Artin groups of euclidean type

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(joint work with Robert Sulway)

Coxeter groups were introduced by Jacques Tits in the 1960s as a natural generalization of the groups generated by reflections which act geometrically (which means properly discontinuously cocompactly by isometries) on spheres and euclidean spaces. And ever since their introduction their basic structure has been reasonably well understood [BB05, Bou02, Dav08]. More precisely, every Coxeter group has a faithful linear representation which preserves a symmetric bilinear form and has an algorithmic solution to its word problem. Moreover, the signature of the quadratic form can be used to coarsely classify Coxeter groups by the type of Riemannian symmetric space on which they naturally act: spherical, euclidean, hyperbolic and higher-rank. For the motivating examples, the spherical Coxeter groups are enumerated by the Dynkin diagrams and the euclidean Coxeter groups are enumerated by the extended Dynkin diagrams.

Artin groups were introduced in the 1970s as a natural class of groups associated to Coxeter groups and are related to them as the braid groups are related to the symmetric groups. More precisely, Artin groups try to capture information about the fundamental group of the quotient of the complexified hyperplane complement by the action of the Coxeter group. To illustrate, the symmetric group acts on $\mathbb{R}^n$ by permuting coordinates, the complexified hyperplane complement is the braid arrangement, its fundamental group is the pure braid group and the fundamental group of the quotient of the complement by the free action by the symmetric group is the full braid group.

The spherical Artin groups (i.e. the ones corresponding to the spherical Coxeter groups) have been well understood since they were introduced [BS72, Del72]. Somewhat surprisingly, the euclidean Artin groups have remained somewhat mysterious outside of a few simple cases investigated by Craig Squier and Fran Digne [Dig, Dig06, Squ87]: it was not known whether they have a solvable word problem, whether they are torsion-free, whether they have trivial center, and whether they have a finite classifying space. These were recently highlighted as four main questions that are open about Artin groups in general and the euclidean Artin groups in particular [GP].

In my talk I discussed progress on these questions with my recent graduate student Robert Sulway. In particular, we prove the following result.

Theorem 1. *Every irreducible euclidean Artin group is a torsion-free centerless group with a solvable word problem and a finite dimensional classifying space.*

Two notes about the statement of the theorem itself. The classifying space we construct is merely finite dimensional but not finite. It is, in fact, locally infinite. And second, we do not know whether or not the classifying space we construct is homotopy equivalent to the usual space associated to these groups constructed from the complexified hyperplane complement. In particular, we do not resolve the classical $K(\pi, 1)$ conjecture for these groups.

The proof proceeds by showing that every irreducible euclidean Artin group has a dual presentation (with infinitely many generators and infinitely many relations) which cannonically embeds as a subgroup of a Garside group. The theory of Garside groups is well-developed and our main results are deduced by working within this larger group.

References

[BB05] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR 2133266 (2006d:05001)

[Bou02] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley. MR 1890629 (2003a:17001)

[BS72] Egbert Brieskorn and Kyoji Saito, *Artin-Gruppen und Coxeter-Gruppen*, Invent. Math. **17** (1972), 245–271. MR 0323910 (48 #2263)

[Dav08] Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR 2360474 (2008k:20091)

[Del72] Pierre Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302. MR 0422673 (54 #10659)

[Dig] François Digne, *A Garside presentation for Artin-Tits groups of type* $\widetilde{C}_n$, Available at `arXiv:1002.4320`.

[Dig06] F. Digne, *Présentations duales des groupes de tresses de type affine* $\widetilde{A}$, Comment. Math. Helv. **81** (2006), no. 1, 23–47. MR 2208796 (2006k:20075)

[GP] Eddy Godelle and Luis Paris, *Basic questions on Artin-Tits groups*, `arXiv:1105.1048`.

[Squ87] Craig C. Squier, *On certain* 3*-generator Artin groups*, Trans. Amer. Math. Soc. **302** (1987), no. 1, 117–124. MR 887500 (88g:20069)