

h -VECTORS OF GENERALIZED ASSOCIAHEDRA AND NONCROSSING PARTITIONS

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ABSTRACT. A case-free proof is given that the entries of the h -vector of the cluster complex $\Delta(\Phi)$, associated by S. Fomin and A. Zelevinsky to a finite root system Φ , count elements of the lattice \mathbf{L} of noncrossing partitions of corresponding type by rank. Similar interpretations for the h -vector of the positive part of $\Delta(\Phi)$ are provided. The proof utilizes the appearance of the complex $\Delta(\Phi)$ in the context of the lattice \mathbf{L} , in recent work of two of the authors, as well as an explicit shelling of $\Delta(\Phi)$.

1. INTRODUCTION

Let Φ be a finite root system of rank n with corresponding finite real reflection group W . The cluster complex $\Delta(\Phi)$ was introduced by S. Fomin and A. Zelevinsky in the context of algebraic Y -systems [13] and cluster algebras [11, 12]. It is a pure $(n - 1)$ -dimensional simplicial complex which is homeomorphic to a sphere [13]. When Φ is crystallographic there is a cluster algebra associated to Φ and $\Delta(\Phi)$ was realized explicitly in [9] as the boundary complex of a simplicial convex polytope, known as a (simplicial) *generalized associahedron*; see [10] for an overview of cluster complexes and generalized associahedra.

The combinatorics of $\Delta(\Phi)$ encodes the exchange of clusters in the corresponding cluster algebra of finite type. The remarkable enumerative properties of $\Delta(\Phi)$ have provided combinatorialists with many interesting puzzles and have suggested connections with a number of seemingly unrelated objects appearing in various areas of mathematics; see [7, 13] and [10, Lecture 5]. One such object is the lattice of noncrossing partitions associated to W (see [1, 3, 5, 6, 14] and Section 2.1). This is a self-dual graded poset of rank n , denoted \mathbf{L}_W , which is a lattice of considerable interest in the topology of finite-type Artin groups. It encodes the combinatorics of reduced decompositions of Coxeter elements of W with respect to the generating set of all reflections in W . At the core of the connection between $\Delta(\Phi)$ and \mathbf{L}_W lies the fact that both the number of facets of $\Delta(\Phi)$ (clusters) and the number of elements of \mathbf{L}_W are given by the expression

$$(1) \quad N(\Phi) = \prod_{i=1}^n \frac{e_i + h + 1}{e_i + 1},$$

known as the *Catalan number* associated to W , where h is the Coxeter number and e_1, \dots, e_n are the exponents of W . Recall (see Section 2.5 for definitions) that the h -vector of a simplicial complex Δ is a fundamental enumerative invariant which

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refines the number of facets of Δ . The following statement has been verified by use of the classification of finite root systems and case by case computations due to various authors; see parts (i) and (ii) of [10, Theorem 5.9].

Theorem 1.1. *Let Φ be a finite root system of rank n with corresponding reflection group W . The entry $h_i(\Delta(\Phi))$ of the h -vector of $\Delta(\Phi)$ is equal to the number of elements of \mathbf{L}_W of rank i for all $0 \leq i \leq n$.*

In particular, the number of facets of $\Delta(\Phi)$ is equal to the total number of elements of \mathbf{L}_W .

It has been one of the challenging problems on cluster combinatorics to find a conceptual proof of the previous theorem. That such a proof may be possible is suggested by the recent work [6] of two of the authors. There, a case-free proof of the lattice property of \mathbf{L}_W is given together with a new characterization of the cluster complex. It is the main objective of this paper to build upon the constructions of [6] and give a case-free proof of Theorem 1.1. More specifically we will describe a shelling of $\Delta(\Phi)$ and a bijection ϕ from the set of facets of $\Delta(\Phi)$ to \mathbf{L}_W such that, for any facet F of $\Delta(\Phi)$, the number of vertices of the restriction face of F with respect to this shelling is equal to the corank of $\phi(F)$ in \mathbf{L}_W . Our shelling of $\Delta(\Phi)$ is a variation of the lexicographic ordering on the facets with respect to the total ordering of the vertices of $\Delta(\Phi)$ considered in [6]. The fact that the map ϕ (see Definition 5.1) is indeed a bijection is the hardest part of the proof and requires a thorough analysis of certain properties of the subcomplexes $X(w)$ of $\Delta(\Phi)$ defined and studied in [6]. For another conceptual approach to Theorem 1.1, see [15, 16].

The entries of the h -vector of $\Delta(\Phi)$ are known as the *Narayana numbers* associated to W . They are the dimensions of the real cohomology groups of the complex projective toric variety associated to the corresponding generalized associahedron and they admit various interesting combinatorial interpretations (one of which appears in Theorem 1.1) and generalizations; see [10, Section 5.2] and the references given there.

The general layout of this paper is as follows. Basic definitions and background related to noncrossing partition lattices and cluster complexes are collected in Section 2. In particular, the realization of $\Delta(\Phi)$ given in [6] and other necessary material from that paper are reviewed. Some results from [6] are also generalized and some new technical results are given. The description [6, Section 6] of the lexicographically first facet of the complex $X(w)$, with respect to the vertex ordering of [6], is recalled in Section 3 and an analogous description for the lexicographically last facet is provided. These constructions are used in establishing some key properties (Theorem 5.2 and Proposition 5.5) of the map ϕ . This map is defined in Section 5 after some further definitions and results related to the structure of faces of $\Delta(\Phi)$ and the complexes $X(w)$ are given in Section 4. The fact that ϕ is a bijection is proved in Section 6. The proof of Theorem 1.1 is completed in Section 7 by combining bijectivity of ϕ with the explicit shelling of $\Delta(\Phi)$ mentioned above. Some combinatorial interpretations of the entries of the h -vector of the positive part of $\Delta(\Phi)$ are also derived. General background on root systems, cluster complexes and noncrossing partition lattices can be found in [10] and references therein.

2. PRELIMINARIES

Throughout this paper W is a finite real reflection group of rank n , generated by orthogonal reflections in \mathbb{R}^n . Unless otherwise stated we assume that W is irreducible. The reflection in the linear hyperplane in \mathbb{R}^n orthogonal to a nonzero vector α is written as $R(\alpha)$. The root system for W which we use is denoted by Φ and consists of the pairs $\{\alpha, -\alpha\}$ of normals to the reflecting hyperplanes. We will further assume that these normals have *unit* length.

2.1. The reflection length order and the lattice \mathbf{L}_W . To any $w \in W$ we can associate two linear subspaces of \mathbb{R}^n : the fixed space $\mathcal{F}(w)$ of w and the orthogonal complement $\mathcal{M}(w)$ of $\mathcal{F}(w)$ in \mathbb{R}^n . The subspace $\mathcal{M}(w)$ is called the *moved space* of w in [6]. The dimension $\ell(w)$ of $\mathcal{M}(w)$ is equal to the smallest integer k such that w can be written as a product of k reflections in W [5, Proposition 2.2]. Note that ℓ is *not* the usual length function associated to a simple system for W . For $a, b \in W$ we let

$$a \preceq b \quad \text{if and only if} \quad \ell(a) + \ell(a^{-1}b) = \ell(b).$$

The relation \preceq turns W into a graded partially ordered set of rank n , which has the identity I as its unique minimal element and rank function ℓ . The following lemma collects some useful properties of this partial order.

Lemma 2.1. *Let $a, b, c \in W$.*

- (i) *If $a \preceq b$ then $\mathcal{M}(a) \subseteq \mathcal{M}(b)$.*
- (ii) *If $\mathcal{M}(a) \subseteq \mathcal{M}(b)$ and a is a reflection then $a \preceq b$.*
- (iii) *If $\mathcal{M}(a) \subseteq \mathcal{M}(b)$ and $a, b \preceq w$ for some $w \in W$ then $a \preceq b$.*
- (iv) *If $a, b \preceq c \preceq w$ for some $w \in W$ and $ab \preceq w$ then $ab \preceq c$.*

Proof. For parts (i), (ii) and (iii) see [5, Section 2]. Part (iv) follows from part (iii) since from the assumptions we have $\mathcal{M}(a), \mathcal{M}(b) \subseteq \mathcal{M}(c)$ and hence $\mathcal{M}(ab) \subseteq \mathcal{M}(a) + \mathcal{M}(b) \subseteq \mathcal{M}(c)$. \square

If γ is a Coxeter element of W then the interval $[1, \gamma]$ in the poset (W, \preceq) , denoted $\mathbf{L}_W(\gamma)$ or $\mathbf{L}(\gamma)$, is a locally self-dual, graded lattice of rank n ; see [1, 3, 5, 6] for further information and interesting properties. The isomorphism type \mathbf{L}_W of $\mathbf{L}_W(\gamma)$ is independent of γ and called the *noncrossing partition lattice* associated to W .

2.2. Coxeter elements and the operator μ . As in [3, 6] our standard choice of γ , which we fix once and for all, is that of a bipartite Coxeter element. Thus we fix an ordered simple system $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ for Φ with the property that there is a $1 \leq s \leq n$ for which $\Pi_1 = \{\alpha_1, \dots, \alpha_s\}$ and $\Pi_2 = \{\alpha_{s+1}, \dots, \alpha_n\}$ are orthonormal sets and we set

$$\gamma = R(\alpha_1)R(\alpha_2) \cdots R(\alpha_n).$$

Following [6] we write

$$\mu(x) = 2(I - \gamma)^{-1}(x)$$

for $x \in \mathbb{R}^n$ and recall from [6, Corollary 3.3] that for $\tau \in \Phi$, $\mu(\tau)$ is the unique vector in the one dimensional space $\mathcal{F}(R(\tau)\gamma)$ satisfying $\mu(\tau) \cdot \tau = 1$. The following lemma is implicit in [6].

Lemma 2.2. *For nonparallel roots σ, τ we have $R(\sigma)R(\tau) \preceq \gamma$ if and only if $\mu(\sigma) \cdot \tau = 0$.*

Proof. The statement $R(\sigma)R(\tau) \preceq \gamma$ is equivalent to $R(\tau) \preceq R(\sigma)\gamma$. By parts (i) and (ii) of Lemma 2.1, this in turn holds if and only if τ lies in the $(n-1)$ -dimensional space $\mathcal{M}(R(\sigma)\gamma)$ or, in other words, if and only if $\mu(\sigma) \cdot \tau = 0$. \square

Lemma 2.3 and part (i) of Lemma 2.4 below generalize [6, Theorem 3.2] and [6, Theorem 3.7 (a)], respectively.

Lemma 2.3. *If $\tau \in \Phi$ and $R(\tau) \preceq w \preceq \gamma$ then $w(\mu(\tau)) = \mu(\tau) - 2\tau$.*

Proof. Observe that $\mu(\tau) \in \mathcal{F}(R(\tau)\gamma) \subseteq \mathcal{F}(w^{-1}\gamma)$, since $w^{-1}\gamma \preceq R(\tau)\gamma$. Hence $w(\mu(\tau)) = \gamma(\mu(\tau)) = \mu(\tau) - 2\tau$. \square

Lemma 2.4. *If $\sigma, \tau \in \Phi$ and $R(\sigma), R(\tau) \preceq w \preceq \gamma$ then*

- (i) $w(\mu(\sigma)) \cdot \tau = -\mu(\tau) \cdot \sigma$,
- (ii) $w(\mu(\sigma)) \cdot \tau = \mu(w(\sigma)) \cdot \tau$,
- (iii) $\mu(w(\sigma)) \cdot w(\tau) = \mu(\sigma) \cdot \tau$.

Proof. For part (i) use Lemma 2.3 to write $\tau = (1/2)[\mu(\tau) - w(\mu(\tau))]$, and similarly for σ , so that

$$\begin{aligned} w(\mu(\sigma)) \cdot \tau &= w(\mu(\sigma)) \cdot (1/2)[\mu(\tau) - w(\mu(\tau))] \\ &= (1/2)[w(\mu(\sigma)) \cdot \mu(\tau) - w(\mu(\sigma)) \cdot w(\mu(\tau))] \\ &= (1/2)[w(\mu(\sigma)) \cdot \mu(\tau) - \mu(\sigma) \cdot \mu(\tau)] \\ &= (1/2)\mu(\tau) \cdot [w(\mu(\sigma)) - \mu(\sigma)] \\ &= -\mu(\tau) \cdot \sigma. \end{aligned}$$

For part (ii) observe that $R(w(\sigma)) \preceq w$ and apply part (i) twice to get

$$\mu(w(\sigma)) \cdot \tau = -w(\mu(\tau)) \cdot w(\sigma) = -\mu(\tau) \cdot \sigma = w(\mu(\sigma)) \cdot \tau.$$

Similarly part (iii) follows by applying part (ii) with τ replaced by $w(\tau)$. \square

2.3. The ordering of roots. Let us denote by Φ^+ the positive system of Φ corresponding to the simple system Π and by N the cardinality of Φ^+ and let us use the notation $\Phi_{\geq -1}$ of [13] for the set $\Phi^+ \cup (-\Pi)$ of almost positive roots in Φ .

As in [6] we set $\rho_i = R(\alpha_1)R(\alpha_2) \cdots R(\alpha_{i-1})(\alpha_i)$ for $i \geq 1$ where the α_i are indexed cyclically modulo n (so that $\rho_1 = \alpha_1$) and $\rho_{-i} = \rho_{2N-i}$ for $i \geq 0$ and recall that

$$\begin{aligned} \{\rho_1, \rho_2, \dots, \rho_N\} &= \Phi^+, \\ \{\rho_{N+i} : 1 \leq i \leq s\} &= \{-\rho_1, \dots, -\rho_s\} = -\Pi_1, \\ \{\rho_{-i} : 0 \leq i < n-s\} &= \{-\rho_{N-i} : 0 \leq i < n-s\} = -\Pi_2. \end{aligned}$$

Thus, as in [6], we can consider the total order $<$ of the set $\Phi_{\geq -1}$ defined by

$$(2) \quad \rho_{-n+s+1} < \cdots < \rho_0 < \rho_1 < \cdots < \rho_{N+s}.$$

Unless otherwise stated, whenever we talk about the lexicographic order on a collection of subsets of $\Phi_{\geq -1}$ it should be understood that this is defined with respect to the total order (2). The set $\{\rho_{N-n+1}, \dots, \rho_N\}$ of the last n positive roots in this order will be denoted by Ω . The following three lemmas from [6] are listed here for easy reference.

Lemma 2.5. ([6, Lemma 3.9]) *For a set $\{\tau_1, \tau_2, \dots, \tau_k\}$ of positive roots satisfying $\tau_1 < \tau_2 < \cdots < \tau_k$ the following are equivalent:*

- (a) $R(\tau_k) \cdots R(\tau_1)$ is an element of $\mathbf{L}(\gamma)$ of length k ,

(b) $\mu(\tau_j) \cdot \tau_i = 0$ for $1 \leq i < j \leq k$. □

Lemma 2.6. ([6, Theorem 3.7 (b) and (d)]) *For $1 \leq i < j \leq N$ we have*

- (i) $\mu(\rho_i) \cdot \rho_j \geq 0$ and □
- (ii) $\mu(\rho_j) \cdot \rho_i \leq 0$. □

Lemma 2.7. ([6, Lemma 5.6]) *Let τ, ρ be distinct positive roots with $R(\tau)R(\rho) \preceq \gamma$.*

- (i) *If $\tau < \rho$ then $\tau \cdot \rho \leq 0$.*
- (ii) *If $\tau > \rho$ then $\tau \cdot \rho \geq 0$.* □

Remark 2.8. Recall from [6, Section 3] that $\gamma^{-1} = R(\rho_i)R(\rho_{i+1}) \cdots R(\rho_{i+n-1})$ and that $\gamma(\rho_i) = \rho_{i+n}$ for $i \geq 1$. In particular, Ω consists of those positive roots ρ such that $\gamma(\rho)$ is a negative root.

The following technical result gives some information about the action of an arbitrary $w \in \mathbf{L}(\gamma)$ on Φ .

Corollary 2.9. (i) *If $R(\rho) \preceq w \preceq \gamma$ and ρ and $w(\rho)$ are positive roots then $\rho < w(\rho)$.*
(ii) *If $R(\rho_i) \preceq w \preceq \gamma$ for some $i \geq 1$ then $w(\rho_i) \notin \{\rho_{i+1}, \rho_{i+2}, \dots, \rho_{i+n-1}\}$.*
(iii) *If $R(\rho) \preceq w \preceq \gamma$ and $\rho \in \Omega$ then $w(\rho)$ is a negative root.*

Proof. (i) Using Lemma 2.4 we compute $\mu(w(\rho)) \cdot \rho = -\mu(\rho) \cdot \rho = -1$, so the result follows from part (i) of Lemma 2.6.

(ii) Suppose that $w(\rho_i) = \rho_j$ with $i < j \leq i + n - 1$. Then $R(\rho_j)R(\rho_i) \preceq \gamma$ by Remark 2.8 and hence $\mu(\rho_j) \cdot \rho_i = 0$ by Lemma 2.2. However $\mu(\rho_j) \cdot \rho_i = \mu(w(\rho_i)) \cdot \rho_i = -1$ as in part (i), which gives a contradiction.

(iii) This follows from parts (i) and (ii). □

2.4. Peripheral elements.

Definition 2.10. *An element $w \in \mathbf{L}(\gamma)$ is called peripheral if $w \preceq \gamma'$ where*

$$\gamma' \in \{R(\alpha_1)\gamma, \dots, R(\alpha_s)\gamma, \gamma R(\alpha_{s+1}), \dots, \gamma R(\alpha_n)\}.$$

Otherwise w is called non-peripheral.

The following characterizations of peripheral elements of $\mathbf{L}(\gamma)$ will be useful. Recall that a standard parabolic subgroup of W is a subgroup generated by a subset of $\{R(\alpha_1), \dots, R(\alpha_n)\}$.

Proposition 2.11. *If $w \in \mathbf{L}(\gamma)$ then the following are equivalent:*

- (i) *w is peripheral,*
- (ii) *w lies in a proper standard parabolic subgroup,*
- (iii) *at least one of the roots in Ω belongs to $\mathcal{M}(w^{-1}\gamma)$.*

Proof. (i) \Rightarrow (ii): If w is peripheral and $R \preceq W$ is a reflection, then R lies in the standard parabolic subgroup with simple system $\Pi \setminus \{\alpha_i\}$ for some $1 \leq i \leq n$. Thus w , which is a product of such reflections, must belong to the same parabolic subgroup.

(ii) \Rightarrow (i): If w belongs to a proper standard parabolic subgroup then it belongs to one with simple system $\Pi \setminus \{\alpha_i\}$ for some $1 \leq i \leq n$ and $\mathcal{M}(w) \subseteq \mathcal{M}(\gamma')$, where

$$\gamma' \in \{R(\alpha_1)\gamma, \dots, R(\alpha_s)\gamma, \gamma R(\alpha_{s+1}), \dots, \gamma R(\alpha_n)\}.$$

By part (iii) of Lemma 2.1 we have $w \preceq \gamma'$.

(i) \Leftrightarrow (iii): Since $R(\alpha_i)\gamma = \gamma R(\gamma^{-1}(\alpha_i))$ for $1 \leq i \leq s$, we have that w is peripheral if and only if $w \preceq \gamma'$ where γ' is one of the elements

$$\gamma R(\gamma^{-1}(\alpha_1)), \dots, \gamma R(\gamma^{-1}(\alpha_s)), \gamma R(\alpha_{s+1}), \dots, \gamma R(\alpha_n).$$

This happens if and only if $R \preceq w^{-1}\gamma$ where R is one of the reflections

$$R(\gamma^{-1}(\alpha_1)), \dots, R(\gamma^{-1}(\alpha_s)), R(\alpha_{s+1}), \dots, R(\alpha_n).$$

By the definition of the total order (2), these are the reflections defined by the last n positive roots, i.e., defined by the elements of Ω . \square

2.5. h -vectors and shellings. We will use the notion of a spherical simplicial complex in \mathbb{R}^n . The faces of such a complex are formed by intersecting the unit sphere S^{n-1} in \mathbb{R}^n with simplicial cones, each pointed at the origin. Thus a polyhedral fan in \mathbb{R}^n is formed. If C is such a cone of dimension n and H runs through the supporting hyperplanes of the facets of C , we will refer to the intersections $S^{n-1} \cap H$ as the *walls* of the spherical simplex $S^{n-1} \cap C$.

Given a finite (abstract, geometric or spherical) simplicial complex Δ of dimension $d-1$, let f_i denote the number of i -dimensional faces of Δ . The h -vector of Δ is the sequence $h(\Delta) = (h_0, h_1, \dots, h_d)$ defined by

$$(3) \quad \sum_{i=0}^d f_{i-1}(x-1)^{d-i} = \sum_{i=0}^d h_i x^{d-i}$$

where $f_{-1} = 1$ unless Δ is empty. The complex Δ is *pure* if all its facets (faces which are maximal with respect to inclusion) have dimension $d-1$. Given a total ordering F_1, F_2, \dots, F_m of the facets of a pure simplicial complex Δ of dimension $d-1$ and $1 \leq j \leq m$ we denote by $\mathcal{R}(F_j)$ the set of vertices x of F_j for which the codimension one face of F_j not containing x is contained in at least one of the facets F_1, F_2, \dots, F_{j-1} . Such an ordering F_1, F_2, \dots, F_m is called a *shelling* of Δ if there are no indices $1 \leq i < j \leq m$ for which $\mathcal{R}(F_j)$ is contained in the vertex set of F_i . In that case $\mathcal{R}(F_j)$ is called the *restriction set* of F_j with respect to this shelling and the entries $h_i = h_i(\Delta)$ of $h(\Delta)$ are nonnegative integers given by

$$(4) \quad h_i = \# \{1 \leq j \leq m : \#\mathcal{R}(F_j) = i\}, \quad 0 \leq i \leq d,$$

where the cardinality of a finite set S is denoted by $\#S$. More information and references on shellability of simplicial complexes can be found in [4, Section 11].

2.6. Cluster complexes and subcomplexes. Let $\Delta(\gamma)$ denote the collection of spherical simplices in \mathbb{R}^n on the vertex set $\Phi_{\geq -1}$ defined by declaring a subset $\{\tau_1, \tau_2, \dots, \tau_k\}$ of $\Phi_{\geq -1}$ satisfying $\tau_1 < \tau_2 < \dots < \tau_k$ to be the vertex set of a simplex in $\Delta(\gamma)$ if and only if

$$R(\tau_k)R(\tau_{k-1}) \cdots R(\tau_1)$$

is an element of $\mathbf{L}(\gamma)$ of rank k . Thus $\Delta(\gamma)$ coincides with the spherical simplicial complex $EX(\gamma)$ of [6]. A simplex in $\Delta(\gamma)$ is said to be *positive* if its vertices are positive roots. For $w \in \mathbf{L}(\gamma)$ let $X(w)$ denote the subcollection of $\Delta(\gamma)$ consisting of those simplices with vertex set contained in $\mathcal{M}(w) \cap \Phi^+$; in particular, $X(\gamma)$ is the subcollection of positive simplices of $\Delta(\gamma)$. The set of vertices (zero dimensional simplices) of $X(w)$ is the positive system induced by Φ^+ on the root system $\Phi(w) = \Phi \cap \mathcal{M}(w)$ and is denoted by $\Phi^+(w)$. By parts (i) and (ii) of Lemma 2.2, this set

coincides with the set of positive roots τ satisfying $R(\tau) \preceq w$. We note that $R(\tau)\rho \in \Phi(w)$ whenever $\tau, \rho \in \Phi(w)$. Finally, let $\Delta_+(\Phi)$ denote the induced subcomplex of $\Delta(\Phi)$ on the vertex set Φ^+ , referred to as the *positive part* of $\Delta(\Phi)$. The following theorem will be crucial in relating the combinatorics of $\Delta(\Phi)$ to that of \mathbf{L}_W .

Theorem 2.12. ([6])

- (i) *The collection $\Delta(\gamma)$ is a spherical simplicial complex of dimension $n - 1$ which is a realization of the cluster complex $\Delta(\Phi)$.*
- (ii) *If $w \in \mathbf{L}(\gamma)$ then the collection $X(w)$ is a spherical simplicial complex of dimension $\ell(w) - 1$.*

In particular, $X(\gamma)$ is a spherical simplicial complex of dimension $n - 1$ which is a realization of $\Delta_+(\Phi)$. \square

It follows that $\Delta(\gamma)$ and $X(\gamma)$ are pure simplicial complexes which are homeomorphic to a sphere and a ball, respectively, of dimension $n - 1$. More generally, for $w \in \mathbf{L}(\gamma)$ the complex $X(w)$ is a triangulation of a spherical simplex of dimension $\ell(w) - 1$ [6, Corollary 7.7] and hence $X(w)$ is a pure simplicial complex which is homeomorphic to a ball of dimension $\ell(w) - 1$. These facts will be used repeatedly in Sections 4 and 7. Figure 1 shows $\Delta(\gamma)$ in the case where W is the group C_3 (or B_3) of symmetries of the cube, with the roots ordered as in (2), so that $\rho_0 = -\rho_9$, while $\rho_{10} = -\rho_1$ and $\rho_{11} = -\rho_2$.

3. LEXICOGRAPHICALLY FIRST AND LAST FACETS

In this section we fix $w \in \mathbf{L}(\gamma)$ of length k and describe the first and last facets of the complex $X(w)$ in the lexicographic order. We let $\Pi(w) = \{\delta_1, \dots, \delta_k\}$ be the simple system for $\Phi^+(w)$, as in [6, Section 5], with $\delta_1 < \delta_2 < \dots < \delta_k$ and $w = R(\delta_1) \cdots R(\delta_k)$.

3.1. The lexicographically first facet. We first recall the description of the first facet of $X(w)$ in the lexicographic order from [6, Section 6]. We provide proofs of statements, somewhat simplified from those of [6, Section 6], to make this paper more self-contained and since proofs of the corresponding statements about the lexicographically last facet will be similar. We define the roots

$$(5) \quad \epsilon_i = R(\delta_1) \cdots R(\delta_{i-1}) \delta_i$$

for $1 \leq i \leq k$, so that $\epsilon_1 = \delta_1$. Since $\epsilon_i - \delta_i$ is a nonnegative linear combination of $\{\delta_1, \dots, \delta_{i-1}\}$, the set $\{\epsilon_1, \dots, \epsilon_k\}$ is a linearly independent subset of $\Phi^+(w)$.

Lemma 3.1. *We have*

$$\mu(\epsilon_i) \cdot \delta_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

In particular $\mu(\epsilon_i) \cdot \tau \geq 0$ for all i and for all $\tau \in \Phi^+(w)$.

Proof. From (5) we deduce that

$$(6) \quad w = R(\epsilon_i)R(\delta_1) \cdots R(\delta_{i-1})R(\delta_{i+1}) \cdots R(\delta_k).$$

Since $w \preceq \gamma$ has length k , it follows that, for $i \neq j$, $R(\epsilon_i)R(\delta_j) \preceq \gamma$ and that $\mu(\epsilon_i) \cdot \delta_j = 0$, by Lemma 2.2.

If $w_i = R(\delta_1) \cdots R(\delta_i)$, then $w_i(\delta_i) = -\epsilon_i$. Since $\epsilon_i, \delta_i \in \mathcal{M}(w_i)$, it follows from Lemma 2.4 that $\mu(\epsilon_i) \cdot \delta_i = -\mu(w_i(\delta_i)) \cdot \delta_i = \mu(\delta_i) \cdot \delta_i = 1$. For the last statement,

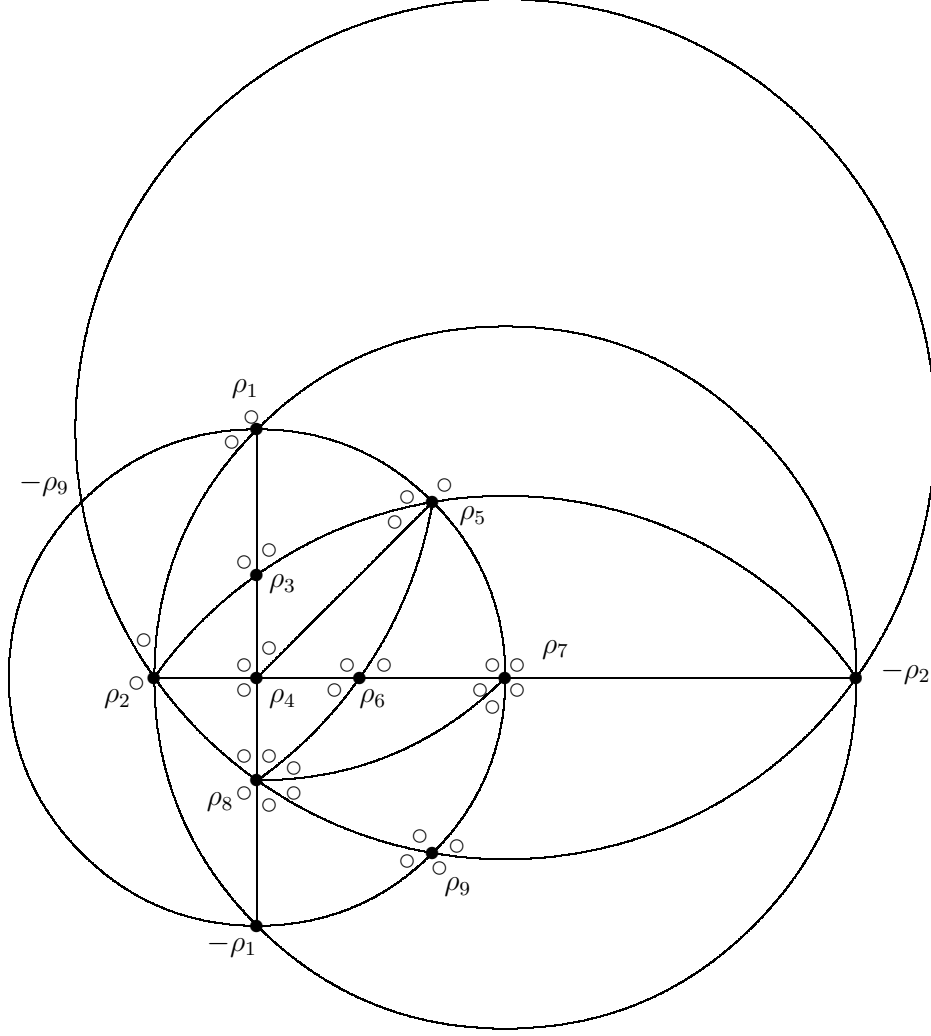


FIGURE 1.

note that any $\tau \in \Phi^+(w)$ can be written as a nonnegative linear combination $\tau = a_1\delta_1 + \cdots + a_k\delta_k$, so that $\mu(\epsilon_i) \cdot \tau = a_i \geq 0$. \square

Lemma 3.2. *If $\tau \in \Phi^+(w)$ satisfies $\tau < \epsilon_i$ then $\mu(\epsilon_i) \cdot \tau = 0$ and $\epsilon_i \cdot \tau \geq 0$.*

Proof. From Lemma 3.1 we have $\mu(\epsilon_i) \cdot \tau \geq 0$. But $\tau < \epsilon_i$ implies $\mu(\epsilon_i) \cdot \tau \leq 0$ by Lemma 2.6 (ii). Hence $\mu(\epsilon_i) \cdot \tau = 0$ or, equivalently, $R(\epsilon_i)R(\tau) \preceq \gamma$. Finally $\epsilon_i \cdot \tau \geq 0$ by applying Lemma 2.7 to this last relation and $\tau < \epsilon_i$. \square

Lemma 3.3. *Let $\tau \in \Phi^+(w)$ and fix $1 \leq i \leq k$. Then $\tau \in \{\epsilon_1, \dots, \epsilon_i\}$ if and only if $R(\delta_i)R(\delta_{i-1}) \cdots R(\delta_1)\tau$ is a negative root. In particular, $\tau \in \{\epsilon_1, \dots, \epsilon_k\}$ if and only if $w^{-1}(\tau)$ is a negative root.*

Proof. If $\tau = \epsilon_j$ with $1 \leq j \leq i$ then $R(\delta_i)R(\delta_{i-1}) \cdots R(\delta_1)\tau = -R(\delta_i) \cdots R(\delta_{j+1})\delta_j$ is a negative root since the $\delta_a \cdot \delta_b \leq 0$. For the converse let $\tau_0 = \tau$ and $\tau_j = R(\delta_j)\tau_{j-1}$

for $1 \leq j \leq i$ and assume that $R(\delta_i)R(\delta_{i-1}) \cdots R(\delta_1)\tau = \tau_i$ is a negative root. Then there exists $1 \leq j \leq i$ such that τ_{j-1} is a positive root but τ_j is negative. Since $\tau_j = R(\delta_j)\tau_{j-1}$ we must have $\tau_{j-1} = \delta_j$. On the other hand $\tau_{j-1} = R(\delta_{j-1}) \cdots R(\delta_1)\tau$, so that $\tau = \epsilon_j$. \square

Lemma 3.4. *If $\tau \in \Phi^+(w)$ satisfies $\tau < \epsilon_i$ and $\tau \notin \{\epsilon_1, \dots, \epsilon_{i-1}\}$ then $\epsilon_i \cdot \tau = 0$.*

Proof. From Lemma 3.2 we have $\epsilon_i \cdot \tau \geq 0$ and $\mu(\epsilon_i) \cdot \tau = 0$. Since $\tau \in \Phi^+(w)$, the last equation implies that τ lies in the positive hull of $\Pi(w) - \{\delta_i\}$. Note that the root $\tau' = R(\delta_{i-1}) \cdots R(\delta_1)\tau$, being positive by Lemma 3.3, lies in the positive hull of $\Pi(w) - \{\delta_i\}$ as well. We conclude that

$$\epsilon_i \cdot \tau = (R(\delta_1) \cdots R(\delta_{i-1})\delta_i) \cdot \tau = \delta_i \cdot \tau' \leq 0$$

and hence that $\epsilon_i \cdot \tau = 0$. \square

Corollary 3.5. *If $i < j$ and $\epsilon_i > \epsilon_j$ then $\epsilon_i \cdot \epsilon_j = 0$.*

Proof. Set $\tau = \epsilon_j$ in Lemma 3.4. \square

Proposition 3.6. *The set $\{\epsilon_1, \dots, \epsilon_k\}$ is the vertex set of the first facet of $X(w)$ in the lexicographic order.*

Proof. It follows from (5) that $R(\epsilon_k) \cdots R(\epsilon_1) = R(\delta_1) \cdots R(\delta_k) = w$. Moreover, by Corollary 3.5 we may relabel so that $\epsilon_1 < \cdots < \epsilon_k$, and hence the set $\{\epsilon_1, \dots, \epsilon_k\}$ is the vertex set of some facet of $X(w)$. Note that Lemma 3.2 continues to hold for this reordered set. Given $1 \leq i \leq k$ we have $\mu(\epsilon_i) \cdot \tau = 0$ for any $\tau \in \Phi^+(w)$ satisfying $\tau < \epsilon_i$ and, by our convention on the ϵ 's, we have $\mu(\epsilon_j) \cdot \tau = 0$ for $j \geq i$ as well. This forces such a root τ into a linear space of dimension $i - 1$, namely the intersection of $\mathcal{M}(w)$ with the hyperplanes $\mu(\epsilon_j)^\perp$ for $i \leq j \leq k$. Thus, for any facet F of $X(w)$, the i th vertex of F cannot precede ϵ_i in the order (2). \square

Corollary 3.7. *Let $\tau \in \Phi^+(w)$. Then $w^{-1}(\tau)$ is a negative root if and only if τ is a vertex of the first facet of $X(w)$ in the lexicographic order.*

Proof. Combine Proposition 3.6 with the last statement of Lemma 3.3. \square

3.2. The lexicographically last facet. We define the roots

$$(7) \quad \zeta_i = R(\delta_k) \cdots R(\delta_{i+1})\delta_i$$

for $1 \leq i \leq k$, so that $\zeta_k = \delta_k$. As with $\{\epsilon_1, \dots, \epsilon_k\}$ in Section 3.1 we see that $\{\zeta_1, \dots, \zeta_k\}$ is a linearly independent subset of $\Phi^+(w)$. From (7) we have

$$(8) \quad w = R(\delta_1) \cdots R(\delta_{i-1})R(\delta_{i+1}) \cdots R(\delta_k)R(\zeta_i)$$

which, combined with (6), gives $R(\epsilon_i)w = wR(\zeta_i)$ or $R(\zeta_i) = w^{-1}R(\epsilon_i)w$ and hence that $\zeta_i = \pm w^{-1}(\epsilon_i)$. Since $w^{-1}(\epsilon_i)$ is a negative root by Lemma 3.3 we deduce that $\zeta_i = -w^{-1}(\epsilon_i)$.

Lemma 3.8. *We have*

$$\mu(\delta_i) \cdot \zeta_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

In particular $\mu(\tau) \cdot \zeta_i \geq 0$ for all i and for all $\tau \in \Phi^+(w)$.

Proof. Using Lemma 2.4 (i) we get

$$\mu(\delta_i) \cdot \zeta_j = -\mu(\delta_i) \cdot w^{-1}(\epsilon_j) = -w(\mu(\delta_i)) \cdot \epsilon_j = \mu(\epsilon_j) \cdot \delta_i$$

so the first statement follows from Lemma 3.1. The second statement follows as in the proof of Lemma 3.1. \square

The proofs of Lemmas 3.9, 3.10 and 3.11 and of Corollary 3.12 below are completely analogous to those of corresponding statements of Section 3.1 and are omitted.

Lemma 3.9. *If $\tau \in \Phi^+(w)$ satisfies $\tau > \zeta_i$ then $\mu(\tau) \cdot \zeta_i = 0$ and $\zeta_i \cdot \tau \geq 0$.* \square

Lemma 3.10. *Let $\tau \in \Phi^+(w)$. Given $1 \leq i \leq k$ we have $\tau \in \{\zeta_i, \zeta_{i+1}, \dots, \zeta_k\}$ if and only if $R(\delta_i)R(\delta_{i+1}) \cdots R(\delta_k)\tau$ is a negative root. In particular, $\tau \in \{\zeta_1, \dots, \zeta_k\}$ if and only if $w(\tau)$ is a negative root.* \square

Lemma 3.11. *If $\tau \in \Phi^+(w)$, $\tau > \zeta_i$ and $\tau \notin \{\zeta_{i+1}, \dots, \zeta_k\}$ then $\zeta_i \cdot \tau = 0$.* \square

Corollary 3.12. *If $i < j$ and $\zeta_i > \zeta_j$ then $\zeta_i \cdot \zeta_j = 0$.* \square

Proposition 3.13. *The set $\{\zeta_1, \dots, \zeta_k\}$ is the vertex set of the last facet of $X(w)$ in the lexicographic order.*

Proof. As in the proof of Proposition 3.6 we have $R(\zeta_k) \cdots R(\zeta_1) = w$ and we may relabel so that $\zeta_1 < \cdots < \zeta_k$ and hence the set $\{\zeta_1, \dots, \zeta_k\}$ is the vertex set of some facet of $X(w)$. Given $1 \leq i \leq k$, Lemma 3.9 implies that for $1 \leq j \leq i$ and any $\tau \in \Phi^+(w)$ with $\tau > \zeta_i$ we have $\mu(\tau) \cdot \zeta_j = 0$. Since μ is an invertible linear transformation, it follows as in the proof of Proposition 3.6 that any such root τ lies in a linear space of dimension $k - i$ and hence that for any facet F of $X(w)$, the i th vertex of F cannot succeed ζ_i in the order (2). \square

Corollary 3.14. *Let $\tau \in \Phi^+(w)$. Then $w(\tau)$ is a negative root if and only if τ is a vertex of the last facet of $X(w)$ in the lexicographic order.*

Proof. Combine Proposition 3.13 with the last statement of Lemma 3.10. \square

Remark 3.15. In the special case $w = \gamma$ the vertices of the last facet of $X(\gamma)$ are the last n positive roots by Remark 2.8.

Remark 3.16. As in the proofs of Propositions 3.6 and 3.13, in the following sections we will denote by $\{\epsilon_1, \epsilon_2, \dots, \epsilon_k\}$ and $\{\zeta_1, \zeta_2, \dots, \zeta_k\}$ the ordered vertex sets of the first and last facet of $X(w)$, respectively, in the lexicographic order. Clearly Lemmas 3.2, 3.4, 3.9 and 3.11, as well as the last statement of Lemmas 3.1 and 3.8, continue to hold under this convention.

Remark 3.17. It follows from Proposition 3.13 that, for $1 < i \leq k$, the set $\{\zeta_i, \dots, \zeta_k\}$ is the vertex set of the last facet of $X(wR(\zeta_1) \cdots R(\zeta_{i-1}))$. Similarly, it follows from Proposition 3.6 that, for $1 \leq i < k$, the set $\{\epsilon_1, \dots, \epsilon_i\}$ is the vertex set of the first facet of $X(R(\epsilon_{i+1}) \cdots R(\epsilon_k)w)$.

4. VERTEX TYPE

In this section F denotes a face of $\Delta(\gamma)$ with ordered vertex set $\{\tau_1, \tau_2, \dots, \tau_k\}$, $w = R(\tau_k)R(\tau_{k-1}) \cdots R(\tau_1) \preceq \gamma$ and $\{\epsilon_1, \epsilon_2, \dots, \epsilon_k\}$ and $\{\zeta_1, \zeta_2, \dots, \zeta_k\}$ are the ordered vertex sets of the first and last facet of $X(w)$, respectively, in the lexicographic order.

Definition 4.1. For $1 \leq i \leq k$ we set

$$\begin{aligned} u_i(F) &= R(\tau_k)R(\tau_{k-1}) \cdots R(\tau_i) \\ v_i(F) &= R(\tau_1)R(\tau_2) \cdots R(\tau_i) \end{aligned}$$

and say that τ_i is a left vertex in F if $u_i(F)\tau_i$ is a positive root. Otherwise we say that τ_i is a right vertex in F .

Lemma 4.2. Any vertex of F which is a negative root is a left vertex in F .

Proof. Suppose that $\tau_i \in -\Pi$ and recall that the first $n - s$ vertices of $\Delta(\gamma)$ in the total order (2) are in $-\Pi_2$ while the last s are in $-\Pi_1$. If $\tau_i \in -\Pi_2$ then $\tau_1, \dots, \tau_{i-1} \in -\Pi_2$ as well. In particular τ_i is orthogonal to each of $\tau_1, \dots, \tau_{i-1}$ and hence $u_i(F)\tau_i = w(\tau_i)$. Since $-\tau_i \in \Pi_2 \subseteq \Omega$, Corollary 2.9 (iii) implies that $w(\tau_i)$ is a positive root, so that τ_i is a left vertex in F . Similarly, if $\tau_i \in -\Pi_1$ then $\tau_{i+1}, \dots, \tau_n \in -\Pi_1$. In particular τ_i is orthogonal to each of $\tau_{i+1}, \dots, \tau_k$ and hence $u_i(F)\tau_i = R(\tau_i)\tau_i = -\tau_i$ is a positive root, so that τ_i is a left vertex in F . \square

Lemma 4.3. The vertex τ_i is a right vertex in F if and only if τ_i is a vertex of the last facet of $X(u_i(F))$ in the lexicographic order. Moreover, if F is positive then τ_i is the first vertex of this facet.

Proof. Since any right vertex in F must be a positive root by Lemma 4.2, the first statement is a direct consequence of the definition and Corollary 3.14. The second statement is obvious since $\{\tau_i, \dots, \tau_k\}$ is the ordered vertex set of a facet of $X(u_i(F))$. \square

Let F be positive, so that F is a facet of $X(w)$. We will describe the wall of F opposite (not containing) the vertex τ_i . To simplify notation, define the roots $\eta_i = u_i(F)\tau_i$ and $\theta_i = v_i(F)\tau_i$. Then

$$R(\eta_i)R(\tau_k) \cdots R(\tau_{i+1})R(\tau_{i-1}) \cdots R(\tau_1) = w \preceq \gamma$$

and

$$R(\tau_k) \cdots R(\tau_{i+1})R(\tau_{i-1}) \cdots R(\tau_1)R(\theta_i) = w \preceq \gamma.$$

Note that $\eta_i, \theta_i \in \Phi(w)$ for all i , although they are not necessarily positive roots. As in the proof of Lemma 3.1 we find that

$$(9) \quad \mu(\eta_i) \cdot \tau_j = \begin{cases} -1, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$

$$(10) \quad \mu(\tau_j) \cdot \theta_i = \begin{cases} -1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

As a result, the wall of F opposite τ_i is the intersection of $S^{n-1} \cap \mathcal{M}(w)$ with $\mu(\eta_i)^\perp$, the linear hyperplane orthogonal to $\mu(\eta_i)$. It follows by linearity from (9) and (10) that $\mu(\eta_i) \cdot x = \mu(x) \cdot \theta_i$ for all $x \in \mathcal{M}(w)$. Note that, since $X(w)$ is homeomorphic to a ball, every codimension one face is contained in either exactly one or exactly two facets of $X(w)$ and the first case occurs if and only if this face is contained in the boundary of $X(w)$.

Proposition 4.4. Suppose that F is positive.

- (i) The root τ_i is a left vertex in F if and only if there exists a vertex $\tau > \tau_i$ of $X(w)$ such that $(V \setminus \{\tau_i\}) \cup \{\tau\}$ is the vertex set of a facet of $X(w)$. Moreover, such a vertex τ is unique.

- (ii) *The root $v_i(F)\tau_i$ is positive if and only if there exists a vertex $\tau < \tau_i$ of $X(w)$ such that $(V \setminus \{\tau_i\}) \cup \{\tau\}$ is the vertex set of a facet of $X(w)$. Moreover, such a vertex τ is unique.*

Proof. The uniqueness of τ in both parts follows from the discussion preceding the statement of the proposition.

(i) Suppose τ_i is a left vertex in F , so that $\eta_i = u_i(F)\tau_i$ is a positive root. Since $\mu(\eta_i) \cdot \tau_i < 0$ by (9) and $\mu(\eta_i) \cdot \eta_i = 1$ we see that $\mu(\eta_i)^\perp$ separates the positive roots τ_i and η_i . It follows that the wall of F opposite τ_i does not lie in the boundary of $X(w)$ and, as a result, there exists a facet of $X(w)$ other than F with vertex set $(V \setminus \{\tau_i\}) \cup \{\tau\}$. Since $\mu(\eta_i)^\perp$ separates τ from τ_i , we must have $\mu(\eta_i) \cdot \tau > 0$. In view of Lemma 2.6, this inequality combined with $\mu(\eta_i) \cdot \tau_i < 0$ gives $\tau > \tau_i$. For the converse suppose that $(V \setminus \{\tau_i\}) \cup \{\tau\}$ is the vertex set of a facet of $X(w)$ for some $\tau > \tau_i$. Then $\mu(\eta_i) \cdot \tau_i < 0$ and $\mu(\eta_i) \cdot \tau > 0$, hence Lemma 2.6 implies that η_i is a positive root, meaning that τ_i is a left vertex in F .

(ii) Suppose that $\theta_i = v_i(F)\tau_i$ is a positive root. Since $\mu(\eta_i) \cdot \tau_i < 0$ by (9) and $\mu(\eta_i) \cdot \theta_i = \mu(\theta_i) \cdot \theta_i = 1$ we see that $\mu(\eta_i)^\perp$ separates the positive roots τ_i and θ_i . As in part (i), it follows that the wall of F opposite τ_i does not lie in the boundary of $X(w)$ and, as a result, there exists a facet of $X(w)$ other than F with vertex set $(V \setminus \{\tau_i\}) \cup \{\tau\}$. Since $\mu(\eta_i)^\perp$ separates τ from τ_i , we must have $\mu(\eta_i) \cdot \tau > 0$. Thus $\mu(\tau) \cdot \theta_i = \mu(\eta_i) \cdot \tau > 0$. In view of Lemma 2.6, this inequality combined with $\mu(\tau_i) \cdot \theta_i = \mu(\eta_i) \cdot \tau_i < 0$ gives $\tau < \tau_i$. The proof of the converse proceeds as in part (i). \square

Corollary 4.5. *If F is a positive face and τ_i is a left vertex in F then $v_i(F)\tau_i$ is a negative root.*

Proof. By part (i) of Proposition 4.4, there exists a vertex $\tau > \tau_i$ of $X(w)$ such that $(V \setminus \{\tau_i\}) \cup \{\tau\}$ is the vertex set of a facet of $X(w)$. Since $X(w)$ is a manifold there cannot be a vertex $\tau' < \tau_i$ of $X(w)$ such that $(V \setminus \{\tau_i\}) \cup \{\tau'\}$ is the vertex set of a facet of $X(w)$. Thus the root $v_i(F)\tau_i$ cannot be positive by part (ii) of Proposition 4.4. \square

Corollary 4.6. *If $\tau \in \Phi^+(w)$ satisfies $\tau \leq \zeta_1$ then there exists a facet of $X(w)$ having τ as its smallest vertex. In particular, τ precedes all vertices of the last facet of $X(wR(\tau))$ in the lexicographic order.*

Proof. The statement is obvious in case $\tau = \zeta_1$, so suppose $\tau < \zeta_1$. Let F be any facet of $X(w)$ having τ as a vertex and let $\{\tau_1, \tau_2, \dots, \tau_k\}$ be the ordered vertex set of F . If $\tau = \tau_1$ there is nothing to prove. Otherwise, since $\tau_1 < \zeta_1$, the vertex τ_1 is a left vertex in F by Corollary 3.14. Proposition 4.4 (i) implies that there exists a vertex τ' of $X(w)$ with $\tau_1 < \tau'$ such that $(V \setminus \{\tau_1\}) \cup \{\tau'\}$ is the vertex set of a facet F' of $X(w)$. Clearly τ is a vertex of F' and the smallest vertex of F' succeeds that of F in the order (2). Therefore, applying the same argument to F' repeatedly, if necessary, we can find a facet of $X(w)$ having τ as its smallest vertex. \square

The following technical fact will be used in Section 6.

Proposition 4.7. *If $\tau \in \Phi^+(w)$ satisfies*

$$\zeta_1 < \dots < \zeta_{i-1} < \tau < \zeta_i < \dots < \zeta_k$$

for some $1 \leq i \leq k$ then

- (i) τ is orthogonal to $\zeta_1, \dots, \zeta_{i-1}$,
- (ii) $R(\tau) \preceq wR(\zeta_1) \cdots R(\zeta_{i-1})$ and
- (iii) the last facet of $X(wR(\tau))$ in the lexicographic order has ordered vertex set

$$\{\zeta_1, \dots, \zeta_{i-1}, \zeta'_{i+1}, \dots, \zeta'_k\},$$

where $\{\zeta'_{i+1}, \dots, \zeta'_k\}$ is the ordered vertex set of the lexicographically last facet of $X(wR(\zeta_1) \cdots R(\zeta_{i-1})R(\tau))$. Moreover $\tau < \zeta'_{i+1}$.

Proof. Part (i) follows from Lemma 3.11. In view of Lemma 2.5 and the fact that $R(\zeta_{i-1}) \cdots R(\zeta_1) \preceq \gamma$ has length $i-1$, we can deduce from Lemma 3.9 that $R(\tau)R(\zeta_{i-1}) \cdots R(\zeta_1) \preceq \gamma$ has length i . Since $R(\tau) \preceq w$ and $R(\zeta_{i-1}) \cdots R(\zeta_1) \preceq w$ by the assumptions, it follows from Lemma 2.1 (iv) that

$$R(\tau)R(\zeta_{i-1}) \cdots R(\zeta_1) \preceq w,$$

which proves (ii). Let $w' = wR(\zeta_1) \cdots R(\zeta_{i-1}) = R(\zeta_k) \cdots R(\zeta_i)$ and observe that $\{\zeta_i, \dots, \zeta_k\}$ is the ordered vertex set of the last facet of $X(w')$ in the lexicographic order by Remark 3.17. Since $R(\tau) \preceq w'$ and $\tau < \zeta_i$, Corollary 4.6 implies that $\tau < \zeta'_{i+1}$ and, in particular, $\zeta_{i-1} < \zeta'_{i+1}$. Since $R(\tau)$ commutes with $R(\zeta_j)$ for $1 \leq j \leq i-1$ by (i), we have

$$R(\zeta'_k) \cdots R(\zeta'_{i+1}) = w'R(\tau) = wR(\tau)R(\zeta_1) \cdots R(\zeta_{i-1})$$

and hence the set $\{\zeta_1, \dots, \zeta_{i-1}, \zeta'_{i+1}, \dots, \zeta'_k\}$ is the ordered vertex set of a facet of $X(wR(\tau))$. To complete the proof of (iii) it suffices to show that ζ_j is the j th vertex of the last facet of $X(wR(\tau))$ in the lexicographic order for $1 \leq j < i$. This follows from the claim that any positive root $\tau' > \zeta_j$ in $\mathcal{M}(wR(\tau))$ lies in the $(k-j-1)$ -dimensional space $\mathcal{M}(wR(\zeta_1) \cdots R(\zeta_j)R(\tau))$. Indeed, we have $R(\tau') \preceq wR(\tau)$, $R(\zeta_j) \cdots R(\zeta_1) \preceq wR(\tau)$ and, by Lemmas 2.5 and 3.9, $R(\tau')R(\zeta_j) \cdots R(\zeta_1) \preceq \gamma$. From Lemma 2.1 (iv) we deduce that $R(\tau')R(\zeta_j) \cdots R(\zeta_1) \preceq wR(\tau)$. \square

5. THE MAP ϕ

Definition 5.1. For a facet F of $\Delta(\gamma)$ with ordered vertex set $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ we define

$$\phi(F) = c(F, \sigma_n)c(F, \sigma_{n-1}) \cdots c(F, \sigma_1)$$

where

$$c(F, \sigma_i) = \begin{cases} R(\sigma_i), & \text{if } \sigma_i \text{ is a right vertex in } F \\ I, & \text{otherwise.} \end{cases}$$

In Figure 1, the right vertices of each facet are indicated with a small circle and the value of ϕ on each facet can be deduced. The vertex types can be verified using Definition 4.1, but the calculations are simplified greatly by using Lemma 4.5, Theorem 5.2 and Lemma 6.2.

Clearly $\phi(F) \in \mathbf{L}(\gamma)$ and thus ϕ is a map from the set of facets of $\Delta(\gamma)$ to $\mathbf{L}(\gamma)$ such that, for any facet F of $\Delta(\gamma)$, the rank of $\phi(F)$ in $\mathbf{L}(\gamma)$ is equal to the number of right vertices of F . It will be shown in Section 6 that ϕ is a bijection. Part of the injectivity of ϕ will be proved in this section. The next theorem gives a fundamental property of ϕ .

Theorem 5.2. Let F be a positive facet of $\Delta(\gamma)$ and let $w = \phi(F)$.

- (i) *The set of right vertices in F is equal to the vertex set of the last facet of $X(w)$ in the lexicographic order.*
- (ii) *The set of left vertices in F is equal to the vertex set of the first facet of $X(w^{-1}\gamma)$ in the lexicographic order.*

For the proof of the theorem we need the following lemma.

Lemma 5.3. *Suppose that F is a positive face of $\Delta(\gamma)$ with ordered vertex set $\{\tau_1, \tau_2, \dots, \tau_k\}$. If τ_i is a right vertex and τ_j is a left vertex in F for some $i < j$ then $\tau_i \cdot \tau_j = 0$.*

Proof. Proceeding by induction on $j - i$, we may assume that the result holds (i) for the face with ordered vertex set $\{\tau_i, \dots, \tau_{j-1}\}$ and (ii) for the face with ordered vertex set $\{\tau_{i+1}, \dots, \tau_j\}$. Furthermore, we may also assume that all roots in $\{\tau_{i+1}, \dots, \tau_{j-1}\}$ are right vertices in F . Indeed, if not then we replace F by the face F' obtained from F by removing all roots in $\{\tau_{i+1}, \dots, \tau_{j-1}\}$ which are left vertices in F and observe that, in view of (i), τ_i and τ_j are still right and left vertices in F' , respectively. Under these assumptions, let $w = u_i(F)$ and $\{\zeta_1, \zeta_2, \dots, \zeta_r\}$ be the ordered last facet of $X(w)$ in the lexicographic order, where $r = k - i + 1$. We have $\tau_i = \zeta_1$ by Lemma 4.3 and

$$R(\tau_j) \preceq u_{i+1}(F) = wR(\tau_i) = R(\zeta_r)R(\zeta_{r-1}) \cdots R(\zeta_2).$$

By Remark 3.17, the last facet of $X(u_{i+1}(F))$ in the lexicographic order has vertex set $\{\zeta_2, \dots, \zeta_r\}$. By our assumptions τ_j is orthogonal to all roots in $\{\tau_{i+1}, \dots, \tau_{j-1}\}$ and hence $u_{i+1}(F)\tau_j = u_j(F)\tau_j$. Since τ_j is a left vertex in F we conclude that $u_{i+1}(F)\tau_j$ is a positive root. It follows from Lemma 3.10 that $\tau_j \notin \{\zeta_2, \dots, \zeta_r\}$. Since $\tau_j \in \mathcal{M}(w)$ and $\tau_j > \zeta_1 = \tau_i$, Lemma 3.11 applies to give $\tau_i \cdot \tau_j = 0$. \square

Proof of Theorem 5.2. Let $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ be the ordered vertex set of F and $\{\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k}\}$ and $\{\sigma_{j_1}, \sigma_{j_2}, \dots, \sigma_{j_\ell}\}$ be the ordered sets of right and left vertices in F , respectively, so that $w = R(\sigma_{i_k}) \cdots R(\sigma_{i_1})$. To prove (i) it suffices to show that σ_{i_p} is the first vertex of the last facet of $X(R(\sigma_{i_k}) \cdots R(\sigma_{i_p}))$ in the lexicographic order for $1 \leq p \leq k$. In view of Corollary 3.14, this is equivalent to the statement that the root $R(\sigma_{i_k}) \cdots R(\sigma_{i_p})\sigma_{i_p}$ is negative. The last statement holds since $R(\sigma_{i_k}) \cdots R(\sigma_{i_p})\sigma_{i_p} = u_{i_p}(F)\sigma_{i_p}$ by Lemma 5.3 and σ_{i_p} is a right vertex in F .

Similarly, by Lemma 5.3 we have $w^{-1}\gamma = R(\sigma_{j_\ell}) \cdots R(\sigma_{j_1})$. To prove (ii) it suffices to show that σ_{j_q} is the last vertex of the lexicographically first facet of $X(R(\sigma_{j_q}) \cdots R(\sigma_{j_1}))$ for $1 \leq q \leq \ell$. By Corollary 3.7 this is equivalent to the statement that $R(\sigma_{j_1}) \cdots R(\sigma_{j_q})\sigma_{j_q}$ is a negative root. Since $R(\sigma_{j_1}) \cdots R(\sigma_{j_q})\sigma_{j_q} = v_{j_q}(F)\sigma_{j_q}$ by Lemma 5.3, this follows from Corollary 4.5. \square

Corollary 5.4. *The restriction of the map ϕ to the set of positive facets of $\Delta(\gamma)$ is injective.*

Proof. If F is a positive facet of $\Delta(\gamma)$ with $w = \phi(F)$, then parts (i) and (ii) of Theorem 5.2 imply that the vertices of F are determined by w and hence F is the unique positive facet whose image under ϕ is w . \square

The image of the facets of $X(\gamma)$ can be characterized in terms of non-peripheral elements. Recall from Section 2.4 that $w \in L(\gamma)$ is peripheral if and only if $R(\rho) \preceq w^{-1}\gamma$ for some $\rho \in \Omega$.

Proposition 5.5. *If F is a facet of $\Delta(\gamma)$ then the element $\phi(F)$ is non-peripheral if and only if F is positive.*

Proof. Let $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ be the ordered vertex set of F . Assume that σ_i is a negative simple root for some $1 \leq i \leq n$, say $\sigma_i = -\alpha_j \in -\Pi_1$ with $1 \leq j \leq s$ (the case $\sigma_i \in -\Pi_2$ is similar). As in proof of Lemma 4.2, $R(\sigma_i)$ commutes with each of $R(\sigma_{i+1}), \dots, R(\sigma_n)$. By the same lemma σ_i is a left vertex in F and hence

$$\phi(F) \preceq R(\sigma_n) \cdots R(\sigma_{i+1}) R(\sigma_{i-1}) \cdots R(\sigma_1) = R(\sigma_i) \gamma = R(\alpha_j) \gamma,$$

which implies that $\phi(F)$ is peripheral.

The converse is proved by contradiction. Let F be positive and assume that $\phi(F)$ is peripheral. Set $w = \phi(F)$, $v = w^{-1}\gamma$ and $r = \ell(v)$. Let $\{\delta'_1, \dots, \delta'_r\}$ be the ordered simple system for $\Phi^+(v)$ and let $\{\epsilon'_1, \dots, \epsilon'_r\}$ be the set of vertices of the first facet of $X(v)$, defined as in equation (5). Recall from [6, Theorem 5.1] that δ'_r is the largest root in $\Phi^+(v)$ with respect to the order (2). Since w is peripheral, the set Ω intersects $\Phi^+(v)$ by Proposition 2.11 and hence $\delta'_r \in \Omega$. Since ϵ'_r is a vertex of the first facet of $X(v)$ in the lexicographic order, ϵ'_r is a left vertex in F by Theorem 5.2 (ii), say $\epsilon'_r = \sigma_i$. Let $u_i = u_i(F) = R(\sigma_n) \cdots R(\sigma_i)$ and $v_i = v_i(F) = R(\sigma_1) \cdots R(\sigma_i)$. By Corollary 3.5 and Proposition 3.6, the root σ_i is orthogonal to all later left vertices in F (if any). It follows from this fact and Lemma 5.3 that $v_i(\sigma_i) = v^{-1}(\sigma_i)$. However $v^{-1}(\sigma_i) = v^{-1}(\epsilon'_r) = -\delta'_r \in -\Omega$ and hence $\gamma(v_i(\sigma_i)) \in -\gamma(\Omega)$ is a positive root by Remark 2.8. On the other hand, $\gamma(v_i(\sigma_i)) = -u_i(\sigma_i)$ since $\gamma = u_i R(\sigma_i) v_i^{-1}$. But $-u_i(\sigma_i)$ (and hence $\gamma(v_i(\sigma_i))$) is a negative root since σ_i is a left vertex in F . This gives the required contradiction. \square

6. BIJECTIVITY OF ϕ

The first step in establishing bijectivity of ϕ is to show that ϕ maps the set of facets of $X(\gamma)$ surjectively onto the set of non-peripheral elements of $\mathbf{L}(\gamma)$.

Lemma 6.1. *If $w \in \mathbf{L}(\gamma)$ is non-peripheral then there exists a facet F of $X(\gamma)$ such that the left vertices in F are precisely the vertices of the first facet of $X(w^{-1}\gamma)$ in the lexicographic order. In particular, $\phi(F) = w$.*

Proof. Let $\{\tau_1, \dots, \tau_k\}$ be the ordered vertex set of the first facet of $X(w^{-1}\gamma)$ in the lexicographic order. For $0 \leq i \leq k$ we define

$$v_i = R(\tau_i) \cdots R(\tau_1) \quad \text{and} \quad w_i = \gamma v_i^{-1},$$

so that $v_0 = I$, $w_0 = \gamma$, $v_k = w^{-1}\gamma$ and $w_k = w$. Note that $v_i = R(\tau_i) v_{i-1}$ and $w_{i-1} = w_i R(\tau_i)$ for $1 \leq i \leq k$. We first claim that $w_{i-1}(\tau_i)$ is a positive root for $1 \leq i \leq k$. Since τ_i is a vertex of the first facet of $X(v_i)$ in the lexicographic order, the root $v_i^{-1}(\tau_i)$ is negative by Corollary 3.7. Therefore $-v_i^{-1}(\tau_i)$ is a positive root, clearly in $\mathcal{M}(v_i)$. Since w , and hence w_i , is non-peripheral we know from Proposition 2.11 that this root cannot be in the set Ω of the last n positive roots. Hence $\gamma(-v_i^{-1}(\tau_i))$ must be a positive root. This proves the claim since $w_{i-1}(\tau_i) = -w_i(\tau_i) = -\gamma(v_i^{-1}(\tau_i))$.

We will show by induction that for each $0 \leq i \leq k$ there exists a facet F_i of $X(\gamma)$ such that the ordered set of left vertices in F_i is equal to $\{\tau_1, \dots, \tau_i\}$. For $i = 0$ observe that the last facet F_0 of $X(\gamma)$ in the lexicographic order, having Ω as its vertex set, has the desired property by Lemma 4.3.

For the inductive step let $1 \leq i \leq k$ and assume that $X(\gamma)$ has a facet F_{i-1} whose ordered set of left vertices is $\{\tau_1, \dots, \tau_{i-1}\}$. Let $\{\zeta_i, \dots, \zeta_n\}$ be the ordered set of right vertices in F_{i-1} . Since $\gamma = w_{i-1}v_{i-1}$, we know from Lemma 5.3 and Theorem 5.2 (ii) that $\phi(F_{i-1}) = w_{i-1}$ and that $\{\zeta_i, \dots, \zeta_n\}$ is the vertex set of the last facet of $X(w_{i-1})$ in the lexicographic order. From $w_{i-1} = w_i R(\tau_i)$ we get $R(\tau_i) \preceq w_{i-1}$. Furthermore, $\tau_i \notin \{\zeta_i, \dots, \zeta_n\}$ by Corollary 3.14, since $w_{i-1}(\tau_i)$ is a positive root by the earlier claim. Thus $\zeta_i < \dots < \zeta_{j-1} < \tau_i < \zeta_j < \dots < \zeta_n$ for some j . Therefore Proposition 4.7 applies to w_{i-1} and τ_i to give

- (i) τ_i is orthogonal to each of $\zeta_i, \dots, \zeta_{j-1}$,
- (ii) $R(\tau_i) \preceq w_{i-1}R(\zeta_i) \dots R(\zeta_{j-1})$,
- (iii) the ordered vertex set of G_i , the last facet of $X(w_{i-1}R(\tau_i))$ in the lexicographic order, is $\{\zeta_i, \dots, \zeta_{j-1}, \zeta'_{j+1}, \dots, \zeta'_n\}$ and
- (iv) $\tau_i < \zeta'_{j+1}$.

Here $\{\zeta'_{j+1}, \dots, \zeta'_n\}$ is the ordered vertex set of the last facet of $X(w')$ in the lexicographic order, where $w' = w_{i-1}R(\zeta_i) \dots R(\zeta_{j-1})R(\tau_i)$.

The union of $\{\tau_1, \dots, \tau_{i-1}, \tau_i\}$ with the vertex set of G_i is the vertex set of a facet F_i of $X(\gamma)$, since the roots in

$$V = \{\tau_1, \dots, \tau_{i-1}\} \cup \{\zeta_i, \dots, \zeta_{j-1}\}$$

can be ordered as they were in F_{i-1} , while τ_i can be positioned between ζ_{j-1} and ζ'_{j+1} by (iv). Since the roots in V are the first $j-1$ roots in both F_{i-1} and F_i and are ordered in precisely the same way, it follows that τ_r remains a left vertex in F_i for all $1 \leq r \leq i-1$. By (i) and (iii) ζ_t remains a right vertex in F_i for all $i \leq t \leq j-1$ and each ζ'_t for $j < t \leq n$ is a right vertex in F_i . Finally, using (i), we have that $w'R(\tau_i)(\tau_i) = w_{i-1}R(\zeta_i) \dots R(\zeta_{j-1})(\tau_i) = w_{i-1}(\tau_i)$ is a positive root by our claim, so that τ_i is a negative vertex in F_i , as desired. Thus $\phi(F_i) = w_i$ and the induction is complete. \square

Let us call an element of $\mathbf{L}(\gamma)$ of the form

$$(11) \quad \gamma' = R(\alpha_{i_1}) \dots R(\alpha_{i_t}),$$

with $1 \leq i_1 < \dots < i_t \leq n$, a *standard parabolic Coxeter element* of W (with respect to the ordered simple system Π). Note that the intersection of the moved spaces of two standard parabolic Coxeter elements is again the moved space of some standard parabolic Coxeter element. It follows from Lemma 2.1 (i) that, given $w \preceq \gamma$, there is a minimum, with respect to the partial order \preceq , standard parabolic Coxeter element γ' satisfying $w \preceq \gamma'$. Clearly w is non-peripheral with respect to γ' and γ' is the unique standard parabolic Coxeter element with this property.

Lemma 6.2. *If F is a face of $\Delta(\gamma)$ and F' is the face of F obtained by removing a set of negative simple roots from the vertex set of F , then the sets of right vertices of F and F' coincide.*

Proof. It suffices to consider the case where F' is obtained from F by removing a single negative simple root. Let $V = \{\tau_1, \dots, \tau_k\}$ be the ordered vertex set of F . By definition of the order (2) there exist integers i and j with $0 \leq i < j \leq k$ such that

$$\{\tau_1, \dots, \tau_i\} \subseteq -\Pi_2, \quad \{\tau_{i+1}, \dots, \tau_j\} \subseteq \Phi^+ \quad \text{and} \quad \{\tau_{j+1}, \dots, \tau_k\} \subseteq -\Pi_1.$$

In view of Lemma 4.2 it suffices to show that for $i + 1 \leq p \leq j$, the type of τ_p is unchanged if a negative simple root is removed from V . Since the type of τ_p is determined by $\{\tau_p, \tau_{p+1}, \dots, \tau_k\}$ we need only consider the removal of a negative simple root in the set $\{\tau_{j+1}, \dots, \tau_k\}$. Suppose that τ_q is such a negative simple root and that the roots $u_p(F)(\tau_p)$ and $u'_p(F)(\tau_p)$ have different signs, where

$$u_p(F) = R(\tau_k) \cdots R(\tau_p) \quad \text{and} \quad u'_p(F) = R(\tau_k) \cdots R(\tau_{q+1})R(\tau_{q-1}) \cdots R(\tau_p).$$

Since Π_1 is an orthonormal set we have $u'_p(F) = R(\tau_q)u_p(F)$ whence, since τ_q is a negative simple root, we conclude that $u'_p(F)(\tau_p) = \pm\tau_q$. However this forces a linear dependence on the set $\{\tau_p, \dots, \tau_k\}$, giving a contradiction. \square

Theorem 6.3. *The map ϕ is a bijection from the set of facets of $\Delta(\gamma)$ to $\mathbf{L}(\gamma)$.*

Proof. To prove surjectivity of ϕ let $w \preceq \gamma$ and choose the unique standard parabolic Coxeter element γ' of the form (11) with respect to which $w \preceq \gamma'$ is non-peripheral. Let ϕ' be the map of Definition 5.1 corresponding to γ' . By Lemma 6.1 we can find a facet F' of $X(\gamma')$ such that $\phi'(F') = w$. Extend F' to a facet F of $\Delta(\gamma)$ by adding the negative simple roots not present in (11) and note that $\phi(F) = w$ by Lemma 6.2.

To prove injectivity of ϕ let $w \preceq \gamma$. We need to show that there is at most one facet F of $\Delta(\gamma)$ with $\phi(F) = w$. Let $\{\tau_1, \dots, \tau_k\}$ be the ordered set of positive vertices of such a facet F , forming the face \tilde{F} of F . Denote by γ' and $\tilde{\gamma}$ the unique standard parabolic Coxeter elements with respect to which $w \preceq \gamma'$ and $R(\tau_k) \cdots R(\tau_1) \preceq \tilde{\gamma}$ are non-peripheral, respectively. By Lemma 6.2 we have $\tilde{\phi}(\tilde{F}) = w$, where $\tilde{\phi}$ is the map of Definition 5.1 corresponding to $\tilde{\gamma}$. Proposition 5.5 implies that w is non-peripheral with respect to $\tilde{\gamma}$ and hence we must have $\tilde{\gamma} = \gamma'$. Thus the set of negative vertices of F is equal to the negative of the set of simple roots not appearing in the expression (11) for γ' and, as a result, this set of negative vertices is uniquely determined by w . Finally, since $\tilde{\phi}(\tilde{F}) = w$, \tilde{F} is uniquely determined by w by Corollary 5.4. \square

Corollary 6.4. *The number of facets of $\Delta(\gamma)$ is equal to the number of elements of $\mathbf{L}(\gamma)$.* \square

Corollary 6.5. *The map ϕ restricts to a bijection from the set of facets of $X(\gamma)$ to the set of non-peripheral elements of $\mathbf{L}(\gamma)$.*

Proof. Combine Theorem 6.3 with Proposition 5.5. \square

In particular, the number of facets of $X(\gamma)$, and hence of $\Delta_+(\Phi)$, is equal to the number of non-peripheral elements of $\mathbf{L}(\gamma)$. This fact was found independently by N. Reading [15, Corollary 9.2]. The number of facets of $\Delta_+(\Phi)$ (positive clusters) is given by a product formula similar to (1); see [13, (3.8)].

7. SHELLINGS AND h -VECTORS

In this section we describe an explicit family of shellings of $\Delta(\gamma)$ and use it to prove Theorem 1.1. We consider the reverse of the lexicographic ordering for the various complexes under consideration instead of the lexicographic ordering itself only because this makes some of the statements technically easier to prove.

Theorem 7.1. *The reverse of the lexicographic ordering on the facets of $X(w)$ is a shelling of $X(w)$ for any $w \in \mathbf{L}(\gamma)$.*

Proof. Let F and F' be two facets of $X(w)$ with vertex sets V and V' , respectively, such that F' succeeds F in the lexicographic order. We need to show that there exists $\rho \in V \setminus V'$ such that $V \setminus \{\rho\}$ is contained in the vertex set of a facet of $X(w)$ which succeeds F in the lexicographic order. We proceed by induction on the length k of w . The statement is trivial for $k \leq 1$, so suppose $k \geq 2$. Let τ and τ' be the smallest elements of V and V' , respectively, so that $\tau \leq \tau'$. If $\tau < \tau'$ then τ is not a vertex of the last facet of $X(w)$ in the lexicographic order and hence $w(\tau)$ is a positive root by Corollary 3.14. This means that τ is a left vertex in F . Clearly $\tau \notin F'$ and the result follows in this case from Proposition 4.4 (i) with $\rho = \tau$. Suppose now that $\tau = \tau'$. Then $V \setminus \{\tau\}$ and $V' \setminus \{\tau\}$ are the vertex sets of facets G and G' , respectively, of $X(wR(\tau))$ such that G' succeeds G in the lexicographic order. By induction G precedes a facet of $X(wR(\tau))$ with vertex set of the form $(V \setminus \{\tau, \rho\}) \cup \{\rho'\}$ for some $\rho \in V \setminus V'$, so that necessarily $\rho < \rho'$. It follows that $(V \setminus \{\rho\}) \cup \{\rho'\}$ is the vertex set of a facet of $X(w)$ which succeeds F . This completes the induction. \square

Let $n(F)$ denote the number of vertices of a face F of $\Delta(\gamma)$ and F_+ (respectively, F_-) denote the face of F whose vertices are the positive (respectively, negative) vertices of F . Define the partial order \trianglelefteq on the set of facets of $\Delta(\gamma)$ as follows. For two such facets F and F' we have $F' \triangleleft F$ if and only if either $n(F'_+) > n(F_+)$ or $n(F'_+) = n(F_+)$ and F_+ precedes F'_+ in the lexicographic order.

Lemma 7.2. *Let V be the vertex set of a facet F of $\Delta(\gamma)$, let $\tau \in V$ and let τ' be the unique vertex of $\Delta(\gamma)$ other than τ such that $(V \setminus \{\tau\}) \cup \{\tau'\}$ is the vertex set of a facet of $\Delta(\gamma)$.*

- (i) *If τ is a negative root then τ' is a positive root.*
- (ii) *Let τ be positive. Then τ is a left vertex in F if and only if τ' is positive and $\tau < \tau'$.*

Proof. (i) Observe that if τ is a negative root then the wall of F opposite τ contains all other negative simple roots.

(ii) Let V_+ denote the set of positive vertices of F and let $w \preceq \gamma$ be such that F_+ is a facet of $X(w)$. By Lemma 6.2 we have that τ is a left vertex in F if and only if it is a left vertex in F_+ . By Proposition 4.4 (i) this happens if and only if there exists a positive root $\sigma > \tau$ such that $(V_+ \setminus \{\tau\}) \cup \{\sigma\}$ is the vertex set of a facet of $X(w)$. To complete the proof observe that for σ positive, $(V_+ \setminus \{\tau\}) \cup \{\sigma\}$ is the vertex set of a facet of $X(w)$ if and only if $(V \setminus \{\tau\}) \cup \{\sigma\}$ is the vertex set of a facet of $\Delta(\gamma)$. \square

Theorem 7.3. (i) *Any linear extension of the partial order \trianglelefteq is a shelling order for $\Delta(\gamma)$.*

- (ii) *The restriction set of a facet F of $\Delta(\gamma)$ with respect to this shelling is equal to the set of left vertices in F .*

Proof. (i) Let F and F' be two facets of $\Delta(\gamma)$ with vertex sets V and V' , respectively, such that $F' \triangleleft F$. It suffices to show that there exists $\rho \in V \setminus V'$ such that $V \setminus \{\rho\}$ is contained in the vertex set of a facet of $\Delta(\gamma)$ which precedes F in the order \trianglelefteq . If $F'_- = F_-$ then the statement follows from Theorem 7.1. Otherwise one can choose as ρ any negative root in $V \setminus V'$ since then, by Lemma 7.2 (i), $V \setminus \{\rho\}$ is contained in the vertex set of a facet of $\Delta(\gamma)$ having one more positive vertex than F .

(ii) This is an immediate consequence of the definition of the restriction set and of Lemmas 4.2 and 7.2. \square

Corollary 7.4. (i) *The entry $h_i(\Delta(\gamma))$ of the h -vector of $\Delta(\gamma)$ is equal to the number of elements of $\mathbf{L}(\gamma)$ of rank i .*

(ii) *The entry $h_i(X(\gamma))$ of the h -vector of $X(\gamma)$ is equal to the number of non-peripheral elements of $\mathbf{L}(\gamma)$ of rank $n - i$.*

Proof. (i) It follows from Theorem 7.3 and (4) that $h_i(\Delta(\gamma))$ is equal to the number of facets of $\Delta(\gamma)$ with i left vertices or, equivalently, to the number of facets of $\Delta(\gamma)$ with $n - i$ right vertices. By Theorem 6.3 this is equal to the number of elements of $\mathbf{L}(\gamma)$ of rank $n - i$, which is equivalent to the statement we want to prove by the self-duality of $\mathbf{L}(\gamma)$.

(ii) The facets of $X(\gamma)$ come first in the shelling of Theorem 7.3 and hence $h_i(X(\gamma))$ is equal to the number of facets of $X(\gamma)$ with i left vertices or, equivalently, to the number of facets of $X(\gamma)$ with $n - i$ right vertices. This is equal to the number of non-peripheral elements of $\mathbf{L}(\gamma)$ of rank $n - i$ by Corollary 6.5. \square

Part (ii) of the previous corollary gives a combinatorial interpretation of the entries of the h -vector of $X(\gamma)$, and hence of $\Delta_+(\Phi)$. Several different combinatorial interpretations to these numbers appear in [2]. The following statement, which is a direct consequence of Proposition 2.11 and part (ii) of Corollary 7.4, provides a similar interpretation.

Corollary 7.5. *The entry $h_i(X(\gamma))$ of the h -vector of $X(\gamma)$ is equal to the number of elements of $\mathbf{L}(\gamma)$ of rank i which are not preceded by any of the reflections corresponding to the last n positive roots.* \square

Remark 7.6. The non-peripheral elements of $\mathbf{L}(\gamma)$ of rank one are the reflections in W which do not lie in any proper standard parabolic subgroup of W . The number f_W of such reflections was studied by F. Chapoton [8] and can be expressed in terms of the exponents and Coxeter number of W [8, Proposition 1.1]. Hence in the special case $i = n - 1$, part (ii) of Corollary 7.4 provides a conceptual explanation of the equality $h_{n-1}(\Delta_+(\Phi)) = f_W$, which can be checked case by case on the basis of the data provided in [2, Section 6] and [8, Section 1].

Proof of Theorem 1.1. Let Φ and W be as in the statement of the theorem. If $\Phi = \Phi_1 \times \Phi_2 \times \cdots \times \Phi_m$ is the decomposition of Φ into irreducible components and $W = W_1 \times W_2 \times \cdots \times W_m$ is the corresponding decomposition of W then $\Delta(\Phi)$ is the simplicial join of the $\Delta(\Phi_i)$ and \mathbf{L}_W is the direct product of the \mathbf{L}_{W_i} and hence the h -polynomial of $\Delta(\Phi)$ (the polynomial with coefficients the entries of the h -vector) and the rank generating polynomial of \mathbf{L}_W are multiplicative with respect to these decompositions. Therefore we may assume that Φ (equivalently, W) is irreducible. Since the combinatorial structure of $\Delta(\Phi)$ is unaffected by rescaling of roots, we may assume further that the elements of Φ are of unit length. Under these assumptions the result follows from Corollary 7.4 (i) and Theorem 2.12 (i). \square

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