Polynomial perturbations of bilinear functionals and Hessenberg matrices

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Abstract

This paper deals with symmetric and non-symmetric polynomial perturbations of symmetric quasi-definite bilinear functionals. We establish a relation between the Hessenberg matrices associated with the initial and the perturbed functionals using LU and QR factorizations. Moreover we give an explicit algebraic relation between the sequences of orthogonal polynomials associated with both functionals.

Key words: Perturbation of a bilinear functional, Hessenberg matrix, LU factorization, QR factorization.
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1 Introduction.

Given a linear functional $\mathbf{S}$ in the linear space $\mathbb{P}$ of polynomials with real coefficients, let us consider the Hankel matrix $M = (\mu_{i+j})_{i,j=0}^{\infty}$, where $\mu_n = \mathbf{S}(x^n)$. This is the Gram matrix of standard moments associated with the bilinear functional $\mathbf{L}(p,q) := \mathbf{S}(pq)$ in terms of the canonical basis $\{x^n\}_{n\in\mathbb{N}}$. If the leading principal submatrices of M are nonsingular [5], $\mathbf{S}$ is said to be *quasi-definite*. Moreover, there exists a sequence of monic polynomials $\{P_n\}$ orthogonal with respect to $\mathbf{S}$, i.e.,

(1) $\deg(P_n) = n, \quad n \geq 0,$
(2) $\mathbf{S}(P_n P_m) = K_n \delta_{n,m}, \quad K_n \neq 0, \quad n, m \geq 0.$

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Every sequence of polynomials orthogonal with respect to a quasi-definite linear functional satisfies a three-term recurrence relation,

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 1, \quad \gamma_n \neq 0, \quad \text{(see [5])}.$$

The previous recurrence relation can be rewritten as

$$xv_p = Jv_p,$$

where $v_p = \left[P_0(x) \; P_1(x) \; P_2(x) \; ... \right]^t$, and

$$J = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & 0 & \cdots \\ 0 & \gamma_2 & \beta_2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix J is said to be the *Monic Jacobi matrix* associated with $\{P_n\}$ (or with $\mathbf{S}$). Notice that the tridiagonal matrix J is associated with the operator multiplication by x with respect to the monic orthogonal family $\{P_n\}$. This matrix gives some information about analytic properties of the sequence $\{P_n\}$, for example the distribution of the zeros of the polynomials as well as certain asymptotic properties.

Given a quasi-definite linear functional $\mathbf{S}$, consider the linear functional $\tilde{\mathbf{S}} := \pi\mathbf{S}$ defined by $(\pi\mathbf{S})(p) := \mathbf{S}(\pi p)$, where π and p denote polynomials. The functional $\pi\mathbf{S}$ is said to be a *polynomial perturbation* of the linear functional $\mathbf{S}$. This kind of perturbation was first considered by Christoffel in 1858 — he used a polynomial of degree one as π. This is the reason why this particular perturbation is known as *Christoffel transformation* in the literature of orthogonal polynomials (see [3], [13], and [14]). It is also called *Darboux transformation without parameter* in the framework of bispectral problems and evolution equations (see [7], [9]). Darboux transformations were rediscovered in the $90's$ because of their multiple applications in several areas, e.g. Quantum mechanics [10], Theory of nonlinear integrable systems [11], Computation of Gauss quadratures with multiple free and fixed knots [6], Bispectral transformations in orthogonal polynomials [7,8], and so on.

Let us consider the matrix interpretation of the Christoffel transformation. Given the monic Jacobi matrix associated with a linear functional $\mathbf{S}$ and the corresponding sequence of monic orthogonal polynomials $\{P_n\}$, if $\alpha \in \mathbb{R}$ and $P_n(\alpha) \neq 0$ for every n, then the unique LU factorization $J - \alpha I = LU$ of $J - \alpha I$ exists, where L denotes a lower triangular and bidiagonal matrix with ones in the main diagonal, and U denotes an upper triangular and bidiagonal matrix. The matrix $J_1 = UL + \alpha I$ is a new tridiagonal matrix, that is, the

Jacobi matrix associated with the sequence of polynomials orthogonal with respect to the Darboux transform $\tilde{\mathbf{S}} = (\mathbf{x} - \alpha)\mathbf{S}$ of $\mathbf{S}$.

While the matrix interpretation of the polynomial perturbation $(\mathbf{x} - \alpha)\mathbf{S}$ is given in terms of the LU factorization, the matrix interpretation of the polynomial perturbation $\mathbf{x^2 S}$ is given in terms of the QR factorization. In [1], it is proven that a single step of the QR method applied to the Jacobi matrix of an orthogonal polynomial system associated with a positive definite linear functional $\mathbf{S}$ corresponds to finding the Jacobi matrix of the orthogonal polynomial system associated with $\mathbf{x^2 S}$.

Let us consider now a general bilinear functional $\mathbf{L}$ defined in the linear space of polynomials with real coefficients. The corresponding Gram matrix of standard moments is given by $(M_L)_{i,j} = \mu_{i-1,j-1}$, $i, j \geq 1$, where $\mu_{i,j} = \mathbf{L}(x^i, x^j)$. If the leading principal submatrices of M_L are nonsingular, $\mathbf{L}$ is said to be quasi-definite. In such a situation there exists a sequence of polynomials $\{P_n\}$ orthogonal with respect to $\mathbf{L}$. In this case, $\{P_n\}$ does not satisfy a three-term recurrence relation anymore. Examples of such a kind of orthogonal polynomials have been considered in [2] and [4]. The matrix associated with the operator multiplication by x is a lower Hessenberg matrix. We will refer to it as the *Hessenberg matrix associated with* $\mathbf{L}$.

The aim of this paper is to define polynomial perturbations of a symmetric quasi-definite bilinear functional that extend the definitions given in the linear case as well as to give an algebraic relation between the Hessenberg matrices associated with the original and the perturbed functional.

The structure of the paper is the following. In section 2 we present some basic concepts related with symmetric bilinear functionals and the corresponding Hessenberg matrices. In section 3 we study the non-symmetric polynomial perturbations of a symmetric quasi-definite bilinear functional. In particular, we show that the Hessenberg matrix associated with the bilinear functional $\tilde{\mathbf{L}}(p, q) := \mathbf{L}((x - \alpha)p, q)$ can be obtained by the application of a Darboux transformation without parameter to the Hessenberg matrix associated with $\mathbf{L}$. Explicit algebraic relations between the sequences of polynomials orthogonal with respect to $\mathbf{L}$ and $\tilde{\mathbf{L}}$ are also obtained. In section 4 we present the symmetric polynomial perturbations of symmetric quasi-definite bilinear functionals. In particular we study the functional $\tilde{\mathbf{L}}$ given by $\tilde{\mathbf{L}}(p, q) = \mathbf{L}(xp, xq)$. We prove that this functional extends the definition of $\mathbf{x^2 S}$, when $\mathbf{S}$ is a linear functional. In the positive definite case, we find the corresponding Hessenberg matrix by the application of one step of the infinite QR algorithm to the Hessenberg matrix associated with the original functional. When the original functional is only quasi-definite a similar transformation is obtained in terms of the hyperbolic QR factorization.

2 Symmetric bilinear functionals and Hessenberg matrices.

In this section we give some basic concepts related with symmetric bilinear functionals and the corresponding Hessenberg matrices, that will be useful in the next sections.

Definition 2.1 *Let* $\boldsymbol{L} : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ *be a function such that*

(1) $\boldsymbol{L}(p + \tilde{p}, q) = \boldsymbol{L}(p, q) + \boldsymbol{L}(\tilde{p}, q)$,
(2) $\boldsymbol{L}(\lambda p, q) = \lambda \boldsymbol{L}(p, q)$,

and such that the two previous properties also hold for the second argument. Then, $\mathbf{L}$ *is said to be a* bilinear functional.

Definition 2.2 *The bilinear functional* $\mathbf{L}$ *is said to be* symmetric *if*

$$\mathbf{L}(p, q) = \mathbf{L}(q, p).$$

In the other case, we say that $\mathbf{L}$ *is* non-symmetric.

Definition 2.3 *Given a bilinear symmetric functional* $\mathbf{L}$*, consider the canonical basis of polynomials* $\{x^n\}$*, and let* $X = [1, x, x^2, \ldots]^t$*. Then, the* standard moments *associated with* $\mathbf{L}$ *are* $\mu_{n,m} = \mathbf{L}(x^n, x^m)$ *and the Gram matrix* M_L *defined by* $(M_L)_{i,j} = \mu_{i-1,j-1}$*,* $i, j \geq 1$*, is said to be the* matrix of standard moments *associated with* $\mathbf{L}$*. If the moments are computed in terms of another basis* $\{P_n\}$ *different from the canonical one, then the* modified moments $u_{n,m}$ *are obtained, where*

$$u_{n,m} = \mathbf{L}(P_n, P_m).$$

The matrix $M_{L,P}$ *defined by*

$$(M_{L,P})_{i,j} = u_{i-1,j-1}, \quad i, j \geq 1,$$

is said to be the matrix of modified moments *associated with* $\mathbf{L}$ *with respect to* $\{P_n\}$*.*

Definition 2.4 *A symmetric bilinear functional* $\mathbf{L}$ *is said to be* quasi-definite *(*$\mathbf{L}$ *generates a pseudo-inner product) if the corresponding matrix of standard moments* M_L *is quasi-definite, i.e., the leading principal submatrices of* M_L *are nonsingular.*

A symmetric bilinear functional L *is said to be* positive definite*(or an inner product) if the leading principal minors of* M_L *are positive.*

Definition 2.5 *Given a symmetric bilinear functional* $\boldsymbol{L}$*, a sequence of monic polynomials* $\{P_n\}$ *is said to be* orthogonal *with respect to* $\boldsymbol{L}$ *if*

(1) $deg(P_n) = n$, *for every* $n \geq 0$,
(2) $\boldsymbol{L}(P_n, P_m) = K_n\delta_{n,m}$, $K_n \neq 0$.

A weaker property than the previous one is defined below.

Definition 2.6 *Given a bilinear functional* $\boldsymbol{L}$ *defined on the space of polynomials with real coefficients, a sequence of polynomials* $\{P_n\}$ *is said to be* left orthogonal *with respect to* $\boldsymbol{L}$ *if*

(1) $deg(P_n) = n$, *for every* $n \geq 0$.
(2) $\boldsymbol{L}(P_n, x^m) = K_n\delta_{n,m}$, *with* $K_n \neq 0$, *for* $m \leq n$.

Remark 2.7 *An equivalent definition for* right orthogonal *polynomials is possible. It is enough to replace condition (2) by*

$$\boldsymbol{L}(x^m, P_n) = K_n\delta_{m,n}, \quad with \quad K_n \neq 0, \quad with\, m \leq n.$$

Proposition 2.8 *Assume that* $\mathbf{L}$ *is a symmetric quasi-definite bilinear functional. Then there exists a sequence of polynomials* $\{P_n\}$ *orthogonal with respect to* $\mathbf{L}$.

PROOF. Let

$$P_n = det \begin{pmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{n0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{n1} \\ \ldots & \ldots & \ldots & \ldots \\ \mu_{0,n-1} & \mu_{1,n-1} & \cdots & \mu_{n,n-1} \\ 1 & z & \ldots & z^n \end{pmatrix}$$

be a polynomial associated with $\mathbf{L}$. It is straightforward to prove that if $\mathbf{L}$ is quasi-definite, then P_n is a polynomial of degree n and

$$\mathbf{L}(P_n, z^j) = det \begin{pmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{n0} \\ \mu_{01} & \mu_{11} & \cdots & \mu_{n1} \\ \ldots & \ldots & \ldots & \ldots \\ \mu_{0,n-1} & \mu_{1,n-1} & \cdots & \mu_{n,n-1} \\ \mu_{0j} & \mu_{1j} & \cdots & \mu_{nj} \end{pmatrix}.$$

Thus, from elementary properties of determinants and taking into account that $\mathbf{L}$ is quasi-definite, it follows that

$$\mathbf{L}(P_n, z^j) = \begin{cases} 0, & \text{for } 0 \le j \le n-1, \\ K_n \neq 0 & \text{for } j = n. \end{cases} \qquad \square$$

Definition 2.9 *Given a symmetric quasi-definite bilinear functional* $\mathbf{L}$ *and the corresponding sequence of orthogonal polynomials* $\{P_n\}$*, consider the semi-infinite lower Hessenberg matrix* H *such that*

$$x v_p = H v_p, \tag{2.1}$$

where $v_p = [P_0(x), P_1(x), \ldots, P_n(x), \ldots]^t$. H *is said to be the* Hessenberg matrix associated with $\{P_n\}$.

Lemma 2.10 *Let* $\mathbf{L}$ *be a symmetric quasi-definite bilinear functional and let* $\{P_n\}$ *be the corresponding sequence of monic orthogonal polynomials, then*

$$\mathbf{L}(v_p, v_p^t) = D_p,$$

where D_p *is a nonsingular diagonal matrix.*

In the particular case when $\mathbf{L}$ is positive definite, there exists a sequence of orthonormal polynomials $\{\tilde{P}_n\}$. Observe that in this case, $\mathbf{L}(v_{\tilde{P}}, v_{\tilde{P}}^t) = I$, where I denotes the identity matrix.

3 Non-symmetric perturbations of symmetric bilinear functionals.

Consider a symmetric quasi-definite bilinear functional $\mathbf{L}$. Next we consider the following perturbations of $\mathbf{L}$:

- The perturbed functional is bilinear but non-symmetric. This situation appears if we perturb just one of the arguments of $\mathbf{L}$, that is, if we consider the functional $\tilde{\mathbf{L}}$ given by either

  $$\tilde{\mathbf{L}}(p,q) := \mathbf{L}(\pi\, p, q), \quad \text{or} \quad \tilde{\mathbf{L}}(p,q) := \mathbf{L}(p, \pi\, q),$$

  where π denotes a polynomial. We can also obtain a non-symmetric bilinear functional when we multiply both arguments of $\mathbf{L}$ by different polynomials, i.e.,

  $$\tilde{\mathbf{L}}(p,q) := \mathbf{L}(\pi_1\, p, \pi_2\, q).$$

  We will refer to this kind of perturbations as *non-symmetric polynomial perturbations of* $\boldsymbol{L}$.

- The perturbed functional is bilinear and symmetric. That is the case if we, for example, multiply both arguments of $\mathbf{L}$ by the same polynomial,

$$\tilde{\mathbf{L}}(p,q) := \mathbf{L}(\pi\, p, \pi\, q).$$

  We will refer to this kind of perturbations as *symmetric polynomial perturbations.*

In this section we study non-symmetric perturbations of symmetric bilinear functionals in which only one of its arguments is modified.

Let $\mathbf{L}$ be a symmetric quasi-definite bilinear functional. We consider the following standard cases:

$$\tilde{\mathbf{L}}_{\mathbf{1}}(p,q) := \mathbf{L}((x-\alpha)p, q), \tag{3.1}$$

where $\alpha \in \mathbb{R}$, and

$$\tilde{\mathbf{L}}_{\mathbf{2}}(p,q) := \mathbf{L}(p, (x-\alpha)q). \tag{3.2}$$

Notice that both functionals extend in a natural way the polynomial perturbation $(\mathbf{x}-\alpha)\mathbf{S}$, where $\mathbf{S}$ is a linear functional, to general bilinear functionals. If $\mathbf{L}$ can be expressed in terms of a linear functional, then

$$\tilde{\mathbf{L}}_{\mathbf{1}}(p,q) = \mathbf{L}((x-\alpha)p, q) = \mathbf{S}((x-\alpha)pq) = (\mathbf{x}-\alpha)\mathbf{S}(pq).$$

The same happens if we consider $\tilde{\mathbf{L}}_{\mathbf{2}}$.

Remark 3.1 *Observe that the bilinear functionals* $\tilde{\mathbf{L}}_1$ *and* $\tilde{\mathbf{L}}_2$ *are non-symmetric in general. Let us consider an example. Assume that* $\mathbf{L}$ *is the following symmetric bilinear functional [2]*

$$\mathbf{L}(p,q) := \int_{\mathbb{R}} pq d\mu_0 + \int_{\mathbb{R}} p'q' d\mu_1.$$

Consider $\alpha = 0$. *Then the functionals* $\tilde{\mathbf{L}}_{\mathbf{1}}$ *and* $\tilde{\mathbf{L}}_{\mathbf{2}}$ *defined as in (3.1) and (3.2) are both non-symmetric:*

$$\tilde{\mathbf{L}}_{\mathbf{1}}(p,q) = \mathbf{L}(xp,q) = \int_{\mathbb{R}} xpq d\mu_0 + \int_{\mathbb{R}} (p+xp')q' d\mu_1 \neq \tilde{\mathbf{L}}_{\mathbf{1}}(q,p).$$

$$\tilde{\mathbf{L}}_{\mathbf{2}}(p,q) = \mathbf{L}(p,xq) = \int_{\mathbb{R}} xpq d\mu_0 + \int_{\mathbb{R}} p'(q+xq') d\mu_1 \neq \tilde{\mathbf{L}}_{\mathbf{2}}(q,p).$$

In the sequel, we analyze the perturbation that generates $\tilde{\mathbf{L}}_{\mathbf{1}}$ from $\mathbf{L}$ and we find an algebraic relation between the Hessenberg matrices associated with both of them. Similar results are obtained for the other case. Let $\{P_n\}$ be the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$ and let H be the corresponding Hessenberg matrix. We also assume that $P_n(\alpha) \neq 0$, for all n.

Lemma 3.2 *Let $\tilde{\mathbf{L}}_\mathbf{1}(v_p, v_p^t)$ be the matrix of modified moments (Definition 2.3) associated with $\tilde{\mathbf{L}}_\mathbf{1}$ with respect to $\{P_n\}$. Then,*

$$\tilde{\mathbf{L}}_\mathbf{1}(v_p, v_p^t) = (H - \alpha I)D_p,$$

where $D_p := \mathbf{L}(v_p, v_p^t)$.

PROOF. Taking into account the definition of $\tilde{\mathbf{L}}_\mathbf{1}$,

$$\tilde{\mathbf{L}}_\mathbf{1}(v_p, v_p^t) = \mathbf{L}((x - \alpha)v_p, v_p^t).$$

From (2.1), we deduce that $(x - \alpha)v_p = (H - \alpha I)v_p$. Then,

$$\tilde{\mathbf{L}}_\mathbf{1}(v_p, v_p^t) = \mathbf{L}((H - \alpha I)v_p, v_p^t) = (H - \alpha I)D_p. \qquad \square$$

Since $\tilde{\mathbf{L}}_\mathbf{1}$ is a non-symmetric functional, there is not a sequence of monic polynomials orthogonal with respect to $\tilde{\mathbf{L}}_\mathbf{1}$. However, the next proposition shows that there exists a sequence of monic polynomials left orthogonal with respect to $\tilde{\mathbf{L}}_\mathbf{1}$

Proposition 3.3 *The sequence of monic polynomials $\{R_n\}$ given by*

$$R_n(x) = \frac{P_{n+1}(x) - \frac{P_{n+1}(\alpha)}{P_n(\alpha)}P_n(x)}{x - \alpha}$$

is left orthogonal with respect to $\tilde{\mathbf{L}}_\mathbf{1}$.

PROOF.

(1) It is obvious that $\deg(R_n) = n$ for $n \geq 0$.
(2) If $0 \leq k < n$, then

$$\tilde{\mathbf{L}}_\mathbf{1}(R_n, x^k) = \mathbf{L}((x - \alpha)R_n, x^k) = \mathbf{L}\left(P_{n+1}(x) - \frac{P_{n+1}(\alpha)}{P_n(\alpha)}P_n(x), x^k\right).$$

Since $\{P_n\}$ is orthogonal with respect to $\mathbf{L}$, from the above expression we deduce

$$\tilde{\mathbf{L}}_\mathbf{1}(R_n, x^k) = 0, \quad \text{for} \quad 0 \leq k < n.$$

(3) On the other hand,

$$\tilde{\mathbf{L}}_{\mathbf{1}}(R_n, x^n) = -\frac{P_{n+1}(\alpha)}{P_n(\alpha)}\mathbf{L}(P_n, x^n) \neq 0. \qquad \square$$

Let H_1 be the Hessenberg matrix associated with $\{R_n\}$. Next we establish an algebraic relation between the Hessenberg matrices H and H_1. Notice that, since $\{P_n\}$ and $\{R_n\}$ are monic polynomial bases, there exists a lower triangular matrix L with ones in the main diagonal such that

$$v_p = L v_r. \tag{3.3}$$

where $v_r = [R_0(x), R_1(x), ..., R_n(x), ...]^t$. In the following proposition we prove that L is also the lower triangular factor obtained from the LU factorization without pivoting of the Hessenberg matrix H associated with $\{P_n\}$.

Proposition 3.4 *Let $\boldsymbol{L}$ be a symmetric quasi-definite bilinear functional and let $\{P_n\}$ be the corresponding sequence of monic orthogonal polynomials. Consider the functional defined in (3.1) as well as the sequence of left orthogonal polynomials given in Proposition 3.3. Let L be the lower triangular matrix with ones in the main diagonal given by (3.3). Then, $H - \alpha I = LU$, where U denotes a nonsingular upper triangular matrix.*

PROOF. From Lemma 3.2,

$$(H - \alpha I)D_p = \tilde{\mathbf{L}}_{\mathbf{1}}(v_p, v_p^t) = \tilde{\mathbf{L}}_{\mathbf{1}}(Lv_r, v_r^t L^t) = L\tilde{\mathbf{L}}_{\mathbf{1}}(v_r, v_r^t)L^t.$$

Therefore,

$$H - \alpha I = L\tilde{\mathbf{L}}_{\mathbf{1}}(v_r, v_r^t)L^t D_p^{-1}.$$

Notice that

(1) L^t is an upper triangular matrix,
(2) The matrix $\tilde{\mathbf{L}}_{\mathbf{1}}(v_r, v_r^t)$ is also an upper triangular matrix, since $\{R_n\}$ is left orthogonal with respect to $\tilde{\mathbf{L}}_{\mathbf{1}}$ and $\tilde{\mathbf{L}}_{\mathbf{1}}(R_n, R_k) = 0$ if $0 \leq k < n$,
(3) D_p^{-1} is a diagonal matrix.

Then, if

$$U := \tilde{\mathbf{L}}_{\mathbf{1}}(v_r, v_r^t)L^t D_p^{-1}, \tag{3.4}$$

the main result follows. The matrix U is a nonsingular matrix since $\tilde{\mathbf{L}}_{\mathbf{1}}(v_r, v_r^t)$, L^t and D_p^{-1} are nonsingular matrices. $\square$

Remark 3.5 *Observe that the LU factorization of $H - \alpha I$ exists since $P_n(\alpha) \neq 0$, for all n. Moreover, it is easy to prove that U is an upper bidiagonal matrix taking into account that $H - \alpha I$ is lower Hessenberg.*

The next theorem shows that the matrix H_1 can be obtained as the Darboux transform without parameter and with shift α of H.

Theorem 3.6 *If $H - \alpha I = LU$ denotes the LU factorization without pivoting of $H - \alpha I$, then,*

$$H_1 = UL + \alpha I.$$

PROOF. Taking into account Definition 2.9

$$\tilde{\mathbf{L}}_{\mathbf{1}}((x-\alpha)v_r, v_r^t) = \tilde{\mathbf{L}}_{\mathbf{1}}((H_1 - \alpha I)v_r, v_r^t) = (H_1 - \alpha I)\tilde{\mathbf{L}}_{\mathbf{1}}(v_r, v_r^t). \qquad (3.5)$$

On the other hand, considering (3.3)

$$\tilde{\mathbf{L}}_{\mathbf{1}}((x-\alpha)v_r, v_r^t) = \tilde{\mathbf{L}}_{\mathbf{1}}((x-\alpha)L^{-1}v_p, v_p^t L^{-t}) = L^{-1}\tilde{\mathbf{L}}_{\mathbf{1}}((x-\alpha)v_p, v_p^t)L^{-t}. \quad (3.6)$$

From Lemma 3.2,

$$L^{-1}\tilde{\mathbf{L}}_{\mathbf{1}}((x-\alpha)v_p, v_p^t)L^{-t} = L^{-1}(H-\alpha I)^2 D_p L^{-t}. \qquad (3.7)$$

Then, comparing (3.5) and (3.6), and replacing (3.7), we get

$$(H_1 - \alpha I)\tilde{\mathbf{L}}_{\mathbf{1}}(v_r, v_r^t) = L^{-1}(H-\alpha I)^2 D_p L^{-t}.$$

Finally, taking into account the LU factorization of $H - \alpha I$ and (3.4), we get

$$H_1 - \alpha I = L^{-1}(H-\alpha I)^2 U^{-1} = L^{-1}LULUU^{-1} = UL. \qquad \square$$

A finite version of Theorem 3.6 is given below. We use the following notation: Given a matrix A, $(A)_n$ denotes the leading principal submatrix of order n of A.

Theorem 3.7 *If $(H - \alpha I)_n = L_n U_n$ denotes the LU factorization without pivoting of $(H - \alpha I)_n$, then*

$$(H_1)_{n-1} = (U_n L_n)_{n-1} + \alpha (I)_{n-1},$$

i.e., the leading principal submatrix of order $n-1$ of H_1 is the leading principal submatrix of order $n-1$ of $U_n L_n + \alpha (I)_n$.

PROOF.

Consider the unique LU factorization of $H - \alpha I$.

$$
H-\alpha I=\left[\begin{array}{cccc|cc}
h_{11}-\alpha & h_{12} & \cdots & 0 & 0 & \cdots \\
h_{21} & h_{22}-\alpha & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
h_{n1} & h_{n2} & \cdots & h_{nn}-\alpha & h_{n,n+1} & \cdots \\
\hline
h_{n+1,1} & h_{n+1,2} & \cdots & h_{n+1,n} & h_{n+1,n+1}-\alpha & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots &
\end{array}\right]=
$$

$$
\left[\begin{array}{cccc|cc}
1 & 0 & \cdots & 0 & 0 & \cdots \\
l_{21} & 1 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
l_{n1} & l_{n2} & \cdots & 1 & 0 & \cdots \\
\hline
l_{n+1,1} & l_{n+1,2} & \cdots & l_{n+1,n} & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots &
\end{array}\right]
\left[\begin{array}{cccc|cc}
u_1 & h_{12} & \cdots & 0 & 0 & \cdots \\
0 & u_2 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & u_n & h_{n,n+1} & \cdots \\
\hline
0 & 0 & \cdots & 0 & u_{n+1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots &
\end{array}\right].
$$

If we reverse the order of the factors L and U, then we get

$$
H_1-\alpha I=\left[\begin{array}{cccc|cc}
u_1 & h_{12} & \cdots & 0 & 0 & \cdots \\
0 & u_2 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & u_n & h_{n,n+1} & \cdots \\
\hline
0 & 0 & \cdots & 0 & u_{n+1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots &
\end{array}\right]
\left[\begin{array}{cccc|cc}
1 & 0 & \cdots & 0 & 0 & \cdots \\
l_{21} & 1 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
l_{n1} & l_{n2} & \cdots & 1 & 0 & \cdots \\
\hline
l_{n+1,1} & l_{n+1,2} & \cdots & l_{n+1,n} & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots &
\end{array}\right].
$$

It is obvious that

$$
(H_1)_n-\alpha(I)_n=U_nL_n+h_{n,n+1}\cdot\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
l_{n+1,1} & l_{n+1,2} & \cdots & l_{n+1,n}
\end{bmatrix}.
$$

Thus, $(H_1)_n-\alpha(I)_n$ and U_nL_n differ only in the last row. Then, deleting the last row and column of the matrices appearing in both sides of the previous expression, we obtain the result. $\square$

Along this section we have considered non-symmetric polynomial perturbations of a symmetric quasi-definite bilinear functional when the polynomial that multiplies one of its arguments has degree one. It is obvious how to extend such results to the general case where the polynomial π has a higher degree. It is enough to take into account that, given a symmetric quasi-definite bilinear functional $\mathbf{L}$, if $\{P_n\}$ denotes the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$ and H is the Hessenberg matrix associated with $\{P_n\}$, then

$$\pi(x)v_p = \pi(H)v_p,$$

where $\pi(H)$ is a matrix polynomial obtained by replacing the variable x in $\pi(x)$ by H. Moreover, if $\deg(\pi) = s$, then it can be considered the sequence $\{R_n\}$ given by

$$R_n(x) = \frac{P_{n+s}(x) + \sum_{i=n}^{n+s-1} a_{n,i}P_i(x)}{\pi(x)},$$

where the coefficients $a_{n,i}$ are chosen so that the zeros of π are also zeros of the polynomial in the numerator. In such a case, $\{R_n\}$ is left (right) orthogonal with respect to the perturbed functional if the perturbation modifies the first (the second) argument of the functional $\mathbf{L}$.

The non-symmetric polynomial perturbations of a symmetric quasi-definite bilinear functional $\mathbf{L}$ when both arguments are perturbed as

$$\tilde{\mathbf{L}}(p,q) := \mathbf{L}((x-\alpha)p, (x-\beta)q), \quad \alpha, \beta \in \mathbb{C}, \quad \alpha \neq \beta$$

are not studied because they are neither left quasi-definite nor right quasi-definite.

4 Symmetric perturbations of symmetric bilinear functionals.

Given the symmetric quasi-definite bilinear functional $\mathbf{L}$, we study the standard perturbation

$$(\mathbf{x^2L})(p,q) = (xp, xq). \tag{4.1}$$

Again, the expression given in (4.1) is a natural extension to bilinear functionals of the perturbation of linear functionals given by $\mathbf{x^2S}$,

$$(\mathbf{x^2L})(p,q) = \mathbf{L}(xp,xq) = \mathbf{S}(x^2pq) = (\mathbf{x^2S})(pq).$$

In the sequel, we distinguish between symmetric positive definite bilinear functionals and symmetric quasi-definite bilinear functionals.

4.0.1 The symmetric functional $\mathbf{x^2L}$ *when* $\mathbf{L}$ *is positive definite*

Let $\mathbf{L}$ be a symmetric positive definite bilinear functional and let $\{\tilde{P}_n\}$ be the corresponding sequence of orthonormal polynomials. Let $\tilde{H}$ be the semi-infinite lower Hessenberg matrix associated with $\{\tilde{P}_n\}$.

Proposition 4.1 *The bilinear functional* $\mathbf{x^2L}$ *is a symmetric positive definite bilinear functional.*

PROOF. Consider the vector $X = [1, x, x^2, \ldots]^t$. Then the matrix of standard moments M_{x^2L} associated with $\mathbf{x^2L}$ is given by

$$M_{x^2L} = (\mathbf{x^2L})(X, X^t) = \mathbf{L}(xX, xX^t) = M_L^{(-1,-1)},$$

i.e., M_{x^2L} is obtained by deleting the first row and column of the matrix of standard moments associated with $\mathbf{L}$. Since M_L is a positive definite matrix, the matrix M_{x^2L} is also positive definite. $\square$

Let $\{\tilde{Q}_n\}$ be the sequence of polynomials orthonormal with respect to $\mathbf{x^2L}$ and let $\tilde{H}_1$ be the semi-infinite Hessenberg matrix associated with $\{\tilde{Q}_n\}$. Next we find an algebraic relation between $\tilde{H}_1$ and $\tilde{H}$. Let us consider the unique factorization

$$\tilde{H} = \tilde{L}\tilde{G}, \tag{4.2}$$

where $\tilde{L}$ is a lower triangular matrix and $\tilde{G}$ is an orthogonal and lower Hessenberg matrix. In more familiar terms, $\tilde{G}^t\tilde{L}^t$ is the unique QR factorization of $\tilde{H}^t$, which exists since $\tilde{H}$ is irreducible. Let us mention that the orthogonal Hessenberg matrices of order n are uniquely determined by n parameters, the so-called *Schur parameters* [12].

Theorem 4.2 *Taking into account the notation introduced above*

(1) $\tilde{v}_p = \tilde{L}\tilde{v}_q$,
(2) $\tilde{H}_1 = \tilde{G}\tilde{L}$.

PROOF. Since $\tilde{v}_p$ and $\tilde{v}_q$ are ordered by increasing degree, there exists a unique nonsingular lower triangular matrix $\hat{L}$ such that

$$\tilde{v}_p = \hat{L}\tilde{v}_q. \tag{4.3}$$

Taking into account that $(\mathbf{x^2L})(\tilde{v}_q, \tilde{v}_q^t) = I$, where I denotes the semi-infinite identity matrix, from (4.3) we get

$$\begin{aligned}(\mathbf{x^2L})(\tilde{v}_p, \tilde{v}_p^t) &= (\mathbf{x^2L})(\hat{L}\tilde{v}_q, \tilde{v}_q^t\hat{L}^t) \\ &= \hat{L}(\mathbf{x^2L})(\tilde{v}_q, \tilde{v}_q^t)\hat{L}^t \\ &= \hat{L}\hat{L}^t. \end{aligned} \tag{4.4}$$

On the other hand, since $\mathbf{L}(\tilde{v}_p, \tilde{v}_p^t) = I$, we obtain

$$\begin{aligned}(\mathbf{x^2L})(\tilde{v}_p, \tilde{v}_p^t) &= \mathbf{L}(x\tilde{v}_p, x\tilde{v}_p^t) \\ &= \mathbf{L}(\tilde{H}\tilde{v}_p, \tilde{v}_p^t\tilde{H}^t) \\ &= \tilde{H}\mathbf{L}(\tilde{v}_p, \tilde{v}_p^t)\tilde{H}^t \\ &= \tilde{H}\tilde{H}^t = \tilde{L}(\tilde{G}\tilde{G}^t)\tilde{L}^t = \tilde{L}\tilde{L}^t. \end{aligned} \tag{4.5}$$

Because of the uniqueness of the Cholesky factorization, from (4.4) and (4.5) we deduce

$$\hat{L} = \tilde{L}.$$

Moreover,

$$\begin{aligned}\tilde{H}_1 &= \tilde{H}_1(\mathbf{x^2L})(\tilde{v}_q, \tilde{v}_q^t) \\ &= (\mathbf{x^2L})(\tilde{H}_1\tilde{v}_q, \tilde{v}_q^t) \\ &= (\mathbf{x^2L})(x\tilde{v}_q, \tilde{v}_q^t) \\ &= (\mathbf{x^2L})(x\tilde{L}^{-1}\tilde{v}_p, \tilde{v}_p^t\tilde{L}^{-t}) \\ &= \tilde{L}^{-1}\tilde{H}(\mathbf{x^2L})(\tilde{v}_p, \tilde{v}_p^t)\tilde{L}^{-t} \\ &= \tilde{G}(\tilde{L}\tilde{L}^t)\tilde{L}^{-t} \\ &= \tilde{G}\tilde{L}, \end{aligned} \tag{4.6}$$

which proves the result. Notice that we have proven that if $\tilde{H}^t = QR$ denotes the unique QR factorization of $\tilde{H}^t$, then $\tilde{H}_1^t = RQ$. $\square$

4.0.2 Explicit algebraic relation between the sequences $\{\tilde{P}_n\}$ and $\{\tilde{Q}_n\}$.

Next we give an explicit algebraic relation between the sequences $\{\tilde{P}_n\}$ and $\{\tilde{Q}_n\}$ associated with $\mathbf{L}$ and $\mathbf{x^2L}$, respectively. Let $\{T_n\}_{n=1}^{\infty}$ be the sequence of polynomials given by $\tilde{G}\tilde{v}_p$. Since $\tilde{G}$ is a lower Hessenberg matrix, $\deg(T_i(x)) = i$ for all $i \geq 1$. If $v_t = [T_1(x), T_2(x), T_3(x), \ldots]^t$, then

$$x\tilde{v}_p = \tilde{H}\tilde{v}_p = \tilde{L}\tilde{G}\tilde{v}_p = \tilde{L}v_t.$$

Taking into account (1) in Theorem 4.2, the following result follows

$$v_t = x\tilde{v}_q. \tag{4.7}$$

Moreover, since $v_t = \tilde{G}\tilde{v}_p$, from (4.7) we obtain the following explicit algebraic relation between the sequences $\{\tilde{P}_n\}$ and $\{\tilde{Q}_n\}$

$$x\tilde{v}_q = \tilde{G}\tilde{v}_p. \tag{4.8}$$

Therefore, the factors $\tilde{L}$ and $\tilde{G}$ given in (4.2) are the matrices that determine the algebraic relations between the sequences $\{\tilde{P}_n\}$ and $\{\tilde{Q}_n\}$, since $\tilde{v}_p = \tilde{L}\tilde{v}_q$ and $x\tilde{v}_q = \tilde{G}\tilde{v}_p$.

4.0.3 The symmetric bilinear functional $\mathbf{x^2L}$ *when* $\boldsymbol{L}$ *is quasi-definite.*

This case is essentially close to the positive definite case but it requires ideas less familiar than QR and Cholesky factorizations. We will use hyperbolic QR factorization and triangular factorization.

The Gram-Schmidt process applied to the columns of a nonsingular matrix X produces the columns of an orthogonal matrix Q such that $X = QR$, where R is an upper triangular matrix with positive main diagonal. When the standard inner product is replaced by the "indefinite inner product" given by Ω, $\langle x, y\rangle_\Omega := x^t\Omega y$, then the Gram-Schmidt process yields an alternative factorization: the hyperbolic QR factorization. We have not found any references about this kind of factorization for semi-infinite matrices. This is the reason why we include the following definition and results.

Definition 4.3 *Let T be a semi-infinite upper Hessenberg matrix, and let $\Omega = diag(\pm 1)$ be a semi-infinite signature matrix. We say that*

$$T = \tilde{Q}\pi R, \tag{4.9}$$

is the hyperbolic QR factorization of T with respect to Ω if $\tilde{Q}$ is a semi-infinite Ω-orthogonal matrix, i.e., it satisfies $\tilde{Q}^t\Omega\tilde{Q} = \Omega$, R is a semi-infinite upper triangular matrix with positive main diagonal, and π is a permutation matrix.

Remark 4.4 *In the previous definition, if we denote by Q the matrix $\tilde{Q}\pi$, then $T = QR$, and Q satisfies $Q^t\Omega Q = \hat{\Omega}$, where $\hat{\Omega}$ is another signature matrix uniquely determined by T and Ω. Moreover, $T^t\Omega T = R^t\hat{\Omega}R$.*

Below we include a theorem that gives conditions for the existence of the hyperbolic QR factorization of a semi-infinite upper Hessenberg matrix.

Lemma 4.5 *Let M be a symmetric matrix such that all its leading principal submatrices are nonsingular. Then, the (unique) triangular factorization of M exists, that is,*

$$M = LDL^t,$$

where L is a lower triangular matrix with ones in the main diagonal, and D

is a diagonal matrix. Moreover, M can be expressed in the following way:

$$M = L|D|^{1/2}\Omega|D|^{1/2}L^t,$$

where $\Omega = diag(\pm 1)$ is a signature matrix and $|D|^{1/2}$ denotes the diagonal matrix given by $(|D|^{1/2})_{i,i} = |D_{i,i}|^{1/2}$, $i \geq 1$. We will refer to the matrix Ω as the tri-signature matrix associated with M.

Definition 4.6 *Let Ω be a signature matrix. If we denote by $i_+(\Omega)$ and $i_-(\Omega)$ the cardinality of the sets $\{(\Omega)_{i,i} : (\Omega)_{i,i} = 1\}$ and $\{(\Omega)_{i,i} : (\Omega)_{i,i} = -1\}$, respectively, then we define the inertia of Ω as*

$$inert(\Omega) := \{i_+(\Omega), i_-(\Omega)\}.$$

Theorem 4.7 *Let T be a semi-infinite nonsingular upper Hessenberg matrix, and let Ω be a signature matrix. The hyperbolic QR factorization of T with respect to Ω exists if and only if all the leading principal submatrices of $T^t\Omega T$ are nonsingular and $inert(\Omega) = inert(\hat{\Omega})$, where $\hat{\Omega}$ is the tri-signature matrix associated with $T^t\Omega T$.*

PROOF.

(1) Assume that the hyperbolic QR factorization of T with respect to Ω exists. Then,

$$T^t\Omega T = R^t\pi^t\tilde{Q}^t\Omega Q\pi R = R^t\tilde{\Omega}R,$$

where $\tilde{\Omega}$ is a signature matrix. For all $n \geq 1$, the leading principal submatrix of order n of $R^t\tilde{\Omega}R$ satisfies

$$(R^t\tilde{\Omega}R)_n = (R)_n^t(\tilde{\Omega})_n(R)_n.$$

Therefore,

$$det((R^t\tilde{\Omega}R)_n) = \pm det^2((R)_n) \neq 0.$$

(2) We assume now that all the leading principal submatrices of $T^t\Omega T$ are nonsingular. Then, the (unique) triangular factorization of $T^t\Omega T$ exists and we get

$$T^t\Omega T = LDL^t = L|D|^{1/2}\hat{\Omega}|D|^{1/2}L^t.$$

Consider the matrix

$$\tilde{Q} = TL^{-t}|D|^{-1/2}. \tag{4.10}$$

Then,

$$\tilde{Q}^t\Omega\tilde{Q} = |D|^{-1/2}L^{-1}T^t\Omega TL^{-t}|D|^{-1/2} = \hat{\Omega}.$$

Since $inert(\Omega) = inert(\hat{\Omega})$, there exists a permutation matrix π such that $\pi^t\hat{\Omega}\pi = \Omega$. Then, we get

$$\left(\pi^t\tilde{Q}^t\right)\Omega\left(\tilde{Q}\pi\right) = \Omega.$$

Multiplying (4.10) by π on the right, we obtain

$$Q := \tilde{Q}\pi = TL^{-t}|D|^{-1/2}\pi,$$

which implies that

$$T = Q\pi^t R,$$

with $R = |D|^{1/2}L^t$. $\square$

The following result is obtained in a straightforward way from the previous theorem.

Corollary 4.8 *Let T be a semi-infinite nonsingular upper Hessenberg matrix, and let Ω be a signature matrix. All the leading principal submatrices of $T^t\Omega T$ are nonsingular if and only if there exists an upper triangular matrix with positive main diagonal R, and a matrix Q such that $T = QR$, where $Q^t\Omega Q = \hat{\Omega}$ and $\hat{\Omega}$ is the tri-signature matrix associated with $T^t\Omega T$.*

We will refer to this factorization as the pseudo-hyperbolic QR factorization of T.

In the sequel, we consider a quasi-definite symmetric bilinear functional $\mathbf{L}$. Let $\{P_n\}$ be the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$, and let H be the semi-infinite Hessenberg matrix associated with $\{P_n\}$. Consider the bilinear functional $\mathbf{x^2L}$ given in Definition 4.1.

Unlike what happens in the positive definite case, $\mathbf{L}$ being quasi-definite does not imply, in general, that $\mathbf{x^2L}$ is a quasi-definite bilinear functional. For instance, assume that the leading principal submatrix of order 3 of the matrix of standard moments associated with $\mathbf{L}$ is

$$(M_L)_3 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is obvious that $(M_L)_3$ is a non-singular matrix while the matrix obtained by deleting its first row and column $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is singular. Therefore, $\mathbf{L}$ must satisfy an extra condition so that $\mathbf{x^2L}$ is quasi-definite. It is easy to prove that a necessary and sufficient condition for $\mathbf{x^2L}$ being quasi-definite is that the matrix $M_L^{(-1,-1)}$ is quasi-definite. However, this condition involves the matrix of standard moments. Later on we will give another condition on the matrix H to assure that the functional $\mathbf{x^2L}$ is quasi-definite. In the sequel, we denote by $\{Q_n\}$ the sequence of monic polynomials orthogonal with respect to $\mathbf{x^2L}$, and

H_1 denotes the corresponding Hessenberg matrix, assuming that the functional $\mathbf{x^2L}$ is quasi-definite.

In the quasi-definite case, the key for the connection between the bilinear functionals $\mathbf{L}$ and $\mathbf{x^2L}$ is the (unique) triangular factorization of $(\mathbf{x^2L})(v_p, v_p^t) = HD_pH^t$, where $D_p = \mathbf{L}(v_p, v_p^t)$, given by

$$HD_pH^t = LDL^t. \tag{4.11}$$

Let us assume that all the leading principal submatrices of HD_pH^t are nonsingular so that its triangular factorization exists.

Theorem 4.9 *Using the notation given above, for a quasi-definite and symmetric bilinear functional* $\mathbf{L}$,

(1) $v_p = Lv_q$,
(2) $H_1 = L^{-1}HL$.

PROOF. Since the entries of v_p and v_q are monic polynomials and they are ordered by increasing degree, there exists a nonsingular lower triangular matrix $\tilde{L}$ with ones in the main diagonal and such that

$$v_p = \tilde{L}v_q. \tag{4.12}$$

Taking into account orthogonality properties $(\mathbf{x^2L})(v_q, v_q^t) = D_q$ is diagonal and

$$\begin{aligned}(\mathbf{x^2L})(v_p, v_p^t) &= (\mathbf{x^2L})(\tilde{L}v_q, v_q^t\tilde{L}^t)\\ &= \tilde{L}(\mathbf{x^2L})(v_q, v_q^t)\tilde{L}^t\\ &= \tilde{L}D_q\tilde{L}^t.\end{aligned} \tag{4.13}$$

On the other hand,

$$\begin{aligned}(\mathbf{x^2L})(v_p, v_p^t) &= \mathbf{L}(xv_p, xv_p^t)\\ &= \mathbf{L}(Hv_p, v_p^tH^t)\\ &= HD_pH^t.\end{aligned} \tag{4.14}$$

Because of the uniqueness of the triangular factorization, from (4.11), (4.13), and (4.14), we deduce that $\tilde{L} = L$ and $D = D_q$, which proves the first result.

On the other hand,

$$
\begin{aligned}
H_1 D_q &= H_1(\mathbf{x^2L})(v_q, v_q^t) \\
&= (\mathbf{x^2L})(H_1 v_q, v_q^t) \\
&= (\mathbf{x^2L})(x v_q, v_q^t) \\
&= (\mathbf{x^2L})(x L^{-1} v_p, v_p^t L^{-t}) \\
&= L^{-1}(\mathbf{x^2L})(x v_p, v_p^t) L^{-t} \\
&= L^{-1} H (\mathbf{x^2L})(v_p, v_p^t) L^{-t} \\
&= L^{-1} H (L D_q L^t) L^{-t} \\
&= L^{-1} H L D_q. \qquad (4.15)
\end{aligned}
$$

Since D_q is nonsingular, the second result is proven. □

In order to recover a QR like transformation, we write

$$D_p = |D_p|^{1/2}\Omega_p|D_p|^{1/2}, \quad D_q = |D_q|^{1/2}\Omega_q|D_q|^{1/2}, \qquad (4.16)$$

where $\Omega_p = sign(D_p)$, and $\Omega_q = sign(D_q)$.

The tool for the connection between the matrices H and H_1 is the pseudo-hyperbolic QR factorization of $|D_p|^{1/2}H^t$ with respect to Ω_p. From Corollary 4.8, this factorization exists if all the leading principal submatrices of

$$(|D_p|^{1/2}H^t)^t\Omega_p|D_p|^{1/2}H^t = HD_pH^t$$

are nonsingular.

Proposition 4.10 *If all the leading principal submatrices of HD_pH^t are nonsingular, the pseudo-hyperbolic QR factorization of $|D_p|^{1/2}H^t$ with respect to Ω_p exists. Moreover,*

$$|D_p|^{1/2}H^t = QR,$$

where $Q^t\Omega_pQ = \Omega_q$ and $R = |D_q|^{1/2}L^t$.

PROOF. The existence of the pseudo-hyperbolic QR factorization of $|D_p|^{1/2}H^t$ with respect to Ω_p is guaranteed by Corollary 4.8. Consider this factorization

$$|D_p|^{1/2}H^t = QR. \qquad (4.17)$$

Then, taking into account that Q is orthogonal with respect to Ω_p and the triangular factorization of HD_pH^t, we get

$$LD_qL^t = HD_pH^t = R^tQ^t\Omega_pQR.$$

Considering (4.16) and Corollary 4.8, we deduce that $R = |D_q|^{1/2}L^t$ and $Q^t\Omega_pQ = \Omega_q$. □

The previous theorem allows us to obtain H_1 from H by the application of a QR-like transformation.

Theorem 4.11 *Given a symmetric quasi-definite bilinear functional* $\mathbf{L}$ *as well as the quantities defined above, if all the leading principal submatrices of* HD_pH^t *are nonsingular, then*

$$H = LG, \quad H_1 = GL,$$

where $G := |D_q|^{1/2}Q^t|D_p|^{-1/2}$.

PROOF. Using transposition in (4.17), we get

$$H = L|D_q|^{1/2}Q^t|D_p|^{-1/2} = LG.$$

From Theorem 4.9,

$$H_1 = L^{-1}HL = GL. \qquad \square$$

Notice that the matrix G is orthogonal with respect to D_p, that is,

$$\begin{aligned} GD_pG^t &= |D_q|^{1/2}Q^t|D_p|^{-1/2}D_p|D_p|^{-1/2}Q|D_q|^{1/2} \\ &= |D_q|^{1/2}Q^t\Omega_pQ|D_q|^{1/2} \\ &= |D_q|^{1/2}\Omega_q|D_q|^{1/2} = D_q. \end{aligned}$$

As an immediate consequence of the previous results we obtain a necessary and sufficient condition for $\mathbf{x^2L}$ to be quasi-definite.

Corollary 4.12 *Let* $\mathbf{L}$ *be a symmetric quasi-definite bilinear functional and let* H *be its corresponding Hessenberg matrix. Then, the symmetric bilinear functional* $\mathbf{x^2L}$ *is quasi-definite if and only if all the leading principal submatrices of* HD_pH^t *are nonsingular.*

Next proposition gives also necessary and sufficient conditions for $\mathbf{x^2L}$ to be quasi-definite.

Proposition 4.13 *Let* $\mathbf{L}$ *be a symmetric quasi-definite bilinear functional and let* $\{P_n\}$ *be the sequence of monic polynomials orthogonal with respect to* $\mathbf{L}$. *Then, the symmetric bilinear functional* $\mathbf{x^2L}$ *is quasi-definite if and only if* $DET_n \neq 0$ *for all* $n \geq 1$, *where*

$$DET_n = \prod_{i=1}^{n} d_i P_n^2(0) + d_{n+1}DET_{n-1},$$

with $DET_0 = 1$ *and* $d_i = \mathbf{L}(P_i, P_i)$ *for all* $i \geq 1$.

PROOF. It is obvious that, for all $n \geq 1$,

$$(HD_pH^t)_n = H_n(D_p)_nH_n^t + d_{n+1}e_n^te_n,$$

where $d_{n+1} = (D_p)_{n+1,n+1}$ and $e_n = [0, ..., 1] \in \mathbb{R}^{1\times n}$. Then, considering some determinant properties,

$$det\left((HD_pH^t)_n\right) = det(H_n(D_p)_nH_n^t) + d_{n+1}det\left((HD_pH^t)_{n-1}\right),$$

where $det\left((HD_pH^t)_0\right) = 1$. Taking into account that $det((H)_n) = (-1)^nP_n(0)$ and that $d_i = \mathbf{L}(P_i, P_i)$,

$$det\left((HD_pH^t)_n\right) = \prod_{i=1}^{n} d_iP_n^2(0) + d_{n+1}det\left((HD_pH^t)_{n-1}\right).$$

Then denoting $DET_n := det\left((HD_pH^t)_n\right)$, and taking into account Corollary 4.12, the result follows. □

Finally, notice that the following explicit algebraic relations between the sequences $\{P_n\}$ and $\{Q_n\}$ can be obtained from Theorem 4.11

$$v_p = Lv_q, \quad \text{and} \quad xv_q = Gv_p.$$

4.0.4 The symmetric bilinear functional $\mathbf{x^{2n}L}$

The results obtained in the previous subsections can easily be generalized. Let $\mathbf{L}$ be a symmetric quasi-definite bilinear functional. It is obvious that the best way to define the symmetric bilinear functional $\mathbf{x^{2n}L}$ is

$$(\mathbf{x^{2n}L})(p, q) := \mathbf{L}(x^np, x^nq). \tag{4.18}$$

Remark 1 *The definition of* $\mathbf{x^{2n}L}$*, for* $n \geq 1$*, is a natural extension of the definition of* $\mathbf{x^{2n}S}$*, where* S *denotes a linear functional.*

It is easy to prove that $\mathbf{x^{2n}L}$ is a positive definite bilinear functional if $\mathbf{L}$ is also positive definite. On the other hand, if $\mathbf{L}$ is quasi-definite, it is necessary to consider an additional condition so that $\mathbf{x^{2n}L}$ is a quasi-definite functional.

Assume that H and H_1 denote, respectively, the Hessenberg matrix associated with $\mathbf{L}$ and $\mathbf{x^{2n}L}$. If $\mathbf{L}$ is positive definite and $(H^n)^t = G^tL^t$ denotes the QR factorization of $(H^n)^t$, then

$$H_1^n = L^{-1}H^nL \quad \text{and} \quad (H_1^n)^t = L^tG^t. \tag{4.19}$$

A similar result is obtained in the quasi-definite case.

The definition in (4.18) can be used to give a general definition of the symmetric bilinear functional $\pi\mathbf{L}$, where π denotes an even polynomial. In the linear case, if $\mathbf{S}$ denotes a linear functional and $\tau(x) = \sum_{i=0}^{n} \alpha_i x^i$, then

$$(\tau\mathbf{S})(p) = \sum_{i=0}^{n} \alpha_i(\mathbf{x^i S})(p).$$

Therefore, as an extension of the above result, if $\pi(x) = \sum_{i=0}^{m} \alpha_i x^{2i}$, it seems to be natural to define

$$(\pi\mathbf{L})(p,q) := \sum_{i=0}^{m} \alpha_i(\mathbf{x^{2i} L})(p,q).$$

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