Braid groups and Curvature
Talk 2: The Pieces

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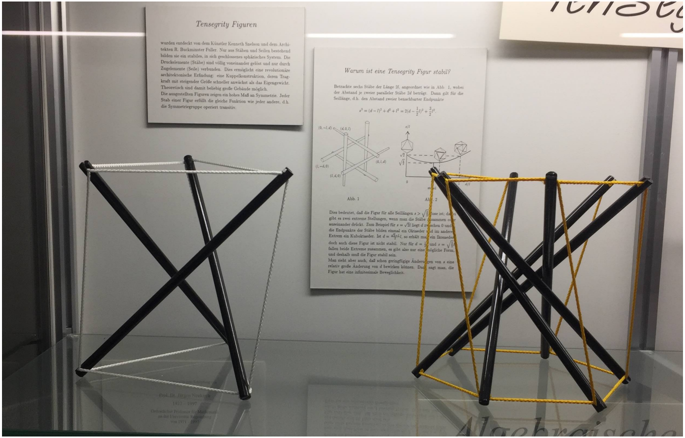
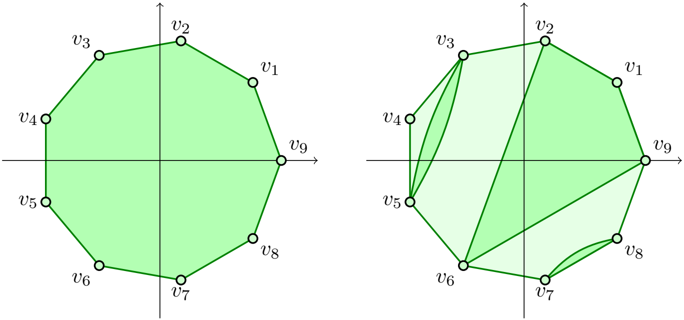
Rotations in Regensburg



Subsets, Subdisks and Rotations

Recall: for each $A \subset [n]$ of size $k > 1$ with $B = [n] - A$ we have defined a subset of vertices V_A, a subdisk P_A, a rotation δ_A and a subgroup $\textsc{Braid}_A = \textsc{Fix}(B)$ isomorphic to $\textsc{Braid}_k$.



Atomic Generators and Relations

When $A = \{i, j\}$ and e is the edge connecting v_i and v_j, we write δ_e for the corresponding rotation of the bigon P_A. The $\binom{n}{2}$ possible edges connecting vertices of P is denoted $\textsc{Edges}(P)$.

Definition (Atomic dual generators)

The set $T = \{\delta_e \mid e \in \textsc{Edges}(P)\}$ generates $\textsc{Braid}_n$ and its elements are the atomic dual generators.

Definition (Atomic dual relations)

When e_1 and e_2 are disjoint, the rotations δ_{e_1} and δ_{e_2} commute. When e_1, e_2 and e_3 form the boundary of a triangle and the subscripts indicate the **clockwise** order of the edges in the boundary, then $\delta_{e_1}\delta_{e_2} = \delta_{e_2}\delta_{e_3} = \delta_{e_3}\delta_{e_1}$. Together these two types of relations are the atomic dual relations.

Birman-Ko-Lee Presentation

Artin's original presentation used a linear ordering of the strands. In the 1990s Birman, Ko and Lee introduced an alternative presentation that used a circular ordering instead.

Definition (Birman-Ko-Lee Presentation)

The atomic dual generators and atomic dual relations form the Birman-Ko-Lee presentation of $\textsc{Braid}_n$.

$$\left\langle \{\delta_e\} \;\middle|\; \begin{array}{cr} \delta_{e_1}\delta_{e_2} = \delta_{e_2}\delta_{e_1} & e_1, e_2 \text{ disjoint} \\ \delta_{e_1}\delta_{e_2} = \delta_{e_2}\delta_{e_3} = \delta_{e_3}\delta_{e_1} & e_1, e_2, e_3 \text{ oriented triangle} \end{array} \right\rangle$$

Remark (Various generating sets)

The Artin generators are contained in the Birman-Ko-Lee generators, which are a subset of the set of all rotations, which are a subset of an even larger generating set.

Noncrossing Partitions

Definition (Partitions)

A partition of $[n]$ is a collection of pairwise disjoint subsets, called blocks, whose union is $[n]$. A singleton block is trivial and trivial blocks are omitted when describing a partition.

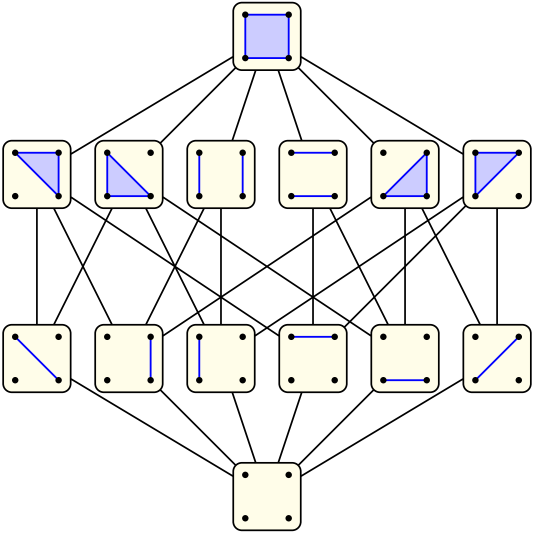
Definition (Noncrossing partitions)

A partition $\Pi = \{A_1, \ldots, A_\ell\}$ of $[n]$ is noncrossing when the convex hulls $\textsc{Conv}(V_{A_i})$ are pairwise disjoint.

Definition (Noncrossing Partition Lattice)

The set of all noncrossing partitions forms a poset under refinement NC_n. In other words, one partition is below another if and only if every block of the first is a subset of a block of the second. In fact, it is a lattice (i.e. meets and joins exists).

Noncrossing Partitions of a Square



Noncrossing Properties

Definition (Properties)

A poset is bounded if it has a unique maximum element $\widehat{\mathbf{1}}$ and a unique minimum element $\widehat{\mathbf{0}}$ and it is graded is every maximal chain has the same length. In a bounded graded poset the rank of an element x is the number of covering relations between x and the minimum element $\widehat{\mathbf{0}}$.

Remark (Noncrossing rank)

The noncrossing partition lattice is bounded and graded and the height of a partition is n minus the number of blocks. The identity has rank 0 and the top element has rank $n - 1$.

Noncrossing Braids

Definition (Noncrossing Braids)

When $\Pi = \{A_1, \ldots, A_\ell\}$ is a noncrossing partition of $[n]$, the subdisks P_{A_i} are pairwise disjoint and the rotations δ_{A_i} pairwise commute. Their product is the noncrossing braid δ_Π.

Remark (Rotations and Irreducible Partitions)

A partition with exactly one nontrivial block is called irreducible. Note that a noncrossing braid δ_Π is a rotation if and only if Π is an irreducible noncrossing partition.

Remark (Noncrossing permutations)

Every noncrossing partition Π also indexes a noncrossing permutation, the permutation associated to the noncrossing braid δ_Π with one nontrivial cycle for each nontrivial block of Π.

Dual Braid Relations

Remark (Atoms)

The atoms in a poset are the elements that cover the unique minimum element. In the noncrossing partition lattice the atoms are the edges which index the atom generators.

Definition (Cayley graphs and the Dual Garside Element)

The Hasse diagram of the noncrossing partition lattice can also be viewed as a portion of the right Cayley grpah of $\textsc{Braid}_n$ with respect to the atomic generating set $T = \{\delta_e\}$. The top element $\delta = \delta_{[n]}$ is called the dual Garside element.

Remark (Dual braid relations)

When $\Pi' \leq \Pi$ in NC_n, then $\delta_\Pi = \delta_{\Pi'}\delta_{\Pi''}$ for some other noncrossing braid Π''. We call this a dual braid relation.

Dual Presentation

The group generated by the noncrossing braids and subject to the dual braid relations is a dual braid presentation of $\textsc{Braid}_n$.

Definition (Dual Braid Presentation)

The noncrossing braids, also known as dual braids, and the dual braid relations form the dual braid presentation of $\textsc{Braid}_n$.

$$\textsc{Braid}_n = \left\langle \{\delta_\Pi\} \mid \delta_{\Pi'}\delta_{\Pi''} = \delta_\Pi \text{ for } \Pi' \leq \Pi \text{ in } \mathrm{NC}_n \right\rangle$$

Definition (Dual Braid Complex)

If we start with the right Cayley graph of $\textsc{Braid}_n$ with respect to the set of all nontrivial dual braid generators and then attach a simplex to each complete (directed) subgraph, then the result is Tom Brady's *n*-strand dual braid complex.

Properties of the Dual Braid Complex

In 2001 Tom Brady proved the following result.

Theorem (Properties of the Dual Braid Complex)

For each $n > 0$, the n-strand dual braid complex is contractible simplicial complex with a free vertex-transitive $\textsc{Braid}_n$*-action, so the quotient complex is a classifying space for* $\textsc{Braid}_n$.

Triangles in the 2-skeleton are labeled by dual braid relations.

Remark (Strong fundamental domain)

The full subcomplex on the vertices labeled by the noncrossing braids is a strong fundamental domain for the $\textsc{Braid}_n$ action on $\textsc{Compl}(\textsc{Braid}_n)$. Its simplicial structure is just the order complex of the noncrossing partition lattice.

The Origin of Orthoschemes

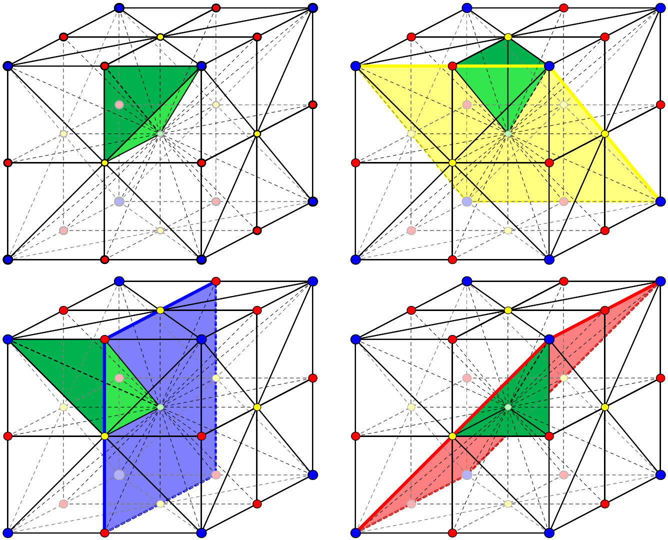
In 2010 Tom and I added a natural metric to the dual braid complex that we call the orthoscheme metric.

Remark (Origin)

The name comes from Coxeter's book on regular polytopes. Roughly speaking an orthoscheme is the type of shape you get when you metrically barycentrically subdivide a regular polytope and a standard orthoscheme is what you get when you barycentrically subdivide a cube of side length 2.

The image on the next slide shows a metric barycentric subdivision of a 3-cube. The image was originally designed to highlight how the B_3 Coxeter group acts on the subdivided cube. The focus here is on the shape of the simplices.

A Subdivided Cube



Boolean lattices and cubes

Let $\mathbb{R}^k$ be a euclidean vector space with a fixed ordered orthonormal basis $e_1, e_2, \ldots, e_k$.

Definition (Boolean lattices and cubes)

The boolean lattice $\textsc{Bool}_k$ is the poset of subsets of $[k]$ under inclusion. The unit k-cube $\textsc{Cube}_k$ in $\mathbb{R}^k$ is the set of vectors where each coordinate is in the interval $[0, 1]$ and its vertices are the vectors where each coordinate is either 0 or 1.

Definition (Special vectors)

There is a bijection between the elements in $\textsc{Bool}_k$ and the vertices of $\textsc{Cube}_k$, that sends $B \subset [k]$ to the vector $\mathbf{1}_B$ that is the sum of the basis vectors indexed by B, i.e. $\mathbf{1}_B = \sum_{i \in B} e_i$. We call these special vectors. At the extremes we write $\mathbf{1} = \mathbf{1}_{[k]} = (1, 1, \ldots, 1)$ and $\mathbf{0} = \mathbf{1}_\varnothing = (0, 0, \ldots, 0)$.

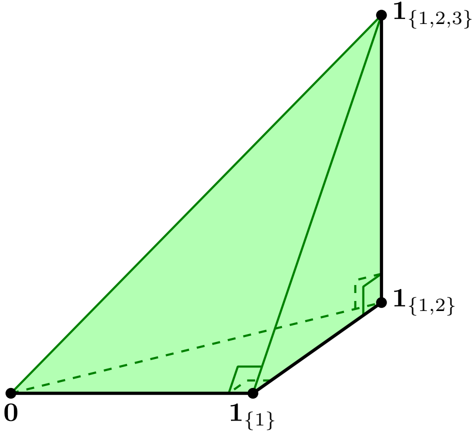
Orthoschemes

Definition (Orthoschemes)

A *k-orthoscheme* is a metric k-simplex formed as the convex hull of a piecewise geodesic path in $\mathbb{R}^k$ where each of the n individual geodesic segments are in pairwise orthogonal directions. When every individual segment has unit length, the shape that results is a standard *k*-orthoscheme.

In the simplicial structure on $\textsc{Cube}_k$, the $k!$ simplices correspond to the various ways to take the k steps from **0** to **1** in the coordinate directions. A 3-orthoscheme from the simplicial structure on $\textsc{Cube}_3$ is shown on the next slide. The edges of the piecewise geodesic path are thicker and darker than the others.

An orthoscheme



Dual Complex for $\mathbb{Z}^k$

Before describing the orthoscheme metric on the dual braid complex, let me describe the orthoscheme metric on the analogous complex for the free abelian group $\mathbb{Z}^k$.

Remark (Dual presentation of $\mathbb{Z}^k$)

Braid groups and free abelian groups are both examples of spherical Artin groups and they have a similar type of structure. We use the boolean lattice instead of the noncrossing partition lattice and the special vectors instead of the dual braids.

Definition (Dual complex for $\mathbb{Z}^k$)

The dual complex for $\mathbb{Z}^k$ starts with the right Cayley graph of $\mathbb{Z}^k$ with respect to the special vector generating set and then attaches a simplex to each complex subgraph. The lengths of the special vectors determine the shape of the euclidean simplices.

Two Types of Hyperplanes in $\mathbb{R}^k$

The result is a subdivision of the natural cubical structure on $\mathbb{R}^k$ with each cube built out of $n!$ standard orthoschemes. Alternatively, the orthoscheme tiling of $\mathbb{R}^k$ can be viewed as the cell structure of a simplicial hyperplane arrangement.

Definition (Two Types of Hyperplanes in $\mathbb{R}^k$)

Consider the hyperplane arrangement consisting of two types of hyperplanes. The first type are defined by the equations

$$x_i = \ell \text{ for all } i \in [k] \text{ and all } \ell \in \mathbb{Z}.$$

The second type are defined by the equations

$$x_i - x_j = \ell \text{ for all } i \neq j \in [k] \text{ and all } \ell \in \mathbb{Z}.$$

Together they partitions $\mathbb{R}^k$ into its standard orthoscheme tiling.

Affine Symmetric Group

The hyperplanes of the first type define the standard cubing of $\mathbb{R}^k$. The hyperplanes of the second type are closely related to the Coxeter complex of the affine symmetric group.

Remark (Affine Symmetric Group)

The affine symmetric group $\widetilde{\textsc{Sym}}_k$ is the euclidean Coxeter group of type $\widetilde{A}_{k-1}$. It is generated by reflections acting on an $(k-1)$-dimensional euclidean space but its action is usually described on $\mathbb{R}^k$ (where its roots and hyperplanes have elegant descriptions) and then restricted to a hyperplane orthogonal to the vector $\mathbf{1} \in \mathbb{R}^k$.

Roots, Hyperplanes and Coxeter Shapes

Definition (Roots and Hyperplanes)

The root system for $\widetilde{\mathrm{SYM}}_k$ is the set $\Phi = \{e_i - e_j \mid i \neq j \in [k]\}$. The span of this set is the hyperplane H orthogonal to the vector $\mathbf{1}$. The affine hyperplanes for this root system are defined by the equations $\langle x, \alpha \rangle = \ell$ for all $\alpha \in \Phi$ and all $\ell \in \mathbb{Z}$. These simplify to the equations of hyperplanes of the second type.

Definition ($\widetilde{A}_{k-1}$ Coxeter shape)

The second type of hyperplanes restricted to H partitions $H \cong \mathbb{R}^{k-1}$ into a reflection tiling by euclidean simplices. The common shape of these simplex is encoded in the extended Dynkin diagram of the type $\widetilde{A}_{k-1}$. We call this shape the Coxeter shape or Coxeter simplex of type $\widetilde{A}_{k-1}$.

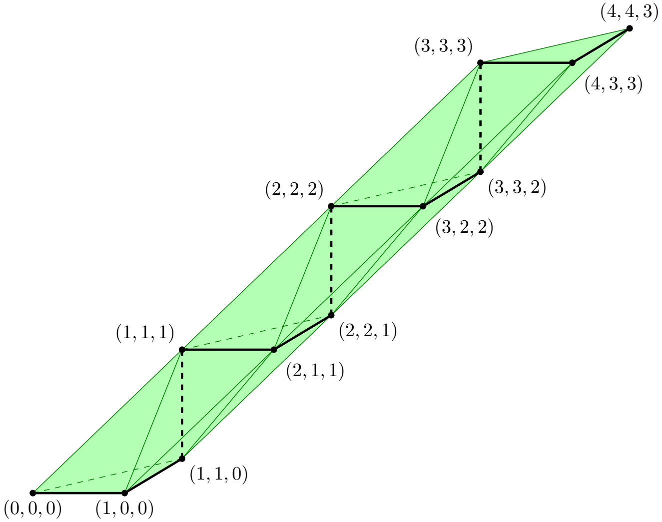
Columns

Definition (Coxeter shapes and columns)

When this hyperplane arrangemet is not restricted to a hyperplane orthogonal to the vector **1**, the closure of a connected component of the complementary region is an unbounded infinite column that is a metric product $\sigma \times \mathbb{R}$ where σ is a Coxeter simplex of type $\widetilde{A}$ and $\mathbb{R}$ is the real line. We call these the columns of $\mathbb{R}^k$.

One consequence of this column structure is that the standard orthoscheme tiling of $\mathbb{R}^k$ partitions the columns of $\mathbb{R}^k$ into a sequence of orthoschemes. We begin with an explicit example in $\mathbb{R}^3$. Let $\mathcal{C}$ be the unique column of $\mathbb{R}^3$ that contains the 3-simplex shown earlier.

A column of orthoschemes



A column in $\mathbb{R}^3$

Example (Column in $\mathbb{R}^3$)

The column $\mathcal{C}$ is defined by the inequalities $x_1 \geq x_2 \geq x_3 \geq x_1 - 1$ and its sides are the hyperplanes defined by the equations $x_1 - x_2 = 0$, $x_2 - x_3 = 0$ and $x_1 - x_3 = 1$.

Example (Vertices in the Column)

The vertices of $\mathbb{Z}^3$ contained in this column form a sequence $\{v_\ell\}_{\ell \in \mathbb{Z}}$ where the order of the sequence is determined by the inner product of these points with the special vector $\mathbf{1} = (1^3) = (1,1,1)$. Concretely the vertex v_ℓ is the unique point in $\mathbb{Z}^3 \cap \mathcal{C}$ such that $\langle v_\ell, \mathbf{1} \rangle = \ell \in \mathbb{Z}$. The vectors in this case are $v_{-1} = (0,0,-1)$, $v_0 = (0,0,0)$, $v_1 = (1,0,0)$, $v_2 = (1,1,0)$, $v_3 = (1,1,1)$, $v_4 = (2,1,1)$ and so on.

Shape of the Column

Example (Spiral of Edges)

Successive points in this list are connected by unit length edges in coordinate directions and this turns the list into a spiral of edges. Traveling up the spiral, the edges cycle through the possible directions in a predictable order: $x, y, z, x, y, z, \ldots$.

Example (Shape of the Column)

Any 3 consecutive edges in the spiral have a standard 3-orthoscheme as its convex hull and the union of these individual orthoschemes is the convex hull of the full spiral, which is also the full column $\mathcal{C}$. Metrically $\mathcal{C}$ is $\sigma \times \mathbb{R}$ where σ is an equilateral triangle also known as the $\widetilde{A}_2$ Coxeter simplex.

Columns in $\mathbb{R}^k$

Columns in $\mathbb{R}^k$ have many of the same properties.

Definition (Columns in $\mathbb{R}^k$)

A column $\mathcal{C}$ in $\mathbb{R}^k$ is defined by inequalities of the form

$$x_{\pi_1} + a_{\pi_1} \geq x_{\pi_2} + a_{\pi_2} \geq \ldots \geq x_{\pi_k} + a_{\pi_k} \ldots \geq x_{\pi_1} + a_{\pi_1} - 1$$

where $(\pi_1, \pi_2, \ldots, \pi_k)$ is a permutation of integers $(1, 2, \ldots, k)$ and $a = (a_1, a_2, \ldots, a_k)$ is a point in $\mathbb{Z}^k$.

Definition (Vertices in a Column in $\mathbb{R}^k$)

The vertices of $\mathbb{Z}^k$ contained in $\mathcal{C}$ form a sequence $\{v_\ell\}_{\ell \in \mathbb{Z}}$ where the order is determined by the inner product with the vector $\mathbf{1} = (1^k) = (1, 1, \ldots, 1)$. Concretely the vertex v_ℓ is the unique point in $\mathbb{Z}^k \cap \mathcal{C}$ such that $\langle v_\ell, \mathbf{1} \rangle = \ell \in \mathbb{Z}$.

Columns are Convex

Definition (Spiral of Edges)

Successive points are connected by unit length edges in coordinate directions and this turns the full list into a spiral of edges. The edges cycle through the possible directions in a predictable order based on the list $(\pi_1, \pi_2, \ldots, \pi_k)$.

Definition (Columns are Convex)

Any k consecutive edges in the spiral have a standard k-orthoscheme as its convex hull and the union of these individual orthoschemes is the convex hull of the full spiral, which is also the full column $\mathcal{C}$.

Columns are CAT(0)

Remark (Columns are CAT(0))

Metrically, $\mathcal{C}$ is $\sigma \times \mathbb{R}$ where σ is a Coxeter simplex of type $\widetilde{A}_{k-1}$. As a convex subset of $\mathbb{R}^k$, the full column is a CAT(0) space. It is also the metric product of the euclidean polytope σ and $\mathbb{R}$.

Definition (Dilated columns)

If the -1 in the final inequality defining a column in $\mathbb{R}^k$ is replaced by a $-\ell$ for some positive integer ℓ, then the shape described is a dilated column.

As a metric space, a dilated column is a metric direct product of the real line and a Coxeter shape of type $\widetilde{A}$ dilated by a factor of ℓ and as a dilation of a CAT(0) space, it is also a CAT(0) space. As a cell complex, a dilated column is the union of ℓ^{k-1} ordinary columns of $\mathbb{R}^k$ tiled by orthoschemes.

Dilated Columns

Some of these dilated columns are of particular interest.

Definition ((k, n)-dilated columns)

Let $n > k > 0$ be positive integers and let $\mathcal{C}$ be the full subcomplex of the orthoscheme tiling of $\mathbb{R}^k$ restricted to the vertices of $\mathbb{Z}^k$ that satisfy the strict inequalities

$$x_1 < x_2 < \cdots < x_k < x_1 + n.$$

We call $\mathcal{C}$ the (k, n)-dilated column in $\mathbb{R}^k$.

A point $x \in \mathbb{Z}^k$ is in $\mathcal{C}$ if and only if its coordinates are strictly increasing in value from left to right and the gap between the first and the last coordinate is strictly less than n. The (k, n)-dilated column $\mathcal{C}$ is a $(n - k)$ dilation of an ordinary column and thus a union of $(n - k)^{k-1}$ ordinary columns.

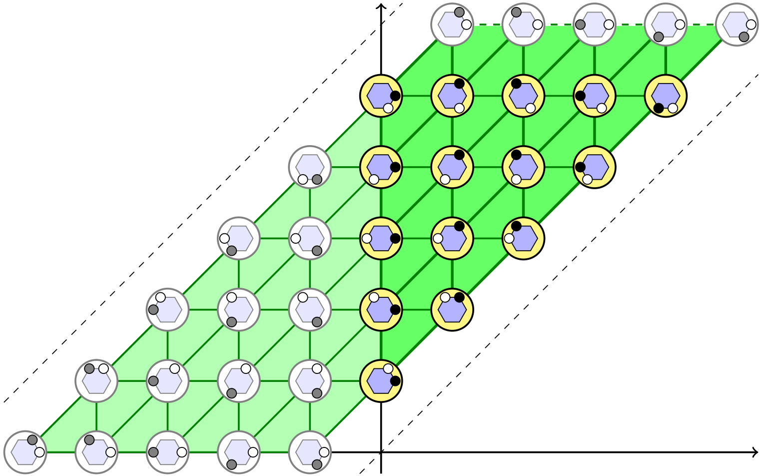
$(2, 6)$-dilated Column

Example ($(2, 6)$-dilated column)

When $k = 2$ and $n = 6$, the defining inequalities are $x < y < x + 6$. and a portion of the $(2, 6)$-dilated column $\mathcal{C}$ is shown on the next slide. Note that metrically $\mathcal{C}$ is an ordinary column dilated by a factor of 4, its cell structure is a union of $(6 - 2)^{2-1} = 4$ ordinary columns, and it is defined by the weak inequalities $x + 1 \leq y$ and $y \leq x + 5$.

The meaning of the vertex labels used in the figure are explained afterwards.

A Dilated Column



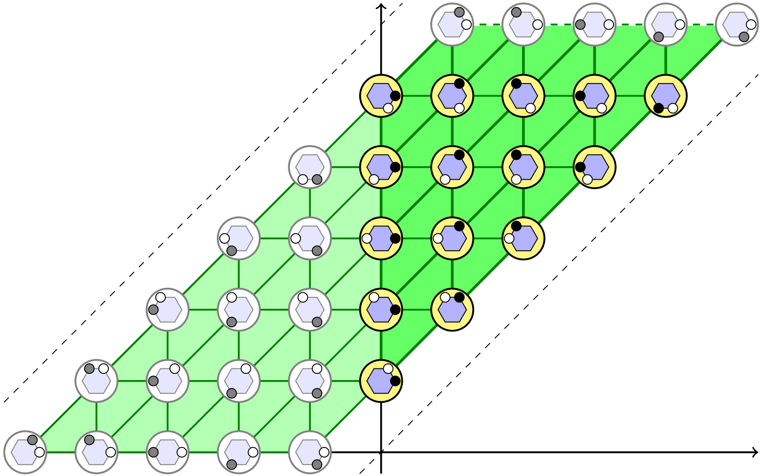
Labeled Robots on a Cycle

Example (Labeled Robots on a Cycle)

When this strip is quotiented by the portion of the $(6\mathbb{Z})^2$-action on $\mathbb{R}^2$ that stabilizes this strip, its vertices can be labeled by two labeled points in a hexagon. The black dot indicates the value of its *x*-coordinate mod 6 and the white dot indicates the value of its *y*-coordinate mod 6. The left vertex of the hexagon corresponds to $0 \mod 6$ and the residue classes proceed in a counter-clockwise fashion. The five hexagons on the *y*-axis, for example have *x*-coordinate equal to $0 \mod 6$ and *y*-coordinate ranging from 1 to $5 \mod 6$.

The answer to Exercise 1 is the annulus formed by identifying the top and bottom edges of the region shown according to their vertex labels.

A Dilated Column



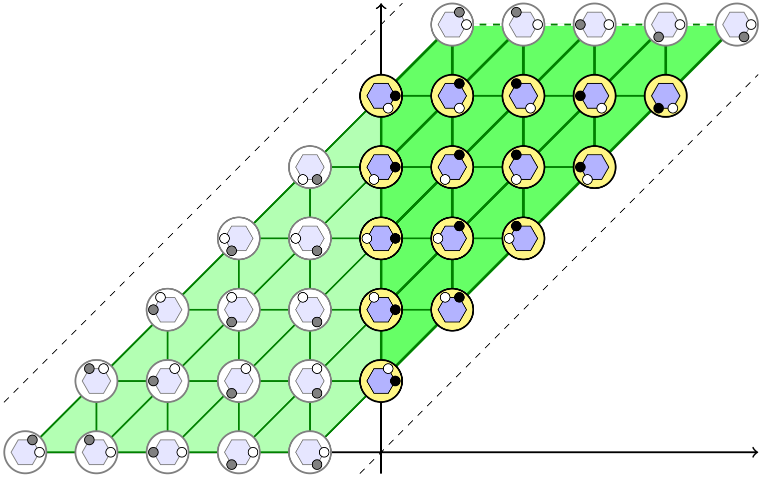
Unlabeled Robots on a Cycle

Example (Unlabeled Robots on a Cycle)

The unlabeled version is formed by further quotienting to remove the distinction between black and white dots. In particular the 5 vertices shown on the horizontal line $y = 6$ are identified with the 5 vertices on the vertical line $x = 0$. This identification can be realized by the glide reflection sending (x, y) to $(y, x + 2)$, a map which also generates the unlabeled stabilizer of the $(2, 6)$-dilated column.

The heavily shaded region is a fundamental domain for this $\mathbb{Z}$-action and the unlabeled orthoscheme configuration space is the formed by identifying its horizontal and vertical edges with a half-twist forming a Möbius strip. The heavily shaded labels represent the vertices in the quotient. This is the answer to Exercise 2.

A Dilated Column



Robots on a Cycle: Universal Covers

The universal covers in Exercise 3 are all of the form $\sigma \times \mathbb{R}$:

Example (Universal Covers)

- When $k = 3$, σ is a dilated equilateral triangle dilated by a factor of 3 and the cover contains $3^2 = 9$ ordinary columns.
- When $k = 4$, σ is an $\widetilde{A}_3$ tetrahedron dilated by a factor of 2 and the cover contains $2^3 = 8$ ordinary columns.
- When $k = 5$, σ is an $\widetilde{A}_4$ shape and the cover contains a single ordinary column ($1^4 = 1$).
- When $k = 6$, the only motion is when all 6 Robots move at once, σ is a point, the cover is just $\mathbb{R}$ with $0^5 = 0$ columns.

Robots on a Cycle: Labeled and Unlabeled

Example (Labeled Robots)

The labeled versions are quotients of the CAT(0) dilated columns that are the universal covers. In each case the $\mathbb{Z}$-action is generated by a pure translation and the quotient is metrically $\sigma \times \mathbb{S}^1$ where the circumference of the circle depends on the value of k.

Example (Unlabeled Robots)

The unlabeled versions are quotients of the CAT(0) dilated columns that are the universal covers. In each case the $\mathbb{Z}$-action is generated by loxodromic isometry that moves every vertex in the spiral up one step. The quotient is a twisted product of σ and a circle.

General Case

Proposition (General Case)

In the general case of k robots on an n-cycle,

- *the universal cover is always a* $\mathrm{CAT}(0)$ *dilated column,*
- *the labeled robot case is always a non-positively curved direct product of a dilated affine symmetric group Coxeter shape and a metric circle, and*
- *the unlabeled robot case is always a non-positively curved twisted direct product of a dliated affine symmetric group Coxeter shape and a metric circle.*

And this is the answer to Exercise 4.