THREE-GENERATOR ARTIN GROUPS OF LARGE TYPE ARE BIAUTOMATIC

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ABSTRACT. In this article we construct a piecewise Euclidean, non-positively curved 2-complex for the 3-generator Artin groups of large type. As a consequence we show that these groups are biautomatic. A slight modification of the proof shows that many other Artin groups are also biautomatic. The general question (whether all Artin groups are biautomatic) remains open.

1. INTRODUCTION

In this article we construct a piecewise Euclidean, non-positively curved 2-complex for the 3-generator Artin groups of large type. Although standard presentations for the Artin groups typically use words which alternate between pairs of the generators, there are other presentations with 'nicer' properties. We shall be employing one of these.

If we let $(a, b)_k$ denote the word of the form $ababa \ldots$ with exactly k letters, then the standard presentation of an Artin group with three generators is given as

$$(1) \quad G_{m,n,p} = \langle\, a, b, c \mid (a,b)_m = (b,a)_m, \;\; (b,c)_n = (c,b)_n, \;\; (c,a)_p = (a,c)_p \,\rangle$$

where m, n, and p are positive integers greater than 1. After introducing an alternative presentation, we will show that the new presentation satisfies the small cancellation conditions $C(3) - T(6)$. As a consequence, a metric of non-positive curvature can be defined on the standard 2-complex of this presentation and by [3] the group it defines is biautomatic. In the final section, we present a generalization of this result.

2. THE PRESENTATION

The non-positively curved 2-complex alluded to above is constructed by altering the standard presentations for Artin groups. Consider the following families of group presentations.
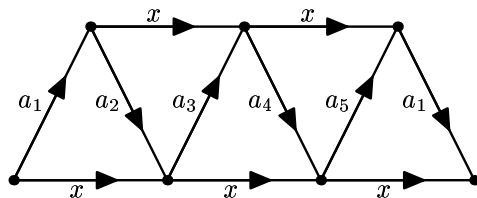
$$(2) \qquad\qquad G_m = \langle\, a_1, a_2 \mid (a_1, a_2)_m = (a_2, a_1)_m \,\rangle$$

$$(3) \qquad\qquad H_{2k} = \langle\, x, a_1 \mid x^k a_1 = a_1 x^k \,\rangle$$

$$(4) \qquad\qquad H_{2k+1} = \langle\, x, a_1 \mid x^{k+1} = a_1 x^k a_1 \,\rangle$$

$$(5) \qquad I_m = \langle\, x, a_1, a_2, \ldots a_m \mid x = a_1 a_2, x = a_2 a_3, \ldots, x = a_m a_1 \,\rangle$$

Note that G_m is the standard presentation for Artin groups with two generators. We will now show that the other presentations define the same group.

FIGURE 1. Relations for H_5 and I_5

Lemma 1. *For all* $m > 1$, $G_m \cong H_m \cong I_m$.

Proof. Start with G_m. If we add a new generator x to G_m with definition $x = a_1 a_2$, and then use this relation to eliminate a_2 we will end up with the presentation H_m. The precise form of the presentation will depend on whether m is even or odd. Since this process can also be reversed, $G_m \cong H_m$.

The relationship between H_m and I_m is even more straight-forward. The relations of I_m are derived by partitioning the unique relation of H_m, while the relation in H_m is formed by arranging the relations of I_m to eliminate the generators a_2, $a_3, \ldots, a_m$. This correspondence is illustrated in Figure 1. $\qquad\square$

Lemma 2. *The Artin group* $G_{m,n,p}$ *can be presented as follows:*

$$I_{m,n,p} = \left\langle \begin{array}{l} x, a, d_3, \ldots d_m \\ y, b, e_3, \ldots, e_n \\ z, c, f_3, \ldots, f_p \end{array} \middle| \begin{array}{l} x = ab,\ x = bd_3,\ x = d_3 d_4, \ldots, x = d_m a \\ y = bc,\ y = ce_3,\ y = e_3 e_4, \ldots, y = e_n b \\ z = ca,\ z = af_3,\ z = f_3 f_4, \ldots, z = f_p c \end{array} \right\rangle$$
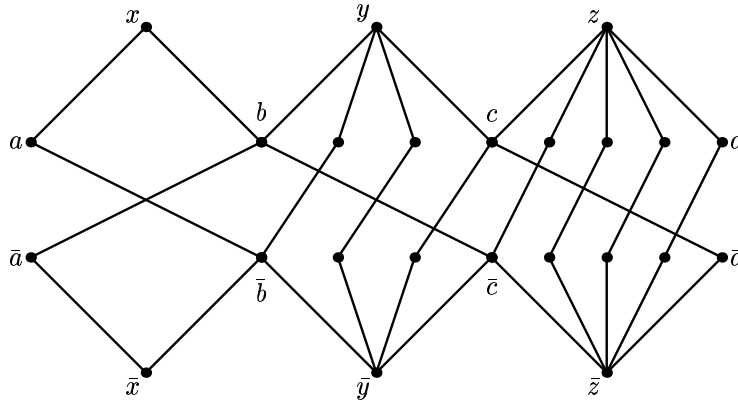
Proof. If the Tietze transformations described in the proof of the previous lemma are applied to each of the sets of relations in this presentation, the result will be the standard presentation (Equation 1). Notice that a and b play the role of d_1 and d_2 in the first set of equations, b and c play the role of e_1 and e_2 in the second set, and c and a play the role of f_1 and f_2 in the third set. $\qquad\square$

3. THE 2-COMPLEX

Let $K_{m,n,p}$ be the standard 2-complex associated with the presentation $I_{m,n,p}$ given in Lemma 2. The complex $K_{m,n,p}$ has a unique 0-cell, $m + n + p$ 1-cells (one for each of the generators), and $m + n + p$ triangular 2-cells (one for each of the relations). Of particular interest is the link the unique 0-cell. Recall that the link of a 0-cell in a 2-complex is the graph obtained by intersecting a small sphere centered at the 0-cell with the surrounding complex. In this case, each 1-cell will contribute two vertices to the link, and each corner of a 2-cell will contribute an edge. The link of $K_{m,n,p}$ will be refered to as $L_{m,n,p}$.

Example. The link $L_{2,4,5}$ is shown in Figure 2, with the exception that the two vertices on the extreme right need to be identified with the two vertices on the extreme left. We have adopted the convention that letters without bars denote the vertex contributed by the head of the oriented 1-cell and letters with bars denote the vertex contributed by the tail.

Remark. Notice that the link is essentially composed of three links of 2-generator Artin groups joined together along pairs of vertices. In the example shown, we see

FIGURE 2. The link of the vertex in $G_{2,4,5}$

the link of I_2, I_4, and I_5 laid side by side. Each of these links is in turn made up of packets of paths of length 3 which are twisted together. The subgraphs corresponding to the 2-generator Artin groups will be called the local pieces of the link. Notice also that these local pieces overlap only at vertices which arise from the standard generators a, b, and c. Another key property of the link is that the vertices separate themselves into 4 distinct levels, with $\bar{x}$, $\bar{y}$, and $\bar{z}$ at the bottom (level 1), and x, y, and z at the top (level 4). The edges connecting levels 2 and 3 will be called middle edges.

In Figure 2 there are two short loops of length 4. Namely the paths connecting $\bar{a} - b - y - c - \bar{a}$ and $\bar{a} - b - \bar{c} - \bar{z} - \bar{a}$. We will now show that if there are no commutation relations, then no short embedded loops can exist.

Lemma 3. *If m, n, and p are all at least 3 then every embedded loop in $L_{m,n,p}$ contains at least 6 edges.*
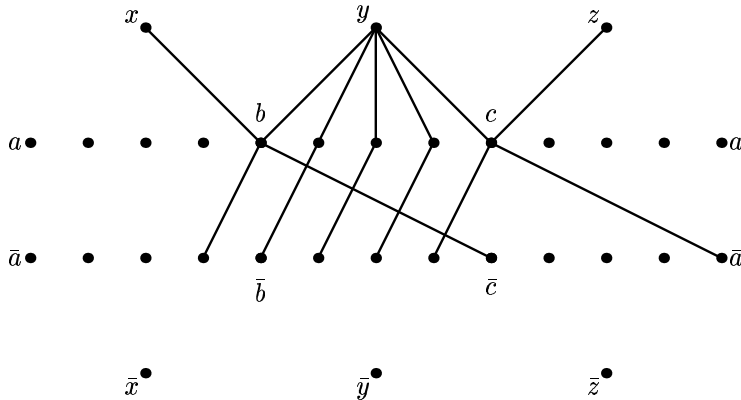
Proof. Since every edge in the link joins a pair of vertices on adjacent levels, we see that the link is bipartite, and thus every embedded loop will have an even length. Furthermore, it should be clear from the example that embedded loops of length 2 cannot occur, so that it only remains to examine possible loops of length 4.

If there was a loop of length 4 which contained a top vertex or a bottom vertex, then the entire loop would be contained in the radius two neighbourhood of this vertex. An examination however shows this neighborhood is a tree. As an example, the radius two neighbourhood of y in $L_{5,5,5}$ is shown in Figure 3. On the otherhand, if there was such a loop which did not pass through a top or bottom vertex then it would be completely composed of middle edges. The subcomplex of these edges however consists of $m + n + p - 9$ disjoint edges and three chains of length 3. This completes the proof. □

In the language of small cancellation theory, this lemma shows that the presentation $I_{m,n,p}$ satisfies the condition $T(6)$.

4. THE MAIN RESULT

The complex $K_{m,n,p}$ can be given a piecewise Euclidean metric by assigning length of 1 to each 1-cell and making each 2-cell an equilateral triangle. This metric

FIGURE 3. The radius 2 neighborhood of y in the link $L_{5,5,5}$

also induces a metric on the link of the 0-cell by assigning the angle at a corner of a 2-cell as the length of the corresponding edge in the link. In this instance, the triangles are all equilateral. Thus all of the edges in the link are assigned a length of $\frac{\pi}{3}$.

Theorem 4. *If m, n, and p are all at least 3, then the metric on $K_{m,n,p}$ is non-positive curved and the group $G_{m,n,p}$ is biautomatic. In particular, all 3-generator Artin groups of large type are biautomatic.*
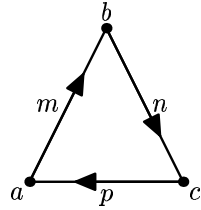
Proof. By Lemma 3, every embedded loop in the link of the 0-cell contains at least six edges, and thus it measures at least 2π. By [1] this proves that the metric on $K_{m,n,p}$ is non-positively curved. That the group is biautomatic follows from a result of Gersten and Short ([3]). In their terminology $K_{m,n,p}$ is an A_2 complex. □

5. GENERALIZATIONS

In order to extend this result Artin groups with more than three generators, additional terminology needs to be introduced.

Definition (G_Γ). An arbitrary Artin group can be defined as follows. Let Γ be a undirected graph with no loops or multiple edges and with a positive integer greater than 1 assigned to each edge. The standard presentation for the Artin group associated to Γ is defined as follows: it has one generator for each of the vertices of Γ and for each edge there is a relation. In particular, if there is an edge connected a vertex a to a vertex b which is labeled by m then we add the relation $(a,b)_m = (b,a)_m$. We will call this presentation G_Γ.

Definition (I_Γ, K_Γ, and L_Γ). In order to convert this presentation into a presentation like that used in Lemma 2, we need to include additional information. There is an ambiguity which stems from the fact that the presentation G_m is symmetric with respect to the generators a_1 and a_2, but the equivalent presentation I_m is not. In particular, notice that in I_m, $a_1 a_2$ occurs as a subword of a relation, but that $a_2 a_1$ does not. Thus to precisely define a presentation of the type we have been considering, we need to require that the defining graph Γ is a directed graph. The direction will indicate which subword of length 2 to be preserved in the rewriting process. The presentation which results will be called I_Γ, the standard 2-complex

FIGURE 4. Defining graph Γ for the presentation $I_{m,n,p}$

for this presentation will be denoted K_Γ, and the link of its unique 0-cell will be denoted L_Γ.

Example. In Figure 4 we have displayed the defining graph Γ which leads to the presentation given in Lemma 2. Although it was not highlighted at the time, the orientations of the edges of Γ were in fact crucial to the argument in the proof of Lemma 3. As we will see in Lemma 5, if one of the orientations had been reversed then embedded loops of length 4 would indeed have been present in the link.

Remark. As in the 3-generator case the link L_Γ is a bipartite graph with 4 distinct levels of vertices, and since it is bipartite, all of its embedded loops will have an even length. The link itself is comprised of various local pieces which look like the link of I_m for some m. Moreover, any two of these local pieces are joined to each other only at a pair of vertices which correspond to a generator of G_Γ that arises from a vertex of Γ. For example, in Figure 2 the local pieces which look like the links of I_4 and I_5 are joined together at the vertices c and $\bar{c}$, and these vertices correspond the generator c in G_Γ that corresponds to the vertex labeled c in Figure 4. The vertices in L_Γ which correspond in this way to the vertices of Γ will be called special vertices and the edges whose endpoints are both special will be called special edges.

Remark. There is another connection between Γ and L_Γ which will be useful in the proof given below. If the vertices of the link are organized so that the vertices fall into four distinct levels, the pairs of vertices are aligned to that the one with the bar is directly below the other, and the local pieces of the link lie in a vertical 2-dimensional surface, then the graph Γ can be thought of as the "top-view" of the link. In this view the pairs of special vertices where the local pieces overlap are seen as the vertices of Γ, the local pieces which are in one-to-one correspondence with the edges of Γ are seen as being projected onto the appropriate edges, and the special edges in the link which allow one to get from a special vertex in level 2 to a special vertex in level 3 in a single step, are seen as the ones which give the orientation to the edge corresponding to its local piece.

Lemma 3 can now be extended to the following, more general, result.

Lemma 5. *The link L_Γ contains an embedded loop of length 4 if and only if the defining graph Γ contains an oriented subgraph isomorphic to one of the graphs in Figure 5. Moreover, these loops of length 4 are the only possible embedded loops of length less than 6.*

Proof. We will begin by showing that all embedded loops must involve at least 3 local pieces. Clearly the local pieces themselves have no embedded loops of length
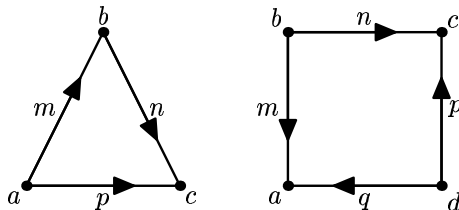
FIGURE 5. Subgraphs which lead to short embedded loops

less than 6, so consider a subgraph of the link consisting of only 2 (connected) local pieces and let a and $\bar{a}$ be the pair of vertices connecting them. Since the shortest path connecting a to $\bar{a}$ has length 3 and the shortest embedded path connecting either vertex to itself has length 6, all of the loops in this subgraph will have length at least 6.

Next, we will establish the forward direction. Assume that L_Γ contains a short embedded loop of length 4. If this loop does not contain a top or a bottom vertex, then it is comprised entirely of middle edges. Since the special vertices are the only vertices in levels 2 or 3 which are connected to more than one middle edge, all four vertices in the loop must be special and as a result all four of the edges are special as well. The vertices which are adjacent in the loop must correspond to distinct vertices in Γ since they connected by a special edge, and the vertices which are not adjacent in the loop must also correspond to distinct vertices in Γ since the loop is embedded. This shows that the subgraph of Γ determined by these four special vertices contains the graph on the righthand of Figure 5 as a subgraph.

If this short loop contained more than one top or bottom vertex then it would be contained in the union of two local pieces since all of the edges connected to a top or bottom vertex belong to the same local piece. We have already shown that the union of two local pieces do not contain short loops, so there are no short loops of this type.

Finally, suppose this short loop contained exactly one top vertex. Since the edges connected to it belong to the same local piece and since we know that at least three local pieces are involved, the other two edges must lie in distinct local pieces. Thus all three of the other vertices are special, and the two edges not connected to the top vertex are special. Since three distinct local pieces are involved it is clear that the three special vertices correspond to distinct vertices in Γ. At this point we know that the subgraph of Γ determined by these these three local pieces is a triangle, and we know the orientations of two of the sides. The orientation of the third side (corresponding to the local piece containing the top vertex) is unknown, but with either orientation, the resulting graph is isomorphic to the graph on the lefthandside of Figure 5. Since the case of exactly one bottom vertex is similar, the proof in the forward direction is complete. The proof in the other direction is essentially immediate. If either of these graphs appears as a subgraph of Γ, then it is easy to exhibit a path of length 4 in the link.   □

The embedded loops of length 4 which result from the graph on the lefthand side of Figure 5 will be called loops of type A while those which result from the one on the righthand side will be called loops of type B.

Theorem 6. *If Γ does not contain any triangles, then K_Γ has a metric of non-positive curvature and the Artin group G_Γ is biautomatic.*

Proof. Since by hypothesis loops of type A cannot occur, we can conclude that all loops of length 4 contain 4 middle edges (since this is true for loops of type B). Next, we claim that all loops of length 6 must contain at least 2 middle edges. The structure of the link guarantees that a loop will contain an even number of middle edges. If it did not contain any middle edges, then it would be contained in either the top or the bottom. Since top and bottom vertices are only connected to edges in the same local piece of the link, the loop itself would be contained in three local pieces. This would implies that a triangle exists in Γ, contradiction.

To complete the proof we assign a metric to K_Γ as follows. The 1-cells corresponding to the top and bottom vertices are assigned a length of $\sqrt{2}$ and all others are assigned a length of 1. The 2-cells are Euclidean triangles with angles of $\frac{\pi}{2}$, $\frac{\pi}{4}$, and $\frac{\pi}{4}$. The conditions shown in the previous paragraph, demonstrate that all embedded loops in the link have a length of at least 2π, and by [1] the metric on K_Γ is non-positively curved. Gersten and Short ([4]) again allow us to conclude that the group G_Γ is biautomatic. In their terminology K is a complex of type B_2. $\square$

This particular result was in some sense already known since Pride showed that the standard presentation of these 'triangle-free' Artin groups satisfies the $C(4) - T(4)$ conditions ([6]), and Gersten-Short showed that $C(4) - T(4)$ groups are biautomatic ([3]). The interest here lies in the alternate method of proof. The following result is new.

Theorem 7. *Let Γ be a graph with every edge labeled by a positive integer greater than 2. If there is a way of orienting the edges in the graph so that neither of the graphs in Figure 5 appear as subgraphs, then K_Γ has a metric of non-positive curvature and the Artin group G_Γ is biautomatic.*

Proof. Since by Lemma 5 there are no embedded loops of length less than 6 in the link L_Γ, the metric which assigns a length of 1 to each 1-cell and gives each 2-cell the metric of a Euclidean equilateral triangle will be non-positively-curved ([1]). And since K_Γ is an A_2 complex, by [3] the corresponding group is biautomatic. $\square$

Example. Let Γ be a graph in which all of the vertices have even degree and assign large integers (each at least 3) to each of its edges. If Γ can be embedded in the plane so that all of its short embedded loops (length 3 or 4) actually bound components of the complement of Γ, then the corresponding Artin group G_Γ will be biautomatic. The proof is an easy application of Theorem 7. In particular, we can use the planarity of the graph to orient the edges. Assign an orientation to each of the regions of the complement so that regions separated by an edge are given opposite orientations. The condition on the degrees of the vertices guarantees that this is possible. Now use these orientations to induce the orientations of the edges. The condition on the embedded loops now guarrantees that neither of the forbidden subgraphs can occur, and by Theorem 7, the group G_Γ is biautomatic.

Remark. Theorem 7 can be extended to include commutation relations under the following restrictions. Since the link of a commutation relation includes special edges in *both* directions, edges labeled by a 2 in Γ must be considered as oriented in both directions. In particular, when searching for a subgraph of type A or B, the edges labeled 2 must be considered as 'wildcards' which can adopt either orientation

as needed. If there is an orientation of the other edges of Γ so that these subgraphs do not appear, then the corresponding complex K_Γ has a metric of non-positive curvature and the group G_Γ is biautomatic.

Using various techniques, researchers have shown that other classes of Artin groups are biautomatic. This is known, for example, for the Artin groups of finite type (R. Charney [2]) and for the Artin groups of extra-large type (D. Peifer [5]). Despite this progress, the following conjecture remains open.

Conjecture 8. *All Artin groups are biautomatic.*

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