quaternion orthogonal to both 1 and q_0 , and let $z = e^{ai}$ with a real be a unit complex number. The

complex spherical map $L_z \circ R_z(x) = zxz$ is a complex reflection because it fixes the complex line $q_1\mathbb{C}$ pointwise and rotates the complex line $\mathbb{C} = 1\mathbb{C}$ through an angle of $2a$ radians. In the notation of Definition 1.3 this map is r_{1,z^2} . To create an arbitrary complex spherical reflection r_{q,z^2} with q a unit quaternion, it suffices to conjugate r_{1,z^2} by L_q since the composition $L_q \circ r_{1,z^2} \circ L_q^{-1}$ defined by the equation $x \mapsto (qzq^{-1})xz$ rotates the complex line $q\mathbb{C}$ through an angle of $2a$ and fixes the unique complex line orthogonal to $q\mathbb{C}$ pointwise.

This explicit description makes complex reflections easy to detect.

Proposition 5.6 (Complex reflections). *Let \mathbb{H}_{q_0} be the quaternions with a complex structure. A complex spherical map $f(x) = qxz$ with q a unit quaternion and z a unit complex number is a complex reflection if and only if $\text{REAL}(q) = \text{REAL}(z)$.*

Proof. Both directions are easy quaternionic exercises. In one direction conjugation by a quaternion does not change its real part so the formula given in Definition 5.5 shows that every complex reflection satisfies $\text{REAL}(q) = \text{REAL}(z)$. In the other direction, whenever there are two unit quaternions q and z with the same real part, there is a third unit quaternion p that conjugates z to q , and once qxz is rewritten as $(pzp^{-1})xz$ it is clear that f is the complex reflection r_{p,z^2} . \square

Once translations are included in the discussion, these results about isometries fixing the origin readily extend to arbitrary isometries of the complex euclidean plane.

Definition 5.7 (Translations). For every quaternion q the *translation* map $t_q(x) = x + q$ is an orientation preserving isometry of the canonical euclidean structure of \mathbb{H} . When a spherical map is combined with translation by an arbitrary quaternion q'' we call the resulting function $f(x) = qxq' + q''$ a *euclidean map*. As was the case with spherical maps, every euclidean map is an orientation preserving euclidean isometry and every orientation preserving euclidean isometry can be represented as a euclidean map in precisely two ways (with the second representation obtained by negating q and q').

Once we allow translations, the quaternions \mathbb{H}_{q_0} with a complex structure can be identified as the complex euclidean plane. The images of the complex lines $q\mathbb{C}$ under translation are called *affine complex lines* and they are sets of the form $q\mathbb{C} + v$. Every translation preserves this complex euclidean structure and Propositions 5.3 and 5.6 extend.

Proposition 5.8 (Complex euclidean isometries). *The euclidean maps that preserve the complex euclidean structure of \mathbb{H}_{q_0} are precisely those*

of the form $x \mapsto qxz + v$ where q is a unit quaternion and z is a unit complex number in the chosen complex subalgebra and v is arbitrary.

Proposition 5.9 (Complex euclidean reflections). *Let \mathbb{H}_{q_0} denote the quaternions with a complex structure. A complex euclidean map $f(x) = qxz + v$ with q a unit quaternion, z a unit complex number and v arbitrary is a complex euclidean reflection if and only if f has a fixed point and $\text{REAL}(q) = \text{REAL}(z)$.*

6. THE 24-CELL

This section describes the regular polytope called the 24-cell and introduces a novel technique for visualizing its structure.

Definition 6.1 (The 24-cell). The convex hull of the unit quaternions

$$\Phi = \{\pm 1, \pm i, \pm j, \pm k\} \cup \left\{ \frac{\pm 1 \pm i \pm j \pm k}{2} \right\}$$

is a 4-dimensional regular polytope known as the 24-cell because it has 24 regular octahedral facets. The centers of these 24 euclidean octahedra are at the points $\frac{i-j}{2} \cdot \Phi$, where $q \cdot \Phi$ denotes a scaled and rotated version of Φ obtained by left multiplying every element of Φ by a quaternion q . In particular, $\frac{i-j}{2} \cdot \Phi$ consists of the 24 quaternions of the form $\frac{\pm u \pm v}{2}$ for $u, v \in \{1, i, j, k\}$ with $u \neq v$. We use Φ for this set because it is the conventional letter used for root systems and the type D_4 root system is the set $\Phi_{D_4} = (i - j) \cdot \Phi$. We also note that the quaternions in Φ form a subgroup of \mathbb{H} .

We use elements in Φ to define a complex structure on \mathbb{H} .

Definition 6.2 (ω and ζ). Let $\omega = \frac{-1+i+j+k}{2}$ and let $\zeta = \frac{1+i+j+k}{2}$, and note that ω is a cube-root of unity, ζ is a sixth-root of unity and $\zeta^2 = \omega$. For the remainder of the article we give the quaternions the complex structure $\mathbb{H}_\omega = \mathbb{H}_\zeta$. Since Φ is a group of order 24 and ζ is an element in Φ of order 6, we can partition Φ into the four cosets $q\langle\zeta\rangle$ with $q \in \{1, i, j, k\}$. Thus every element in Φ is of the form $q\zeta^\ell$ with $q \in \{1, i, j, k\}$ and ℓ an integer mod 6 and Φ is contained in the union of the four complex lines $1\mathbb{C}$, $i\mathbb{C}$, $j\mathbb{C}$ and $k\mathbb{C}$.

In 2007 John Meier and the second author developed a technique for visualizing the regular 4-dimensional polytopes as a union of spherical lenses that has been very useful for understanding the various groups that act on these polytopes. To our knowledge this is the first time that this technique has appeared in print.

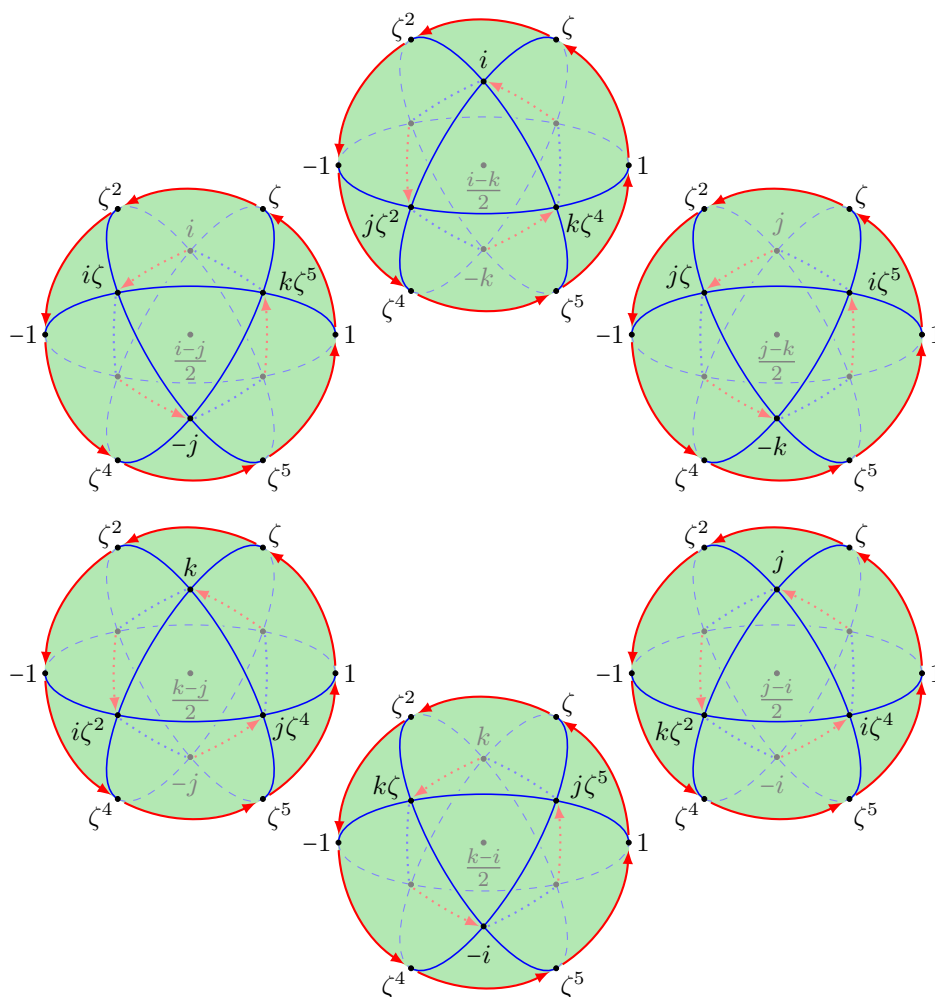


FIGURE 2. Six lenses that together display the structure of the 24-cell. Each figure represents a one-sixth lens in the 3-sphere with dihedral angle $\frac{\pi}{3}$ between its front and back hemispheres. They are arranged so that every front hemisphere is identified with the back hemisphere of the next one when ordered in a counter-clockwise way.

Definition 6.3 (Lunes and Lenses). A *lune* is a portion of a 2-sphere bounded by two semicircular arcs with a common 0-sphere boundary and its shape is completely determined by the angle at which these semicircles meet. A *lens* is a 3-dimensional analog of a lune. Concretely, a *lens* is a portion of the 3-sphere determined by two hemispheres sharing a common great circle boundary and the shape of a lens is completely

determined by the dihedral angle between these hemispheres along the great circle where they meet.

In the same way that lunes can be used to display the map of a 2-sphere such as the earth in \mathbb{R}^2 with very little distortion, lenses can be used to display a map of the 3-sphere in \mathbb{R}^3 with very little distortion.

Definition 6.4 (6 lenses). To visualize the structure of the 24-cell it is useful to use the 6 lenses displayed in Figure 2. Each of the six figures represents one-sixth of the 3-sphere. The outside circle is a great circle in \mathbb{S}^3 , the solid lines live in the hemisphere that bounds the front of the lens, the dashed lines live in the hemisphere that bounds the back of the lens and the dotted lines live in the interior of the lens. The dihedral angle between the front and back hemispheres, along the outside boundary circle is $\frac{\pi}{3}$ and all the edges are length $\frac{\pi}{3}$. The six lenses are arranged so that the front hemisphere of each lens is identified with the back hemisphere the next one in counter-clockwise order. Each lens contains one complete octahedral face at its center and six half octahedra, three bottom halves corresponding to the squares in the front hemisphere and three top halves corresponding to the squares in the back hemisphere. The label at the center of each lens is the coordinate of the center of the euclidean octahedron spanned by the six nearby vertices. The arrows in Figure 2 indicate how the 24 vertices move under the map R_ζ which right multiplies by ζ . The arrows glue together form four oriented hexagons with vertices $q\langle\zeta\rangle$ that live in the four complex lines $q\mathbb{C}$ where q is 1, i , j or k .

7. THE GROUP $\text{REFL}(G_4)$

In this section the complex spherical reflection group $\text{REFL}(G_4)$ is defined and its natural action on the 24-cell is investigated.

Definition 7.1 (The group $\text{REFL}(G_4)$). The group $\text{REFL}(G_4)$ is defined to be the complex spherical reflection group generated by the order 3 reflections $r_{1,\omega}(x) = \zeta x \zeta$ and $r_{i,\omega}(x) = \zeta^i x \zeta$, where $\zeta^i = (-i)\zeta i = i\zeta(-i) = \zeta^{-i}$ is the conjugation of ζ by $\pm i$. For simplicity we abbreviate these as $r_1 = r_{1,\omega}$ and $r_i = r_{i,\omega}$. The resulting group also includes the order 3 reflections $r_j = r_{j,\omega}(x) = \zeta^j x \zeta$ and $r_k = r_{k,\omega}(x) = \zeta^k x \zeta$ as well as the reflections $r_q^2 = r_{q,\omega^2}$ for $q \in \{1, i, j, k\}$. It turns out that this group includes the map which (left or right) multiplies by -1 , so it also includes the negatives of these eight reflections, which are no longer reflections. Finally, $\text{REFL}(G_4)$ contains elements which left multiply by $\pm q$ with $q \in \{1, i, j, k\}$. Thus the full list of all 24 elements in $\text{REFL}(G_4)$ is $\{\pm L_q\} \cup \{\pm r_q\} \cup \{\pm r_q^2\}$ with $q \in \{1, i, j, k\}$.

Remark 7.2 (Binary tetrahedral group). The group formed by the elements in Φ is called the *binary tetrahedral group* and it can be identified with the group of left multiplications L_q with $q \in \Phi$ acting freely on Φ , preserving the complex structure \mathbb{H}_ω . Its name derives from the fact that it is the inverse image of the rotation group of the regular tetrahedron under the Hopf fibration. Note that although the binary tetrahedral group and the complex spherical reflection group $\text{REFL}(G_4)$ both have size 24 and both act freely on the set Φ , their actions are distinct since every element of the former has a fixed-point free action on all of \mathbb{S}^3 while the reflections in the latter pointwise fix complex lines. Both groups can be viewed as index 3 subgroups of the group of size 72 that stabilizes Φ setwise and preserves the complex structure, or as subgroups of the full isometry group of the 24-cell of size 1152, also known as the Coxeter group of type F_4 .

We use the lens diagram in Figure 2 to understand the points in the 24-cell that are fixed by some reflection in $\text{REFL}(G_4)$.

Remark 7.3 (Fixed points). Consider the reflection r_1 in the group $\text{REFL}(G_4)$. It rotates the complex line $1\mathbb{C}$ through an angle of $\frac{2\pi}{3}$ and pointwise fixes the orthogonal complex line, which in this case is the line $(i-j)\mathbb{C}$. In Figure 2, each of the six lenses is stabilized and rotated. In the top lens, for example, 1 goes to ζ^2 , which goes to ζ^4 , which goes to 1 and i goes to $j\zeta^2$, which goes to $k\zeta^4$, which goes to i . The fixed portion of each lens is the line segment connecting the center of the back hemisphere to the center of the front hemisphere through the center of the octahedron. The six fixed arcs in the six lenses glue together to form a single fixed circle or a single fixed hexagon, depending on whether this figure is viewed as representing the 3-sphere through the points Φ or as a slight distortion of portions of the piecewise euclidean boundary of the 24-cell with vertices Φ , respectively. The other reflections r_q with $q \in \{i, j, k\}$, being conjugates of r_1 , are geometrically similar but their action is slightly harder to see. Basically, r_q rotates the complex line $q\mathbb{C}$ and it cyclically permutes the other three complex lines. Every octahedron contains parts of three complex lines in its 1-skeleton and the six octahedra that contain parts of the three other lines form a solid ring or *necklace*, overlapping on triangles, which contains the circle/hexagon orthogonal to the line $q\mathbb{C}$ in its interior as in Figure 3. Concretely, the fixed hyperplanes for the reflections r_1, r_i, r_j and r_k are $(i-j)\mathbb{C}$, $(1+k)\mathbb{C}$, $(1-k)\mathbb{C}$ and $(i+j)\mathbb{C}$, respectively.

The portion of the 24-cell that avoids the fixed hyperplanes of the reflections in $\text{REFL}(G_4)$ is of particular interest.

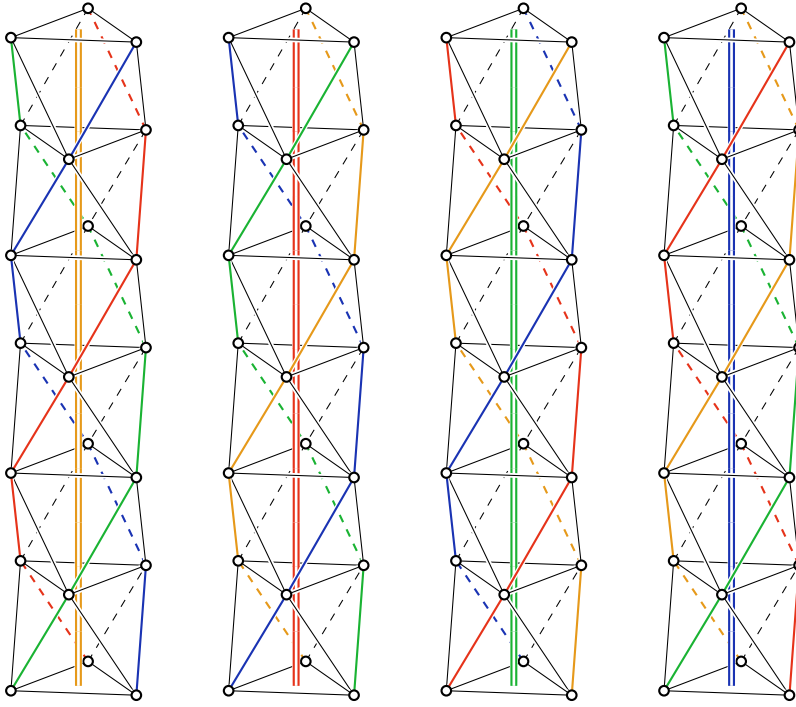


FIGURE 3. The 4 octahedral necklaces centered around the fixed orthogonal circles/hexagons are created by identifying the top and bottom triangle in each pillar. The triangles in the boundaries of the necklaces can be pairwise identified to form the boundary of the 24-cell homomorphic to a 3-sphere.

Definition 7.4 (The complement complex K_0). Let P be the 24-cell whose vertices are the quaternions in Φ and let K_0 be the cell complex formed by the union of the faces of P that do not intersect the fixed hyperplanes of the reflections in $\text{REFL}(G_4)$. The interior, all 24 octahedral facets, and some of the equilateral triangles are removed while the entire 1-skeleton and some of the triangles remain. From the description of the fixed hyperplanes given in Remark 7.3 we see that a triangular face of P is excluded precisely when all three of its vertices belong to distinct complex lines and it is included when two of the vertices belong to the same complex line. In Figure 2 the included triangles can be characterized as those which contain an arrow (representing right multiplication by ζ) as one of its edges.

The complex K_0 has a number of nice properties including being non-positively curved.

Remark 7.5 (Non-positive curvature). We have chosen not to include a detailed review of the notions of CAT(0) and non-positive curvature because we only need an easily described special case of the theory. In any piecewise euclidean 2-complex, the link of a vertex is the metric graph of points distance ϵ from the vertex for some small ϵ that is then rescaled so that the length of each arc is equal to the radian measure of the angle at the corner of the polygon to which it corresponds. Such a metric graph is said to be CAT(1) when it does not contain any simple loop of length strictly less than 2π and a piecewise euclidean 2-complex is called *non-positively curved* when every vertex link is CAT(1). The universal cover of a non-positively curved 2-complex is contractible and it satisfies the definition of being a complete CAT(0) space. Finally, a group that acts geometrically on a complete CAT(0) space is called a CAT(0) *group*.

Theorem 7.6 (The complement complex K_0). *The hyperplane complement of $\text{REFL}(G_4)$ deformation retracts onto a non-positively curved piecewise euclidean 2-complex K_0 contained in the boundary of the 24-cell in which every 2-cell is an equilateral triangle and every vertex link is a subdivided theta graph.*

Proof. The deformation retraction from the hyperplane complement to K_0 comes from our description of how the fixed hyperplanes of the reflections in $\text{REFL}(G_4)$ intersect the 24-cell. More explicitly, since the origin belongs to all 4 fixed hyperplanes, we can radially deformation retract the hyperplane complement onto the boundary of the 24-cell, away from the origin (and from ∞) and the missing hyperplanes correspond to four missing hexagons running through the centers of the four solid rings formed out of six octahedra each. See Figure 3. The second step radially deformation retracts from these missing hexagons onto the 2-complex K_0 . The punctured triangles retract onto their boundary and the pierced octahedra retract on the annulus formed by the six triangles which contain an arrow as an edge. Finally, each vertex of K_0 is part of 9 triangles and its link is a theta-graph consisting of three arcs of length π sharing both endpoints, subdivided into subarcs of length $\frac{\pi}{3}$. Since the vertex links contain no simple loops of length less than 2π , the complex itself is non-positively curved. \square

As a corollary of Theorem 7.6 we get a detailed description of the corresponding braid group $\text{BRAID}(G_4)$.

Corollary 7.7 (The group $\text{BRAID}(G_4)$). *The group $\text{BRAID}(G_4)$ is a CAT(0) group isomorphic to the three-strand braid group and it is defined by the presentation $\langle a, b, c, d \mid abd, bcd, cad \rangle$.*

Proof. By Steinberg's Theorem the hyperplane complement is the same as the space of regular points in this case and by Theorem 7.6 the quotient of K_0 by the action of $\text{REFL}(G_4)$ is homotopy equivalent to the space of regular orbits for $\text{REFL}(G_4)$. In particular, the fundamental group of the quotient is isomorphic to $\text{BRAID}(G_4)$. The quotient of the 2-complex K_0 by the free action of $\text{REFL}(G_4)$ yields a one vertex complex with four edges and three equilateral triangles. The presentation is read off from this quotient with the three relations corresponding to the three triangles and, once d is solved for and eliminated, the relations reduce to $ab = bc = ca$ which is the dual presentation for the three-strand braid group. Finally, since K_0 is non-positively curved, so is its quotient and its universal cover is $\text{CAT}(0)$. The free and cocompact action of $\text{BRAID}(G_4)$ on \tilde{K}_0 shows that it is a $\text{CAT}(0)$ group. \square

The fact that the braid group of $\text{REFL}(G_4)$ is isomorphic to the 3-strand braid group is well-known [Ban76, BMR95, BMR98]. The novelty of our presentation is that we use an explicit piecewise euclidean 2-complex in the 2-skeleton in the boundary of the 24-cell to establish this connection.

8. THE GROUP $\text{REFL}(\tilde{G}_4)$

This section defines the group $\text{REFL}(\tilde{G}_4)$ and establishes key facts about its translations, its reflections and their fixed hyperplanes and intersections, as well as the structure of its Voronoi cells.

Definition 8.1 (The group $\text{REFL}(\tilde{G}_4)$). Let $\text{REFL}(\tilde{G}_4)$ denote the group generated by the reflections r_1 , r_i and $r'_1 = t_{1+k} \circ r_1 \circ t_{1+k}^{-1} = t_2 \circ r_1$. The first two generate $\text{REFL}(G_4)$ as before and the third, $r'_1(x) = \zeta x \zeta + 2$, is a complex euclidean reflection whose action on \mathbb{H}_ω is a translated version of r_1 . The first equation shows that r'_1 is a complex euclidean reflection fixing $1+k$ and the equality of the two is an easy computation.

One can also write $r'_1 = t_{1+i} \circ r_1 \circ t_{1+i}^{-1}$. Our choice of t_{1+k} as the conjugating translation is motivated by the following computation.

Example 8.2 (An isolated fixed point). The sets $\text{FIX}(r_1) = (i-j)\mathbb{C}$, $\text{FIX}(r_i) = (1+k)\mathbb{C}$ and $\text{FIX}(r'_1) = (1+k) + (i-j)\mathbb{C}$ can be described as

$$\text{FIX}(r_1) = \{a + bi + cj + dk \mid a = 0, b + c + d = 0\},$$

$$\text{FIX}(r_i) = \{a + bi + cj + dk \mid b = 0, a + c - d = 0\},$$

and

$$\text{FIX}(r'_1) = \{a + bi + cj + dk \mid a = 1, b + c + d = 1\}.$$

Solving these equations, one finds that $1 + k \in \Phi_{D_4}$ is the unique point in the intersection $\text{FIX}(r'_1) \cap \text{FIX}(r_i)$. Thus r'_1 and r_i generate a copy of $\text{REFL}(G_4)$ that uses $1 + k$ as its origin.

The complex spherical reflection group $\text{REFL}(G_4)$ acts on the root system Φ and the complex euclidean reflection group $\text{REFL}(\tilde{G}_4)$ acts on the Hurwitzian integers they generate.

Definition 8.3 (Hurwitzian integers). The \mathbb{Z} -span of Φ inside \mathbb{H} is the set Λ of *Hurwitzian integers*. It consists of all quaternions of the form $\frac{a+bi+cj+dk}{2}$ where a, b, c and d are all even integers or all odd integers. Our notation is derived from the theory of Coxeter groups. The \mathbb{Z} -span of a root system Φ is its *root lattice* Λ and, as with Φ , we write $q \cdot \Lambda$ for the \mathbb{Z} -span of $q \cdot \Phi$ and $\Lambda_{D_4} = \{(a, b, c, d) \in \mathbb{Z}^4 \mid a + b + c + d \in 2\mathbb{Z}\}$ for the \mathbb{Z} -span of Φ_{D_4} . We note that $2 \cdot \Lambda \subset \Lambda_{D_4} \subset \Lambda$ and that each is an index 4 subset of the next.

The Hurwitzian integers Λ has many nice properties including that they form a subring of the quaternions with Φ as its group of units, every element has an integral norm, and it satisfies a noncommutative version of the euclidean algorithm. See [CS03, Chapter 5] for details. The remainder of the section records basic facts about the action of $\text{REFL}(\tilde{G}_4)$ on \mathbb{H}_w .

Fact 8.4 (Translations). *The translations in the group $\text{REFL}(\tilde{G}_4)$ are those of the form t_q with $q \in 2 \cdot \Lambda$.*

Proof. The element $t_2 = r'_1 \circ r_1^{-1}$ is a translation in $\text{REFL}(\tilde{G}_4)$ and conjugating t_2 by elements of $\text{REFL}(G_4)$ shows that all the translations t_q for all $q \in 2 \cdot \Phi$ are also in $\text{REFL}(\tilde{G}_4)$. Thus $\text{REFL}(\tilde{G}_4)$ contains the abelian subgroup T that they generate and this consists of all translations of the form $2 \cdot \Lambda$. The subgroup T is normal since it is stabilized by the generating set and, because the quotient by T is $\text{REFL}(G_4)$, the elements in T are the only translations in $\text{REFL}(\tilde{G}_4)$. \square

Fact 8.5 (Isolated fixed points). *There is a copy of $\text{REFL}(G_4)$ inside $\text{REFL}(\tilde{G}_4)$ fixing a point v for each v in the lattice Λ_{D_4} . In particular, every point in Λ_{D_4} is an intersection of fixed hyperplanes of complex reflections in $\text{REFL}(\tilde{G}_4)$.*

Proof. By Example 8.2 this holds for $v = 1 + k$ and if we conjugate the copy of $\text{REFL}(G_4)$ fixing $1 + k$ by an element of the copy fixing the origin we find copies fixing v for all $v \in \Phi_{D_4}$. Next, conjugating the copy at the origin by elements in the copies fixing the points in Φ_{D_4} shows that there is a copy fixing every point that is a sum of two elements in Φ_{D_4} .

Continuing in this way shows that there is a copy fixing any point that is a finite sum of elements in Φ_{D_4} , a set equal to Λ_{D_4} . \square

Fact 8.6 (Reflections). *For every element $v \in \Lambda_{D_4}$ and for every $q \in \{1, i, j, k\}$, the primitive complex reflection $t_v \circ r_q \circ t_v^{-1}$ is in $\text{REFL}(\tilde{G}_4)$. In fact, these are the only primitive complex reflections in $\text{REFL}(\tilde{G}_4)$.*

Proof. The first assertion is an immediate consequence of Fact 8.5. When $r' = t_v \circ r_q \circ t_v^{-1}$ for some v and for $q \in \{1, i, j, k\}$, we say that r' is *parallel* to r_q . For the second assertion we note that every primitive complex reflection r' in $\text{REFL}(\tilde{G}_4)$ must be parallel to one of the primitive reflections r_q with $q \in \{1, i, j, k\}$ in $\text{REFL}(G_4)$. It is then straight-forward to show that if there were an r' in $\text{REFL}(\tilde{G}_4)$ parallel to r_q other than the ones listed, then $r' \circ r_q^{-1}$ would be a translation in $\text{REFL}(\tilde{G}_4)$ that violates Fact 8.4. \square

Fact 8.7 (Fixed hyperplanes). *The fixed hyperplanes of the complex reflections in $\text{REFL}(\tilde{G}_4)$ of the form $t_v \circ r_q \circ t_v^{-1}$ with $v \in \Lambda_{D_4}$ and $q \in \{1, i, j, k\}$ can be described as follows*

$$\begin{aligned} \text{FIX}(t_v \circ r_1 \circ t_v^{-1}) &= \{a + bi + cj + dk \mid a = \ell, b + c + d = m\} \\ \text{FIX}(t_v \circ r_i \circ t_v^{-1}) &= \{a + bi + cj + dk \mid b = \ell, a + c - d = m\} \\ \text{FIX}(t_v \circ r_j \circ t_v^{-1}) &= \{a + bi + cj + dk \mid c = \ell, a + d - b = m\} \\ \text{FIX}(t_v \circ r_k \circ t_v^{-1}) &= \{a + bi + cj + dk \mid d = \ell, b + c - a = m\} \end{aligned}$$

where ℓ and m are the unique integers so that v satisfies the equations.

Proof. Direct computation. \square

Once the reflections and their fixed hyperplanes have been computed, it is easy to show that the isolated fixed points listed in Fact 8.5 are the only points that arise as intersections of fixed hyperplanes. This set then determines the structure of the Voronoi cells.

Fact 8.8 (Voronoi cells). *In the Voronoi cell structure around the set of isolated hyperplane intersections for the group $\text{REFL}(\tilde{G}_4)$ the Voronoi cell around the origin is the standard 24-cell with vertices Φ , the other Voronoi cells are translates of the 24-cell by vectors in Λ_{D_4} and the link of each vertex in the Voronoi cell structure is a 4-dimensional cube.*

Proof. The Voronoi cells for the D_4 root lattice is a standard computation. See [CS99, Section 7.2] for details. \square

Fact 8.9 (Vertices). *The vertices of the Voronoi cell structure are located at the points in $\Lambda \setminus \Lambda_{D_4}$ and the group $\text{REFL}(\tilde{G}_4)$ acts transitively on this set.*

Proof. Since the translates of the 24-cell are centered at the elements of Λ_{D_4} , every vertex of the Voronoi cell structure can be described as $u + v$ with $u \in \Phi$ and $v \in \Lambda_{D_4}$. After noting that Λ contains Λ_{D_4} as a sublattice, it is easy to check that every element of Λ that is not in Λ_{D_4} differs from an element of Λ_{D_4} by an element in Φ . To see transitivity, note that the 1-skeleton of the Voronoi cell structure is connected, each edge is in the boundary of one of the 24-cells, and the local copy of $\text{REFL}(G_4)$ fixing each 24-cell acts transitively on its vertices. \square

The following key fact is another easy computation.

Fact 8.10 (Intersections). *If the fixed hyperplane H of a complex reflection in the group $\text{REFL}(\tilde{G}_4)$ non-trivially intersects is one of the closed Voronoi cells, then H contains the point at the center of that Voronoi cell.*

9. PROOFS OF MAIN THEOREMS

In this section we prove our three main results. We begin by defining the complement complex K .

Definition 9.1 (Complement complex K). The *complement complex* K is the portion of the Voronoi cell structure for the group $\text{REFL}(\tilde{G}_4)$ that is disjoint from the union of the fixed hyperplanes of its complex reflections. By Fact 8.10, around each fixed hyperplane intersection point the portion of K in the boundary of this particular 24-cell is a copy of the 2-complex K_0 defined in Definition 7.4. Thus K can be viewed as a union of local copies of K_0 .

The vertex links in K are isomorphic to a well-known graph.

Definition 9.2 (Möbius-Kantor graphs). The link of a vertex in the complement complex K is the portion of the 1-skeleton of the 4-cube shown in Figure 4. This is a 16 vertex 3-regular graph known as the *Möbius-Kantor graph*. The 8 removed edges correspond to the equilateral triangles whose center lies in one of the fixed hyperplanes. The portion of this graph that lives in one of the eight 3-cubes in the 4-cube is the subdivided theta graph that is the link of this vertex inside the corresponding copy of K_0 inside a particular 24-cell.

At this point, the proof of our first main theorem is straight-forward.

Theorem A (Complement complex). *The hyperplane complement of $\text{REFL}(\tilde{G}_4)$ deformation retracts onto a non-positively curved piecewise euclidean 2-complex K in which every 2-cell is an equilateral triangle and every vertex link is a Möbius-Kantor graph.*

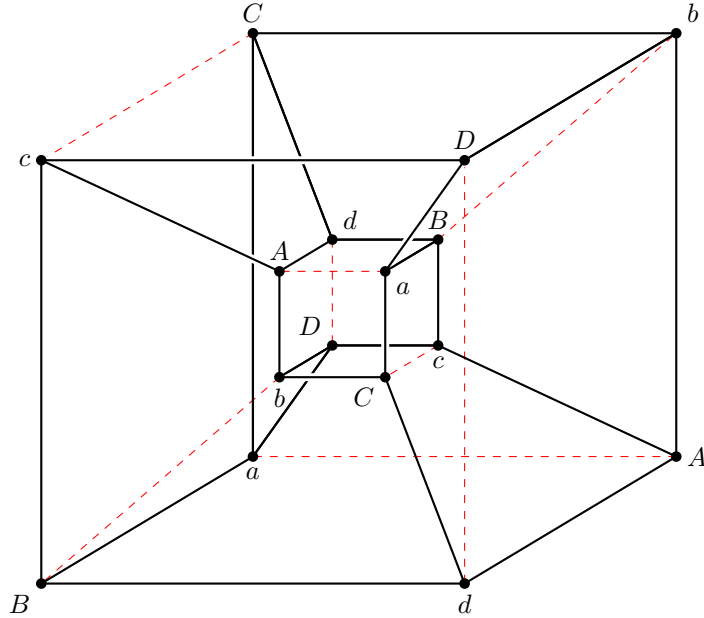


FIGURE 4. The Möbius-Kantor graph as a subgraph of the 1-skeleton of a 4-cube with 8 edges removed.

Proof. The proof is essentially the same as that of Theorem 7.6 but with the local deformations combined into a global deformation. The first step is to radially deformation retract from the removed isolated fixed point at the center of each Voronoi cell to its boundary, which can be carried out because of Fact 8.10. Next, the secondary deformations applied to the punctured equilateral triangles and the skewered octahedra are compatible regardless of which Voronoi cell one views them as belonging to. Finally, every edge in every vertex link has length $\frac{\pi}{3}$ and since Möbius-Kantor graphs have no simple cycles of combinatorial length less than 6, there are no simple loops of length less than 2π , the vertex links are CAT(1) and K is non-positively curved. \square

Remark 9.3 (Other examples). We should note that when we have attempted to extend our main theorems to other complex euclidean reflections groups acting on \mathbb{C}^2 , it is the analog of Fact 8.10 where those attempts have failed. It is apparently quite common for a fixed hyperplane to intersect the boundary of a Voronoi cell without passing through its center. Unless this intersection happens to be contained in a different fixed hyperplane that does pass through the center of the Voronoi cell, this missing boundary prevents the initial deformation retraction onto a portion of the 3-skeleton of the Voronoi cell structure.

We now prove a stronger result that immediately implies Theorem B.

Theorem 9.4 (Isolated fixed points). *The points in \mathbb{H}_ω stabilized by a non-trivial element of the group $\text{REFL}(\tilde{G}_4)$ are those in the union of the fixed hyperplanes of its complex reflections together with all of the vertices of the complement complex K .*

Proof. Let T be the set of translations in $\text{REFL}(\tilde{G}_4)$, let T_0 the images of the origin under the translations in T and let FP_A be the set of points fixed by some element in $\text{REFL}(\tilde{G}_4)$ whose linear part is the antipodal map. As in Remark 3.6, the simplification $-(x - v) + v = -x + 2v$ shows that $2 \cdot FP_A = T_0$. Since $T_0 = 2 \cdot \Lambda$, $FP_A = \Lambda$ and by Fact 8.9 there is an element of order 2 fixing each vertex of the complement complex K . Since the remaining points in Λ are contained in the fixed hyperplanes (Fact 8.5), all points fixed by an element whose linear part is the antipodal map have been accounted for. To see that the vertices of K are the only isolated points with non-trivial stabilizers, suppose that x is a point with a non-trivial stabilizer s . If x does not lie in fixed hyperplane, the linear part of s must be something other than a complex reflection. The possibilities for its linear part are the antipodal map L_{-1} , $\pm L_q$ with $q \in \{i, j, k\}$ or $-r_q$ or $-r_q^2$ with $q \in \{1, i, j, k\}$ but all of these have a power equal to the antipodal map L_{-1} : the second power of $\pm L_q$ is the antipodal map and the third power of $-r_q$ and of $-r_q^2$ is the antipodal map. In particular, x must be stabilized by an element whose linear part is the antipodal map and thus it is one of the ones already identified. \square

Since the braid group of a group action is defined as the fundamental group of the space of regular orbits, and the vertices of K are not regular points, the complement complex K needs to be modified before it can be used to investigate the group $\text{BRAID}(\tilde{G}_4)$.

Definition 9.5 (Modified complement complex K'). Let K_1 be the union of the complement complex K and the set of small closed balls of radius $\epsilon > 0$ centered at each of the vertices of K . Next, let K_2 be the metric space obtained by removing from K_1 the points corresponding to the vertices of K . Finally, let K' be the space obtained by removing from K_1 the open balls of radius ϵ centered at each of the vertices of K . We call K' the *modified complement complex*.

In the same way that K is homotopy equivalent to the hyperplane complement, K' is homotopy equivalent to the space of regular points.

Proposition 9.6 (Homotopy equivalences). *The spaces K , K_1 and the hyperplane complement are homotopy equivalent as are the spaces K' , K_2 and the space of regular points.*

Proof. It should be clear that K and K_1 are homotopy equivalent as are K_2 and K' . Moreover, the deformation retractions used to show that the hyperplane complement deformation retracts to K can be modified to show that it deformation retracts to K_1 instead by simply stopping the retraction whenever a point is distance ϵ from a vertex. This modified deformation retraction can then be combined with the radial deformation retraction from K_2 to K' to show that the space of regular points (which removes the fixed hyperplanes and the vertices of K) is homotopy equivalent to the modified complex K' . \square

The action of $\text{REFL}(\tilde{G}_4)$ on K' is now free and the fundamental group of the quotient is, by definition, the group $\text{BRAID}(\tilde{G}_4)$.

Definition 9.7 (Quotient complex). Let $G = \text{REFL}(\tilde{G}_4)$. Although the action of G on K is not free, we can still investigate the properties of the orbifold quotient. Since the action is proper and cellular with trivial stabilizers for every cell of positive dimension, the quotient remains a 2-complex. In this case, the quotient K/G has one vertex, four edges and four triangles and it corresponds to the presentation 2-complex of the presentation $\langle a, b, c, d \mid abd, bcd, cad, cba \rangle$. The group defined by this presentation is the binary tetrahedral group and the universal cover of the orbifold quotient is the 2-skeleton of the 24-cell. Note that selecting any 3 of the 4 relations produces an infinite group isomorphic to the 3-strand braid group. The *modified quotient complex* K'/G is the quotient of K' by the free action of $\text{REFL}(\tilde{G}_4)$. To see its structure consider K_1/G and K_2/G . The former is a modification of K/G where the neighborhood of the unique vertex becomes a cone on an $\mathbb{R}P^2$ with the vertex as its cone point, and the latter is this space with the cone point removed. Thus K'/G is a copy of K/G with a neighborhood of the vertex removed and a real projective plane attached in its place.

The universal cover of the quotient K'/G is the same as the universal cover of K' and because 2-spheres are simply connected, the universal cover of K' is essentially a modified version of the universal cover of K , where the modifications around each vertex are locally identical to the ones described in Definition 9.5. As a consequence we have the following result.

Theorem 9.8 (Universal cover). *The group $\text{BRAID}(\tilde{G}_4)$ acts geometrically on \tilde{K} , the $\text{CAT}(0)$ universal cover of the complement complex K and the vertex stabilizers have size 2.*

Proof. There is a natural free and isometric action of $\text{BRAID}(\tilde{G}_4)$ on \tilde{K}' , the universal cover of K' by deck transformations, which leads to a proper isometric action of $\text{BRAID}(\tilde{G}_4)$ on \tilde{K} , the $\text{CAT}(0)$ universal cover of K . The only non-trivial stabilizers are, of course, order 2 and they only occur at the vertices of \tilde{K} . Finally, the action is cocompact because the quotient of \tilde{K} by the action of $\text{BRAID}(\tilde{G}_4)$ is equal to the quotient of K by $\text{REFL}(\tilde{G}_4)$ which is a compact 2-complex with one vertex, four edges and four triangles. \square

And this proves our third main result.

Theorem C (Braid group). *The group $\text{BRAID}(\tilde{G}_4)$ is a $\text{CAT}(0)$ group and it contains elements of order 2.*

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