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Eigenvectors and minimal bases for some families of Fiedler-like linearizations

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In this paper we obtain formulas for the left and right eigenvectors and minimal bases of some families of Fiedler-like linearizations of square matrix polynomials. In particular, for the families of Fiedler pencils, generalized Fiedler pencils, and Fiedler pencils with repetition. To do this, we introduce the notion of left and right eigencolumn, which allows us to relate the eigenvectors and minimal bases of the linearizations with the ones of the polynomial. Since eigenvectors appear in the standard formula for the condition number of eigenvalues of matrix polynomials, these formulas may be used to compare the condition number of eigenvalues of the linearizations within these families, and also with the condition number of eigenvalues in the matrix polynomial.

Keywords: polynomial eigenvalue problem, Fiedler pencils, matrix polynomials, linearizations, eigenvector, minimal bases, symmetric matrix polynomials

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1. Introduction

In the present paper we are concerned with eigenvectors and minimal bases of linearizations of square matrix polynomials over the complex field $\mathbb{C}$. A square $n \times n$ matrix polynomial over $\mathbb{C}$

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \,, \quad A_0, \ldots, A_k \in \mathbb{C}^{n\times n}, \quad A_k \neq 0 \,, \tag{1}$$

is said to be *regular* if the determinant of $P(\lambda)$ is not the identically zero polynomial. The matrix polynomial $P(\lambda)$ is *singular* otherwise. The *finite eigenvalues* and associated *eigenvectors* of a regular matrix polynomial (1) are defined as those values $\lambda_0 \in \mathbb{C}$ and nonzero vectors $v \in \mathbb{C}^n$, respectively, such that $P(\lambda_0)v = 0$. They are of relevance in several applied problems where matrix polynomials arise (see, for instance, [20] for a survey on quadratic polynomials, and [17, 18, 22] for recent examples of applications of higher degree polynomials). The problem of the computation of eigenvalues and eigenvectors of regular matrix polynomials, which is know as the Polynomial Eigenvalue Problem (PEP), has attracted the attention of many researchers in numerical linear algebra. When the matrix polynomial is singular, instead of the eigenvectors we are interested in *minimal bases*, which are particular bases of the right and left nullspaces of $P(\lambda)$ and are also relevant in applications [2, 9].

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The standard way to numerically solve the PEP for regular polynomials is through the use of *linearizations*. These are essentially matrix pencils $H(\lambda) = \lambda X + Y$, with $X, Y \in \mathbb{C}^{nk \times nk}$, sharing certain information with the polynomial $P(\lambda)$, in particular, the *invariant polynomials*, which include the eigenvalues and its associated *partial multiplicities* (see [11] for the definition of these notions). However, the eigenvectors of $H(\lambda)$ and $P(\lambda)$ are not the same, and actually they can never be the same because the sizes of $H(\lambda)$ and $P(\lambda)$ are different. Similarly, for singular matrix polynomials, minimal bases are not usually preserved by linearization. Then, the problem of relating the eigenvectors and minimal bases of $P(\lambda)$ with the ones of $H(\lambda)$ becomes essential in numerical computations.

An important issue to determine the errors in the numerical computation of eigenvalues is the *eigenvalue condition number*. The standard formula for the condition number of eigenvalues of a matrix polynomial $P(\lambda)$ involves the associated left and right eigenvectors of $P(\lambda)$ [19]. When using linearizations to compute eigenvalues of $P(\lambda)$, we have to consider the eigenvalue condition numbers corresponding to the linearization $H(\lambda)$, which are, in general, larger than the ones of the polynomial $P(\lambda)$. Actually, these condition numbers involve the eigenvectors of $H(\lambda)$, instead of the eigenvectors of $P(\lambda)$. Hence, in order to compare the condition numbers of the eigenvalues corresponding to $H(\lambda)$ with the condition numbers corresponding to $P(\lambda)$, the knowledge of the left and right eigenvectors of $H(\lambda)$ is relevant. Moreover, it would be desirable to know the relationship between these eigenvectors and the eigenvectors of $P(\lambda)$.

The classical linearizations of matrix polynomials used in practice have been the *first* and *second (Frobenius) companion forms* [11]. However, during the last decade several new families of linearizations have been introduced by different authors [1, 3, 7, 16, 21], some of them extending other known families, like the one introduced back in the 1960's in [14]. The natural subsequent step is to analyze the advantages or disadvantages of these new families and, in particular, to study their numerical features. In connection with the problems mentioned in the previous paragraphs, a natural first step for this would be:

(P1) Find recovery formulas for eigenvectors and minimal bases of $P(\lambda)$ from the ones of the linearizations.

(P2) Obtain explicit formulas for the eigenvectors and minimal bases of the linearizations in terms of the eigenvectors and minimal bases of $P(\lambda)$.

We want to stress that solving (P2) implies solving (P1), but the converse is not true.

For the families of linearizations introduced in [16], Problem (P1) has been solved in [5, 12, 16], but (P2) has been only partially solved. For the family of *Fiedler pencils*, introduced in [3] (and named later in [6]), both (P1) and (P2) have been completely solved in [6] for square matrix polynomials and in [8] for rectangular polynomials. For the family of *generalized Fiedler pencils*, also introduced in [3] (though named in [4]) (P1) has been solved in [4], but (P2) remains open. The present paper deals with problem (P2). Our main goal is to obtain formulas for the eigenvectors and minimal bases of the generalized Fiedler pencils and the *Fiedler pencils with repetition*, which is the family recently introduced in [21]. These formulas will be given in terms of the eigenvectors and minimal bases of the polynomial. We will also provide a simpler expression of the formula obtained in [6] for the eigenvectors of Fiedler pencils. In order to get our formulas for the left and right eigenvectors, we introduce the notion of *right* and *left eigencolumn*. This will allow us to get formulas for left and right minimal bases as well.

The paper is organized as follows. In Section 2 we introduce basic notation and definitions, and we recall the families of linearizations that we have mentioned above. In Section 3 we introduce the notion of eigencolumn of linearizations and explain how it is related to eigenvectors and minimal bases. In Section 4 we present the main results of the paper, namely, formulas for the left and right eigencolumns of the families of Fiedler pencils, proper generalized Fiedler pencils and Fiedler pencils with repetition. Section 5 is devoted to the proofs of these results, and in Section 6 we summarize the main contributions of the paper and we pose some open problems that appear as a natural continuation of this work. The case of non-proper generalized Fiedler pencils is addressed in Appendix A, because this is a very particular case which deserves a separate treatment. Finally, in Appendix B we obtain formulas for left and right eigenvectors associated with the infinite eigenvalue of regular polynomials. This case is also addressed in a final appendix because the techniques employed in this case have nothing to do with the main techniques of the paper, and even the formulas for this case are very specific.

2. Basic definitions

Along the paper we will use the following notation: I_m will denote the $m \times m$ identity matrix. When no subindex appear in this identity, we will assume it to be n, which is the size of the matrix polynomial in (1). We will also deal with block-partitioned matrices with blocks of size $n \times n$. For these matrices, we will use the following operation.

Definition 2.1m: If $A = [A_{ij}]$ is a block $r \times s$ matrix consisting of block entries A_{ij} with size $n \times n$, then its block transpose is a block-partitioned $s \times r$ matrix $A^{\mathcal{B}}$ whose (i, j) block is $(A^{\mathcal{B}})_{ij} = A_{ji}$.

By $\mathbb{C}[\lambda]$ we will denote the ring of polynomials in the variable λ with complex coefficients, and $\mathbb{C}(\lambda)$ will denote the field of rational functions in the variable λ with complex coefficients. Accordingly, $\mathbb{C}[\lambda]^n$ is the set of vectors whose n coordinates are polynomials in $\mathbb{C}[\lambda]$ and $\mathbb{C}(\lambda)^n$ is the vector space of dimension n with coordinates in $\mathbb{C}(\lambda)$.

Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are said to be *equivalent* if there are two matrix polynomials with constant nonzero determinant, $U(\lambda)$ and $V(\lambda)$ (such matrix polynomials are known as *unimodular*), such that $Q(\lambda) = U(\lambda)P(\lambda)V(\lambda)$. If $U(\lambda)$ and $V(\lambda)$ are constant matrices, then $P(\lambda)$ and $Q(\lambda)$ are said to be *strictly equivalent*.

The *reversal* of the matrix polynomial $P(\lambda)$ is the matrix polynomial obtained by reversing the order of the coefficient matrices, that is

$$\operatorname{rev} P(\lambda) := \sum_{i=0}^{k} \lambda^i A_{k-i}.$$

We use in this paper the classical notion of linearization for square $n \times n$ polynomials (see [11] and [10] for regular matrix polynomials and [5] for singular ones).

Definition 2.2m: A matrix pencil $H(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{C}^{nk \times nk}$ is a *linearization* of an $n \times n$ matrix polynomial $P(\lambda)$ of degree k if there exist two unimodular $nk \times nk$ matrices $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda)H(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}, \tag{2}$$

or, in other words, if $H(\lambda)$ is equivalent to $\operatorname{diag}(I_{(k-1)n}, P(\lambda))$. A linearization $H(\lambda)$ is called a *strong linearization* if $\operatorname{rev} H(\lambda)$ is also a linearization of $\operatorname{rev} P(\lambda)$.

In Section 2.3 we will introduce the families of linearizations which are the subject of the present paper. They are constructed using the following $nk \times nk$ matrices, partitioned into $k \times k$ blocks of size $n \times n$. Here and hereafter, A_i denotes the ith coefficient of the matrix polynomial (1).

$$M_{-k} := \begin{bmatrix} A_k & \\ & I_{(k-1)n} \end{bmatrix}, \quad M_0 := \begin{bmatrix} I_{(k-1)n} & \\ & -A_0 \end{bmatrix}, \tag{3}$$

and

$$M_i := \begin{bmatrix} I_{(k-i-1)n} & & & \\ & -A_i & I & \\ & I & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \quad i = 1, \ldots, k-1. \tag{4}$$

The M_i matrices in (4) are always invertible, and the inverses are given by

$$M_i^{-1} = \begin{bmatrix} I_{(k-i-1)n} & & & \\ & 0 & I & \\ & I & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}. \tag{5}$$

However, note that M_0 and M_{-k} are invertible if and only if A_0 and A_k, respectively, are.
We will also use the notation

$$M_{-i} := M_i^{-1}, \qquad \text{for } i = 0, 1, ..., k-1, \qquad \text{and} \quad M_k := M_{-k}^{-1}.$$

The notation for M_{-k} differs from the standard one used in [3, 4, 6]. The reason for this change here is that, for all but one of the families of linearizations considered in this paper (and this last one is addressed only in Appendix A), M_{-k} will appear in the leading term of the linearization, and we follow the convention of using negative indices for the matrices in this term. We want to emphasize also that $M_{-0} := M_0^{-1}$. For this reason, we will use along this paper both 0 and -0, with different meanings.

It is easy to check the commutativity relations

$$M_i M_j = M_j M_i \quad \text{for} \quad ||i| - |j|| \neq 1. \tag{6}$$

For $0 \leq i \leq k$ we will make use along the paper of the polynomial

$$P_i(\lambda) = A_{k-i} + \lambda A_{k-i+1} + \cdots + \lambda^i A_k.$$

This polynomial is known as the *ith Horner shift of* $P(\lambda)$, with $P(\lambda)$ as in (1). Notice that $P_0(\lambda) = A_k$, $P_k(\lambda) = P(\lambda)$ and $\lambda P_i(\lambda) = P_{i+1}(\lambda) - A_{k-i-1}$, for $0 \leq i \leq k-1$.

2.1. *Index tuples, column standard form, and the SIP property*

In this paper we are concerned with pencils constructed from products of M_i and M_{-i} matrices. In our analysis, the order in which these matrices appear is relevant. For this reason, we will associate an index tuple with each of these products to simplify our developments. We also introduce some additional concepts defined in [21] which are related to this notion. We will use boldface letters, namely $\mathbf{t}, \mathbf{q}, \mathbf{z} \ldots$, for ordered tuples of indices (or *index tuples* in the following).

Definition 2.3m: Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple containing indices from $\{0, 1, \ldots, k, -0, -1, \ldots, -k\}$. We say that $\mathbf{t}$ is simple if $i_j \neq i_l$ for all $j, l \in \{1, 2, \ldots, r\}$ with $j \neq l$.

Definition 2.4m: Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple containing indices from $\{0, 1, \ldots, k, -0, -1, \ldots, -k\}$. Then,

$$M_{\mathbf{t}} := M_{i_1} M_{i_2} \cdots M_{i_r}. \tag{7}$$

We want to insist on the fact that 0 and -0 are different. We include -0 along this section for completeness and symmetry in definitions and developments, though the only case where it is relevant is the one addressed in Appendix A, where matrix M_{-0} appears.

Unless otherwise stated, the matrices M_i, $i = 0, \ldots, k$, and $M_{\mathbf{t}}$ refer to the matrix polynomial $P(\lambda)$ in (1). When necessary, we will explicitly indicate the dependence on a certain polynomial $Q(\lambda)$ by writing $M_i(Q)$ and $M_{\mathbf{t}}(Q)$.

Definition 2.5m: Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples containing indices from $\{0, 1, \ldots, k, -0, -1, \ldots, -k\}$. We say that $\mathbf{t}_1$ is equivalent to $\mathbf{t}_2$, and we will write $\mathbf{t}_1 \sim \mathbf{t}_2$, if $M_{\mathbf{t}_1} = M_{\mathbf{t}_2}$.

Notice that this is an equivalence relation and that if $M_{\mathbf{t}_2}$ can be obtained from $M_{\mathbf{t}_1}$ by the repeated application of the commutativity relations (6), then $\mathbf{t}_1$ is equivalent to $\mathbf{t}_2$.

We will refer to an index tuple consisting of consecutive integers as a *string*. We will use the notation $(q : l)$ for the string of integers from q to l, that is

$$(q : l) := \begin{cases} (q, q+1, \ldots, l), & \text{if } q \leq l \\ \emptyset, & \text{if } q > l \end{cases}.$$

Observe that if $q_1 \neq q_2$, with $q_1 > l$ and $q_2 > l$, then both $(q_1 : l)$ and $(q_2 : l)$ correspond to the empty index tuple. This creates an ambiguity that can be avoided by using the notation $(\infty : l)$ for any tuple of the form $(q : l)$ with $q > l$. We shall also say that $M_{\emptyset} = I_{nk}$.

Definition 2.6m: Given an index tuple $\mathbf{t} = (i_1, \ldots, i_r)$, we define the reverse tuple of $\mathbf{t}$, denoted by rev $\mathbf{t}$, as $\operatorname{rev} \mathbf{t} := (i_r, \ldots, i_1)$.

Definition 2.7m: Given an index tuple $\mathbf{t} = (i_1, \ldots, i_r)$, we define the tuple $-\mathbf{t} := (-i_1, \ldots, -i_r)$.

The following two notions are introduced for tuples of nonnegative integers (later we will consider the case of negative indices).

Definition 2.8m: [21] Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple with elements from $\{0, 1, \ldots, k-1\}$. Then $\mathbf{t}$ is said to satisfy the Successor Infix Property (SIP) if for every pair of indices $i_a, i_b \in \mathbf{t}$ with $1 \leq a < b \leq r$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ such that $a < c < b$.

Definition 2.9m: [21] Let $\mathbf{t}$ be an index tuple containing indices from $\{0, 1, \ldots, k-1\}$. Then $\mathbf{t}$ is said to be in column standard form if

$$\mathbf{t} = (c_{k-1} : k-1, c_{k-2} : k-2, \ldots, c_2 : 2, c_1 : 1, c_0 : 0), \quad c_i \in (0 : i) \cup \{\infty\}.$$

By removing the empty strings of the form $(\infty : j)$ in $\mathbf{t}$, it can be seen that $\mathbf{t}$ is in column standard form if and only if

$$\mathbf{t} = (a_s : b_s, a_{s-1} : b_{s-1}, \ldots, a_2 : b_2, a_1 : b_1),$$

with $k - 1 \geq b_s > b_{s-1} > \cdots > b_2 > b_1 \geq 0$ and $0 \leq a_j \leq b_j$, for all $j = 1, \ldots, s$.

The connection between the column standard form and the SIP property of an index tuple is shown in the following result.

Lemma 2.10m: [21] *Let* $\mathbf{t} = (i_1, \ldots, i_r)$ *be an index tuple containing indices from* $\{0, 1, \ldots, k-1\}$*. Then* $\mathbf{t}$ *satisfies the SIP if and only if* $\mathbf{t}$ *is equivalent to a tuple in column standard form.*

Note that, in particular, if $\mathbf{t}$ is simple, then $\mathbf{t}$ satisfies the SIP and, therefore, is equivalent to a tuple in column standard form. In the more particular case of a permutation we can obtain an expression for $\mathbf{t}$ in column standard form that will be used in further developments.

Lemma 2.11m: *Let* $\mathbf{t}$ *be a permutation of* $\{h_0, h_0+1, \ldots, h\}$*, with* $0 \leq h_0 \leq h \leq k-1$*. Then* $\mathbf{t}$ *is in column standard form if and only if*

$$\mathbf{t} = (t_{s-1}+1 : h, t_{s-2}+1 : t_{s-1}, \ldots, t_2+1 : t_3, t_1+1 : t_2, h_0 : t_1)$$

for some positive integers $h_0 \leq t_1 < t_2 < \cdots < t_{s-1} < h$*.*

Denote $t_0 = h_0 - 1$ *and* $t_s = h$*. We call each sequence of consecutive integers* $(t_{i-1}+1 : t_i)$*, for* $i = 1, \ldots, s$*, a* string *in* $\mathbf{t}$*.*

The proof of Lemma 2.11 is straightforward and is left to the reader.

The previous concepts can be extended to tuples of negative indices. In particular, below we extend definitions 2.8 and 2.9, and also Lemma 2.11.

Definition 2.12m: Let $\mathbf{t}' = (i_1, i_2, \ldots, i_r)$ be an index tuple with elements from $\{-k, -k+1, \ldots, -1\}$. Then $\mathbf{t}'$ is said to satisfy the SIP if for every pair of indices $i_a, i_b \in \mathbf{t}$ with $1 \leq a < b \leq r$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ such that $a < c < b$.

Definition 2.13m: Let $\mathbf{t}'$ be an index tuple containing indices from $\{-k, -k+1, \ldots, -1\}$. Then $\mathbf{t}'$ is said to be in column standard form if

$$\mathbf{t}' = (c_{-1} : -1, c_{-2} : -2, \ldots, c_{-k+1} : -k+1, c_{-k} : -k), \quad c_i \in (-k : i) \cup \{\infty\}.$$

By removing the empty strings of the form (∞, j) in $\mathbf{t}'$ we see that $\mathbf{t}'$ is in column standard form if and only if $\mathbf{t}'$ is of the form

$$\mathbf{t}' = (-a_r : -b_r, -a_{r-1} : -b_{r-1}, \ldots, -a_2 : -b_2, -a_1 : -b_1),$$

with $1 \leq b_r < b_s < \cdots < b_2 < b_1 \leq k$ and $k \geq a_j \geq b_j \geq 1$, for all $j = 1, \ldots, r$.

Lemma 2.14m: *Let* $\mathbf{t}'$ *be a permutation of* $\{-q_0, -q_0+1, \ldots, -q-2, -q-1\}$*, where* $1 \leq q+1 \leq q_0$*. Then* $\mathbf{t}'$ *is in column standard form if and only if*

$$\mathbf{t}' = (-t'_{r-1}+1 : -q-1, -t'_{r-2}+1 : -t'_{r-1}, \ldots, -t'_1+1 : -t'_2, -q_0 : -t'_1),$$

for some positive integers $q_0 \geq t'_1 > t'_2 > \ldots > t'_{r-1} > q+1$*.*

Denote $t'_0 = q_0 + 1$ *and* $t'_r = q+1$*. We call each sequence of consecutive integers* $(-t'_{i-1}+1 : -t'_i)$*, with* $i = 1, \ldots, r$*, a* string *in* $\mathbf{t}'$*.*

Lemma 2.15m: [21] *Let* $\mathbf{t}' = (i_1, \ldots, i_r)$ *be an index tuple containing indices from* $\{-k, -k+1 \ldots, -1\}$*. Then* $\mathbf{t}'$ *satisfies the SIP if and only if* $\mathbf{t}'$ *is equivalent to a tuple in column standard form.*

2.2. *Consecutions and inversions of simple index tuples*

Here we recall some definitions introduced in [6] which are key in the formulas for the eigencolumns.

Definition 2.16m: Let $\mathbf{q}$ be a simple index tuple with all its elements from $\{0, 1, \ldots, k-1\}$ or all from $\{-k, -k+1, \ldots, -1, -0\}$.

(a) We say that $\mathbf{q}$ has a consecution at $j \neq -0$ if both $j, j+1 \in \mathbf{q}$ and j is to the left of $j+1$ in $\mathbf{q}$. We say that $\mathbf{q}$ has an inversion at $j \neq -0$ if both $j, j+1 \in \mathbf{q}$ and j is to the right of $j+1$ in $\mathbf{q}$.
(b) We say that $\mathbf{q}$ has $\mathfrak{c}_j$ (resp. $\mathfrak{i}_j$) consecutions (resp. inversions) at $j \neq -0$ if $\mathbf{q}$ has consecutions (resp. inversions) at $j, j+1, \ldots, j+\mathfrak{c}_j-1$ (resp. at $j, j+1, \ldots, j+\mathfrak{i}_j-1$) and $\mathbf{q}$ has not a consecution (resp. inversion) at $j+\mathfrak{c}_j$ (resp. $j+\mathfrak{i}_j$).
(c) We say that $\mathbf{q}$ has $\mathfrak{c}_{-0}$ (resp. $\mathfrak{i}_{-0}$) consecutions (resp. inversions) at -0 if $-\mathfrak{c}_{-0}, \ldots, -1, -0$ (resp $-0, -1, \ldots, -\mathfrak{c}_{-0}$) appear in this order in $\mathbf{q}$, and $-\mathfrak{c}_{-0}-1$ is to the right of $-\mathfrak{c}_{-0}$ (resp. to the left of $-\mathfrak{c}_{-0}$) in $\mathbf{q}$.

We insist again that part (c) of Definition 2.16 will be only used in Appendix A.

Example 2.17 Let $\mathbf{q} = (11:13, 10, 6:9, 5, 4, 0:3)$. This tuple has consecutions at $0, 1, 2, 6, 7, 8, 11$ and 12. Moreover, $\mathbf{q}$ has three consecutions at 0, it has two consecutions at 1, and just one consecution at 2.

2.3. *Fiedler pencils, generalized Fiedler pencils, and Fiedler pencils with repetition*

In this section we recall the families of Fiedler pencils, generalized Fiedler (GF) pencils, and Fiedler pencils with repetition (FPR) of a given matrix polynomial, and some of their properties. The Fiedler and GF families were introduced in [3] for regular matrix polynomials (although the authors did not assign any specific name to these pencils). They were also studied, and named, in [6] and [4], respectively, for square singular polynomials. The Fiedler pencils have been addressed recently in [8] for rectangular matrix polynomials. Finally, the FPR have been introduced in [21]. It is worth to mention also that the GF pencils have been used in the construction of structured linearizations, like symmetric [3] and, more recently, palindromic [7].

Definition 2.18m: (Fiedler pencils) Let $P(\lambda)$ be the matrix polynomial in (1). Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, k-1\}$ and $M_{\mathbf{q}}$ be the matrix in (7). Then the Fiedler pencil of $P(\lambda)$ associated with $\mathbf{q}$ is

$$F_{\mathbf{q}}(\lambda) = \lambda M_{-k} - M_{\mathbf{q}}.$$

Next we introduce GF pencils. In the following, if $\mathcal{E} = \{i_1, \ldots, i_r\}$ is a set of indices, then $-\mathcal{E}$ denotes the set $\{-i_1, \ldots, -i_r\}$.

Definition 2.19m: (GF and PGF pencils). Let $P(\lambda)$ be the matrix polynomial in (1) and let M_i, for $i = 0, 1, \ldots, k-1, -k$, be the matrices defined in (3)-(4). Let $\{C_0, C_1\}$ be a partition of $\{0, 1, \ldots, k\}$ and $\mathbf{q}$, $\mathbf{m}$ be permutations of C_0 and $-C_1$, respectively. Then the generalized Fiedler (GF) pencil of $P(\lambda)$ associated with $(\mathbf{m}, \mathbf{q})$ is the $nk \times nk$ pencil

$$K(\lambda) := \lambda M_{\mathbf{m}} - M_{\mathbf{q}}.$$

If $0 \in C_0$ and $k \in C_1$, then the pencil $K(\lambda)$ is said to be a proper generalized Fiedler (PGF) pencil of $P(\lambda)$.

If, in Definition 2.19 we admit $C_0 = \emptyset$, then $M_{\mathbf{q}} = I_{nk}$ and, if $C_1 = \emptyset$ then $M_{\mathbf{m}} = I_{nk}$.
It is obvious that any Fiedler pencil $F_{\mathbf{q}}(\lambda)$ of $P(\lambda)$ is a particular case of a GF pencil with $C_0 = \{0, 1, \ldots, k-1\}$ and $C_1 = \{k\}$. We stress that GF pencils that are not proper are defined only if A_k and/or A_0 are nonsingular.

It is proved in [4, Theorem 2.2] that the GF pencils are strong linearizations of $P(\lambda)$. We state this result here for completeness.

Theorem 2.20 m: *Let $P(\lambda)$ be an $n \times n$ matrix polynomial. Then any GF pencil of $P(\lambda)$ is a strong linearization for $P(\lambda)$.*

Theorem 2.20 is true for both regular and singular polynomials $P(\lambda)$, but in this last case we recall that the only GF pencils that are defined are the PGF pencils.

Now we recall the notion of FPR, recently introduced in [21].

Definition 2.21 m: (FPR). Let $P(\lambda)$ be the matrix polynomial in (1), where A_0 and A_k are nonsingular matrices. Let $0 \leq h \leq k-1$, and let $\mathbf{q}$ and $\mathbf{m}$ be permutations of $\{0, 1, \ldots, h\}$ and $\{-k, -k+1, \ldots, -h-1\}$, respectively. Assume that $\mathbf{l}_q$ and $\mathbf{r}_q$ are index tuples with elements from $\{0, 1, \ldots, h-1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Similarly, let $\mathbf{l}_m$ and $\mathbf{r}_m$ be index tuples with elements from $\{-k, -k+1, \ldots, -h-2\}$ such that $(\mathbf{l}_m, \mathbf{m}, \mathbf{r}_m)$ satisfies the SIP. Then, the pencil

$$L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$$

is a Fiedler pencil with repetition (FPR) associated with $P(\lambda)$.

Remark 1 m: The constraint A_0 and A_k being nonsingular can be relaxed. We need A_0 to be nonsingular only if 0 is an index in $\mathbf{l}_q$, or $\mathbf{r}_q$, or both. Similarly with A_k and the index $-k$ in $\mathbf{l}_m$ and $\mathbf{r}_m$.

Notice that if $\mathbf{l}_q$, $\mathbf{r}_q$, $\mathbf{l}_m$, and $\mathbf{r}_m$ are all the empty index tuple in Definition 2.21, then $L(\lambda)$ is a GF pencil (actually, a PGF pencil). Note also that not all GF pencils are FPR.

We have the analogue of Theorem 2.20 for FPR.

Theorem 2.22 m: [21] *Let $P(\lambda)$ be an $n \times n$ matrix polynomial. Then every FPR of $P(\lambda)$ is a strong linearization of $P(\lambda)$.*

The requirement that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_m, \mathbf{m}, \mathbf{r}_m)$ satisfy the SIP in Definition 2.21 is introduced in order to keep the product of the M_i matrices defining $L(\lambda)$ *operation free* [21]. As a consequence, the coefficients of $L(\lambda)$ are block-partitioned matrices, whose $n \times n$ blocks are of the form $0, \pm I$, or $\pm A_i$ (that is, no products of A_i blocks appear). This requirement implies some constraints in the strings of $\mathbf{l}_q$, $\mathbf{r}_q$, $\mathbf{l}_m$ and $\mathbf{r}_m$ when they are expressed in column standard form. In the following, we focus on $\mathbf{r}_q$ and $\mathbf{r}_m$ because they are the only relevant strings for the right eigencolumns (as we will see in Section 5.3).

Lemma 2.23 m: *Let h be a nonnegative integer and $\mathbf{q} = (\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$ be a permutation of $\{0, 1, \ldots, h\}$ in column standard form, where $\mathbf{b}_i = (t_{i-1}+1 : t_i)$, for $i = 1, \ldots, s$, with $t_0 = -1$ and $t_s = h$. Let $\mathbf{r}_q = (h_1 : h_2)$ be a string with $0 \leq h_1 \leq h_2 < h$ and such that $(\mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Then, either*

- $t_{d-1}+1 = h_1 \leq h_2 < t_d$, *for some $s \geq d \geq 1$; or*
- $t_{d-1}+1 < h_1 \leq h_2 < t_d$, *for some $s \geq d \geq 1$.*

Proof: If $h_1 = t_{d-1}+1$ for some $s \geq d \geq 1$, then $(\mathbf{q}, \mathbf{r}_q) \sim (t_{s-1}+1 : h, \ldots, t_{d-1}+1 : t_d, t_{d-2}+1 : t_{d-1}, t_{d-1}+1 : h_2, \ldots, 0 : t_1)$. Since $(\mathbf{q}, \mathbf{r}_q)$ satisfies the SIP, then $h_2 < t_d$.

If $h_1 \neq t_{i-1}+1$ for all $1 \leq i \leq s$, since $\mathbf{q}$ is a permutation of $\{0, 1 \ldots, h\}$ and the elements of $\mathbf{r}_q$ are in $\{0, 1, \ldots, h-1\}$, there is a string $\mathbf{b}_d$ in $\mathbf{q}$ containing h_1, that is, $t_{d-1}+1 < h_1 \leq t_d$. Then $(\mathbf{q}, \mathbf{r}_q)$ is equivalent to the following index tuple in column standard form

$$(t_{s-1}+1 : h, \ldots, t_d+1 : t_{d+1}, t_{d-1}+1 : t_d, h_1 : h_2, t_{d-2}+1 : t_{d-1}, \ldots, 0 : t_1).$$

Then it must be $h_2 < t_d$ because, otherwise, this tuple would not satisfy the SIP. □

We have the analogue of Lemma 2.23 for tuples of negative integers. We omit the proof because it is similar to the one of Lemma 2.23.

Lemma 2.24$\mathbf{m}$: *Let h be a nonnegative integer with $0 \leq h \leq k-1$ and $\mathbf{m} = (\mathbf{b}'_r, \mathbf{b}'_{r-1}, \ldots, \mathbf{b}'_1)$ be a permutation of $\{-k, -k+1, \ldots, -h-1\}$ in column standard form, where $\mathbf{b}'_i = (-t'_{i-1}+1 : -t'_i)$, for $i = 1, \ldots, r$, with $-t'_0 + 1 = -k$ and $-t'_r = -h-1$. Let $\mathbf{r}_m = (-h'_1 : -h'_2)$ be a string with $-k \leq -h'_1 \leq h'_2 < -h-2$ and such that $(\mathbf{m}, \mathbf{r}_m)$ satisfies the SIP. Then, either*

- $-t'_{d-1} + 1 = -h'_1 \leq -h'_2 < -t'_d$, *for some $r \geq d \geq 1$; or*
- $-t'_{d-1} + 1 < -h'_1 \leq -h'_2 < -t'_d$, *for some $r \geq d \geq 1$.*

Lemmas 2.23 and 2.24 motivate the following definition.

Definition 2.25$\mathbf{m}$: (Type 1 strings). Let h be a nonnegative integer with $0 \leq h \leq k-1$ and $\mathbf{q} = (\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$ be a permutation of $\{0, 1, \ldots, h\}$ in column standard form, with $\mathbf{b}_i = (t_{i-1}+1 : t_i)$, for $i = 1, \ldots, s$. Let $\mathbf{r}_q = (h_1 : h_2)$ be a string with $0 \leq h_1 \leq h_2 < h$ and such that $(\mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Then $\mathbf{r}_q$ is said to be a type 1 string relative to $\mathbf{q}$ if $h_1 = t_{d-1} + 1$, for some $d = 1, \ldots, s$.

Similarly, let $\mathbf{m} = (\mathbf{b}'_r, \mathbf{b}'_{r-1}, \ldots, \mathbf{b}'_1)$ be a permutation of $\{-k, -k+1, \ldots, -h-1\}$ in column standard form, with $\mathbf{b}'_i = (-t'_{i-1}+1 : -t'_i)$, for $i = 1, \ldots, r$. Let $\mathbf{r}_m = (-h'_1 : -h'_2)$ be a string with $h+2 \leq h'_2 \leq h'_1 \leq k$ such that $(\mathbf{m}, \mathbf{r}_m)$ satisfies the SIP. Then $\mathbf{r}_m$ is said to be a type 1 string relative to $\mathbf{m}$ if $h'_1 = t'_{d-1} - 1$, for some $d = 1, \ldots, r+1$.

Definition 2.26$\mathbf{m}$: (Associated simple tuple for one string). Let h be a nonnegative integer and $\mathbf{q} = (\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$ be a permutation of $\{0, 1, \ldots, h\}$ in column standard form, with $\mathbf{b}_i = (t_{i-1}+1 : t_i)$, for $i = 1, \ldots, s$. Let $\mathbf{r}_q = (h_1 : h_2)$ be a type 1 string relative to $\mathbf{q}$. Set

$$t_{d-1} + 1 = h_1 \leq h_2 < t_d,$$

for some $1 \leq d \leq s$. Then the simple tuple associated with $(\mathbf{q}, \mathbf{r}_q)$ is the simple tuple

$$\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) := \left(\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_{d+1}, \widetilde{\mathbf{b}}_d, \widetilde{\mathbf{b}}_{d-1}, \mathbf{b}_{d-2}, \ldots, \mathbf{b}_1\right),$$

where:

(a) If $d > 1$, then

$$\widetilde{\mathbf{b}}_{d-1} = (t_{d-2}+1 : h_2) \quad \text{and} \quad \widetilde{\mathbf{b}}_d = (h_2+1 : t_d).$$

(b) If $d = 1$, then

$$\widetilde{\mathbf{b}}_{d-1} = (0 : h_2) \quad \text{and} \quad \widetilde{\mathbf{b}}_d = (h_2+1 : t_1).$$

Now we extend recursively definitions 2.25 and 2.26 to tuples with more than one string.

Definition 2.27$\mathbf{m}$: (Type 1 tuples and associated simple tuple) Let h be a nonnegative integer and let $\mathbf{q} = (\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$ be a permutation of $\{0, 1, \ldots, h\}$ in column standard form, with $\mathbf{b}_i = (t_{i-1}+1 : t_i)$, for $i = 1, \ldots, s$. Let $\mathbf{r}_q$ be an index tuple such that $(\mathbf{q}, \mathbf{r}_q)$ satisfies the SIP (so, in particular, $\mathbf{r}_q$ satisfies the SIP). Let $\mathbf{r}_q \sim (\mathbf{c}_1, \ldots, \mathbf{c}_g)$ in column standard form, with $\mathbf{c}_1, \ldots, \mathbf{c}_g$ strings. Then $\mathbf{r}_q$ is a type 1 tuple relative to $\mathbf{q}$ if the following conditions hold:

(i) $\mathbf{c}_1$ is a type 1 string relative to $\mathfrak{s}_0 := \mathbf{q}$.

(ii) For each $i = 1, \ldots, g-1$, $\mathbf{c}_{i+1}$ is a type 1 string relative to $\mathfrak{s}_i$, which is the simple tuple associated with $(\mathfrak{s}_{i-1}, \mathbf{c}_i)$.

The simple tuple associated with $(\mathbf{q}, \mathbf{r}_q)$ is the simple tuple associated with $(\mathfrak{s}_{g-1}, c_g)$. We will denote it by $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q)$.

Similarly, let $\mathbf{m} = (\mathbf{b}'_r, \mathbf{b}'_{r-1}, \ldots, \mathbf{b}'_1)$ be a permutation of $\{-k, -k+1, \ldots, -h-1\}$ in column standard form, with $\mathbf{b}'_i = (-t'_{i-1} + 1 : -t'_i)$, for $i = 1, \ldots, r$. Let $\mathbf{r}_m$ be an index tuple such that $(\mathbf{m}, \mathbf{r}_m)$ satisfies the SIP. Let $\mathbf{r}_m \sim (\mathbf{c}'_1, \ldots, \mathbf{c}'_g)$ in column standard form, with $\mathbf{c}'_1, \ldots, \mathbf{c}'_g$ strings. Then $\mathbf{r}_m$ is a type 1 tuple relative to $\mathbf{m}$ if the following conditions hold:

(i) $\mathbf{c}'_1$ is a type 1 string relative to $\mathfrak{s}_0 := \mathbf{m}$.
(ii) For each $i = 1, \ldots, g-1$, $\mathbf{c}'_{i+1}$ is a type 1 string relative to $\mathfrak{s}_i$, which is the simple tuple associated with $(\mathfrak{s}_{i-1}, \mathbf{c}'_i)$.

The simple tuple associated with $(\mathbf{m}, \mathbf{r}_m)$ is the simple tuple associated with $(\mathfrak{s}_{g-1}, c'_g)$. We will denote it by $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m)$.

Example 2.28 Let $h = 16$, $\mathbf{q} = (16, 11 : 15, 7 : 10, 6, 2 : 5, 0 : 1)$ and $\mathbf{r}_q = (11, 12, 2, 7, 13, 8, 9, 10, 11, 3, 12)$. It is immediate to check that $(\mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Also, $\mathbf{r}_q \sim (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ in column standard form, with $\mathbf{c}_1 = (11 : 13)$, $\mathbf{c}_2 = (7 : 12)$ and $\mathbf{c}_3 = (2 : 3)$ (where we have removed the empty strings). Then, following the notation in Definition 2.26, we have $\mathfrak{s}_0 = \mathbf{q}$, so $\mathbf{c}_1$ is a type 1 string relative to $\mathfrak{s}_0$; $\mathfrak{s}_1 = (16, 14 : 15, 7 : 13, 6, 2 : 5, 0 : 1)$, so $\mathbf{c}_2$ is a type 1 string relative to $\mathfrak{s}_1$; $\mathfrak{s}_2 = (16, 14 : 15, 13, 6 : 12, 6, 2 : 5, 0 : 1)$, so $\mathbf{c}_3$ is also a type 1 string relative to $\mathfrak{s}_2$. Finally, $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) = (16, 14 : 15, 13, 6 : 12, 4 : 5, 0 : 3)$.

3. Eigenvalues and eigenvectors, minimal indices and minimal bases. Right and left eigencolumns of linearizations

The right and left *eigenspaces* of an $n \times n$ regular matrix polynomial $P(\lambda)$ at $\lambda_0 \in \mathbb{C}$ are the right and left null spaces of $P(\lambda_0)$, i.e.,

$$\mathcal{N}_r(P(\lambda_0)) := \{x \in \mathbb{C}^n : P(\lambda_0)x = 0\} ,$$
$$\mathcal{N}_\ell(P(\lambda_0)) := \left\{y \in \mathbb{C}^n : P(\lambda_0)^T y = 0\right\} .$$

If $P(\lambda)$ is a regular matrix polynomial and $\mathcal{N}_r(P(\lambda_0))$ (or, equivalently, $\mathcal{N}_\ell(P(\lambda_0))$) is nontrivial, then λ_0 is said to be a (*finite*) *eigenvalue*, and a vector $x \neq 0$ (respectively, $y \neq 0$) in $\mathcal{N}_r(P(\lambda_0))$ (resp. $\mathcal{N}_\ell(P(\lambda_0))$) is a *right* (resp. *left*) *eigenvector of P associated with* λ_0. Matrix polynomials may also have infinite eigenvalues. In this work we will focus on finite eigenvalues. Infinite eigenvalues are considered only in Appendix B, because the techniques used for this case are completely different (though simpler) than the ones employed for finite eigenvalues.

In the case of $P(\lambda)$ being a square singular $n \times n$ matrix polynomial, the previous notion of eigenvalue (and eigenvector) makes no sense, because with this definition all complex values would be eigenvalues of $P(\lambda)$. In this case we are interested, instead of eigenvectors, in minimal bases of $P(\lambda)$. This notion is related to the *right* and *left nullspaces* of $P(\lambda)$, which are, respectively, the following subspaces:

$$\mathcal{N}_r(P) := \{x(\lambda) \in \mathbb{C}(\lambda)^n : P(\lambda)x(\lambda) \equiv 0\} ,$$
$$\mathcal{N}_\ell(P) := \left\{y(\lambda) \in \mathbb{C}(\lambda)^n : P(\lambda)^T y(\lambda) \equiv 0\right\} .$$

A *polynomial basis* of a vector space over $\mathbb{C}(\lambda)$ is a basis consisting of polynomial vectors (that is, vectors whose coordinates are polynomials in λ). The *order* of a polynomial basis is the sum of the degrees of its vectors. Here the *degree* of a polynomial vector is the maximum degree of its components. A *right* (respectively, *left*) minimal basis of $P(\lambda)$ is a polynomial basis of $\mathcal{N}_r(P)$ (resp. $\mathcal{N}_\ell(P)$) such that the order is minimal among all polynomial bases of $\mathcal{N}_r(P)$ (resp. $\mathcal{N}_\ell(P)$) [9].

In order to relate eigenvectors and minimal bases of $P(\lambda)$ with the ones of a given linearization of $P(\lambda)$ we introduce the following notion, which is valid even for rectangular matrix polynomials although, for simplicity, we restrict ourselves here to square matrix polynomials.

Definition 3.1 m: **(Right and left eigencolumn)** Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k and $H(\lambda)$ be a linearization of $P(\lambda)$. Then, a right eigencolumn (resp., a left eigencolumn) of $H(\lambda)$ is a block-column matrix polynomial $\mathcal{R}_H(\lambda) \in \mathbb{C}[\lambda]^{nk\times n}$ (resp. $\mathcal{L}_H(\lambda) \in \mathbb{C}[\lambda]^{nk\times n}$) partitioned into k blocks of size $n \times n$ and such that

(a) $\operatorname{rank} \mathcal{R}_{\rm H}(\mu) = {\rm n}$ (resp. $\operatorname{rank} \mathcal{L}_{\rm H}(\mu) = {\rm n}$), for all $\mu \in \mathbb{C}$;
(b) there is a nonnegative integer $\kappa(P)$ such that, for all polynomial vector $v(\lambda) \in \mathcal{N}_r(P)$ (resp $w(\lambda) \in \mathcal{N}_\ell(P)$), $\deg \mathcal{R}_H(\lambda)v(\lambda) = \deg v(\lambda) + \kappa(P)$ (resp. $\deg \mathcal{L}_H(\lambda)w(\lambda) = \deg w(\lambda) + \kappa(P)$); and
(c) $H(\lambda)\mathcal{R}_H(\lambda) = \widetilde{U}(\lambda)P(\lambda)$ for some matrix polynomial $\widetilde{U}(\lambda) \in \mathbb{C}[\lambda]^{nk\times n}$ (resp. $H(\lambda)^T\mathcal{L}_H(\lambda) = \widetilde{V}(\lambda)P(\lambda)^T$), for some matrix polynomial $\widetilde{V}(\lambda) \in \mathbb{C}[\lambda]^{nk\times n}$).

The motivation for introducing Definition 3.1 is given in Lemma 3.3. To prove it, we need the following result, which deals with rectangular matrix polynomials.

Lemma 3.2 m:

(a) *Let $Q(\lambda)$ be an $m \times n$ matrix polynomial with $m \geq n$ such that $Q(\mu)$ has full-column rank for all $\mu \in \mathbb{C}$. Then, there exists an $m \times m$ unimodular matrix polynomial $U(\lambda)$ such that*

$$Q(\lambda) = U(\lambda)\begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

that is, $Q(\lambda)$ is formed by a subset of the columns of a certain unimodular matrix.
(b) *Let $Q(\lambda)$ be an $n \times n$ matrix polynomial with $m \leq n$ such that $Q(\mu)$ has full-row rank for all $\mu \in \mathbb{C}$. Then, there exists an $n \times n$ unimodular matrix polynomial $V(\lambda)$ such that*

$$Q(\lambda) = \begin{bmatrix} I_m & 0 \end{bmatrix} V(\lambda),$$

that is, $Q(\lambda)$ is formed by a subset of the rows of a certain unimodular matrix.

Proof : We prove only part (a), since part (b) follows from (a) via transposition. The condition "$Q(\mu)$ has full-column rank for all $\mu \in \mathbb{C}$" implies that all the invariant polynomials of $Q(\lambda)$ are equal to one. Therefore, a Smith canonical factorization [11] of $Q(\lambda)$ is

$$Q(\lambda) = \widetilde{U}(\lambda)\begin{bmatrix} I_n \\ 0 \end{bmatrix} V(\lambda),$$

with $\widetilde{U}(\lambda)$ and $V(\lambda)$ unimodular. Now simply observe that

$$Q(\lambda) = \widetilde{U}(\lambda)\begin{bmatrix} V(\lambda) \\ 0 \end{bmatrix} = \widetilde{U}(\lambda)\begin{bmatrix} V(\lambda) & \\ & I_{m-n} \end{bmatrix}\begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

and define

$$U(\lambda) := \widetilde{U}(\lambda) \begin{bmatrix} V(\lambda) & \\ & I_{m-n} \end{bmatrix},$$

which is unimodular, because the product of unimodular matrices is unimodular. □

Notice that both $\mathcal{R}_H(\lambda)$ and $\mathcal{L}_H(\lambda)$ in Definition 3.1 are matrix polynomials, so it makes sense to talk about their degrees.

Lemma 3.3m: *Let $H(\lambda)$ be a linearization of a matrix polynomial $P(\lambda)$, and let $\mathcal{R}_H(\lambda)$ and $\mathcal{L}_H(\lambda)$ be, respectively, a right and a left eigencolumn of $H(\lambda)$. Then:*

(a) The maps

$$\begin{array}{rccc} \mathcal{R}_H : & \mathcal{N}_r(P) & \longrightarrow & \mathcal{N}_r(H) \\ & v(\lambda) & \longmapsto & \mathcal{R}_H(\lambda)v(\lambda) \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathcal{L}_H : & \mathcal{N}_\ell(P) & \longrightarrow & \mathcal{N}_\ell(H) \\ & w(\lambda) & \longmapsto & \mathcal{L}_H(\lambda)w(\lambda) \end{array}$$

are isomorphisms of $\mathbb{C}(\lambda)$-vector spaces. Moreover, $\mathcal{R}_H v(\lambda) \in \mathbb{C}[\lambda]^n$ (resp. $\mathcal{L}_H w(\lambda) \in \mathbb{C}[\lambda]^n$) if and only if $v(\lambda) \in \mathbb{C}[\lambda]^n$ (resp. $w(\lambda) \in \mathbb{C}[\lambda]^n$), and $\mathcal{R}_H$, $\mathcal{L}_H$ are maps with a uniform degree-shift between polynomial vectors equal to $\kappa(P)$.

(b) If $P(\lambda)$ is regular and $\lambda_0 \in \mathbb{C}$ is a finite eigenvalue of $P(\lambda)$, the maps

$$\begin{array}{rccc} \mathcal{R}_H^0 : & \mathcal{N}_r(P(\lambda_0)) & \longrightarrow & \mathcal{N}_r(H(\lambda_0)) \\ & v & \longmapsto & \mathcal{R}_H(\lambda_0)v \end{array}$$

and

$$\begin{array}{rccc} \mathcal{L}_H^0 : & \mathcal{N}_\ell(P(\lambda_0)) & \longrightarrow & \mathcal{N}_\ell(H(\lambda_0)) \\ & w & \longmapsto & \mathcal{L}_H(\lambda_0)w \end{array}$$

are isomorphisms of $\mathbb{C}$-vector spaces.

Proof : We will only prove the statement for the right eigencolumns, because the arguments for the left ones are similar. Let us begin with $\mathcal{R}_H$ in part (a). Clearly, the map $\mathcal{R}_H$ is a linear map. Let $\mathcal{R}_H(\lambda)$ be as in Definition 3.1. Then, given $v(\lambda) \in \mathcal{N}_r(P)$, we have

$$H(\lambda)\mathcal{R}_H(\lambda)v(\lambda) = \widetilde{U}(\lambda)P(\lambda)v(\lambda) \equiv 0,$$

so $\mathcal{R}_H(\lambda)v(\lambda) \in \mathcal{N}_r(H)$, and the map is well-defined. Now, notice that, as a consequence of (a) or (b) in Definition 3.1, the columns of $\mathcal{R}_H(\lambda)$ are linearly independent over $\mathbb{C}(\lambda)$. Hence the map is injective. Since the dimension of $\mathcal{N}_r(P)$ and $\mathcal{N}_r(H)$ coincide (as a consequence of the definition of linearization), the map $\mathcal{R}_H$ is an isomorphism.

Now, let $v(\lambda) \in \mathbb{C}[\lambda]^n$. Since $\mathcal{R}_H(\lambda)$ is a matrix polynomial, $\mathcal{R}_H(\lambda)v(\lambda)$ is also a polynomial vector. Conversely, let $v(\lambda) \in \mathbb{C}(\lambda)^n$ be such that $\mathcal{R}_H(\lambda)v(\lambda) \in \mathbb{C}[\lambda]^{nk}$. By Lemma 3.2 we have

$$\mathcal{R}_H(\lambda) = U(\lambda) \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

for some unimodular matrix polynomial $U(\lambda) \in \mathbb{C}[\lambda]^{nk \times nk}$. Then

$$\begin{bmatrix} v(\lambda) \\ 0 \end{bmatrix} = U(\lambda)^{-1}\mathcal{R}_H(\lambda)v(\lambda),$$

with $U(\lambda)^{-1}$ being a matrix polynomial. Hence $v(\lambda) \in \mathbb{C}[\lambda]^n$. The fact that $\mathcal{R}_H$ is a degree-shifting map with constant shift equal to $\kappa(P)$ is a direct consequence of (b) in Definition 3.1.

Now let us prove (b). Let λ_0 and v be as in the statement. Clearly the map $\mathcal{R}_H^0$ is linear. We have $P(\lambda_0)v = 0$ and, by definition of right eigencolumn, we get $H(\lambda_0)\mathcal{R}_H^0(v) = H(\lambda_0)\mathcal{R}_H(\lambda_0)v = \widetilde{U}(\lambda_0)P(\lambda_0)v = 0$, so the map is well defined. Now, set $v \neq 0$. By property (a) in Definition 3.1, $\mathcal{R}_H(\lambda_0)$ is of full column rank over $\mathbb{C}$. Hence $\mathcal{R}_H^0(v) \neq 0$, so $\mathcal{R}_H^0$ is injective. The fact that the map $\mathcal{R}_H^0$ is an isomorphism follows from the fact that the dimensions of both $\mathcal{N}_r(P(\lambda_0))$ and $\mathcal{N}_r(H(\lambda_0))$ coincide. □

Remark 1 m: The fact that $\mathcal{R}_H$ is an isomorphism implies that all bases of $\mathcal{N}_r(H)$ are of the form $\{\mathcal{R}_H v_1(\lambda), \ldots, \mathcal{R}_H v_p(\lambda)\}$, with $\{v_1(\lambda), \ldots, v_p(\lambda)\}$ a basis of $\mathcal{N}_r(P)$. The same happens with $\mathcal{L}_H$ and bases of $\mathcal{N}_\ell(P)$ and $\mathcal{N}_\ell(H)$, and also with $\mathcal{R}_H^0$ and $\mathcal{L}_H^0$ and right and left eigenspaces associated with λ_0.

In Lemma 3.3 we are using the same number $\kappa(P)$ for both the left and right eigencolumns of $H(\lambda)$. However, if $\mathcal{R}_H(P)$ and $\mathcal{L}_H(P)$ are, respectively, a right and a left eigencolumn associated with a given linearization $H(\lambda)$ of $P(\lambda)$, and $\kappa_1(P), \kappa_2(P)$ are the corresponding nonnegative integers in part (b) of Definition 3.1, we may have $\kappa_1(P) \neq \kappa_2(P)$.

As an immediate consequence of Lemma 3.3 we get the following result, which shows the relevance of eigencolumns to get formulas for the left and right eigenvectors and minimal bases of linearizations in terms of the corresponding magnitudes of the original polynomial.

Theorem 3.4 m: *Let $H(\lambda)$ be a linearization of an $n \times n$ matrix polynomial $P(\lambda)$ of degree k, and let $\mathcal{R}_H(\lambda) \in \mathbb{C}[\lambda]^{nk\times n}$ and $\mathcal{L}_H(\lambda) \in \mathbb{C}[\lambda]^{nk\times n}$ be a right and a left eigencolumn of $H(\lambda)$, respectively.*

(a) **(Right and left eigenvectors using eigencolumns)** *If λ_0 is a finite eigenvalue of $P(\lambda)$ and v, w are, respectively, a right and a left eigenvector of $P(\lambda)$ associated with λ_0, then $\mathcal{R}_H(\lambda_0)v$ and $\mathcal{L}_H(\lambda_0)w$ are, respectively, a right and a left eigenvector of $H(\lambda)$ associated with λ_0.*

(b) **(Right and left minimal bases using eigencolumns)** *Let $\{v_1(\lambda), \ldots, v_p(\lambda)\}$ and $\{w_1(\lambda), \ldots, w_p(\lambda)\}$ be a right and a left minimal basis of $P(\lambda)$, respectively. Then $\{\mathcal{R}_H(\lambda)v_1(\lambda), \ldots, \mathcal{R}_H(\lambda)v_p(\lambda)\}$ and $\{\mathcal{L}_H(\lambda)w_1(\lambda), \ldots, \mathcal{L}_H(\lambda)w_p(\lambda)\}$ are a right and a left minimal basis of $H(\lambda)$, respectively.*

Proof : We will only address the proof for the right eigenvectors and minimal bases, since the proof for the left ones is similar.

Claim (a) is an immediate consequence of (b) in Lemma 3.3.

For claim (b), let $\mathcal{B}_P = \{v_1(\lambda), \ldots, v_p(\lambda)\}$ be a right minimal basis of $P(\lambda)$, where $p = \dim \mathcal{N}_r(P)$. By Lemma 3.3, $\{\mathcal{R}_H(\lambda)v_1(\lambda), \ldots, \mathcal{R}_H(\lambda)v_p(\lambda)\}$ is a basis of $\mathcal{N}_r(H)$ consisting of polynomial vectors. It remains to show that this basis is minimal. We proceed by contradiction. Let us assume that $\{\mathcal{R}_H(\lambda)v_1(\lambda), \ldots, \mathcal{R}_H(\lambda)v_p(\lambda)\}$ is not minimal. Then there is a right minimal basis $\mathcal{B}_H = \{\widetilde{v}_1(\lambda), \ldots, \widetilde{v}_p(\lambda)\}$ of $H(\lambda)$ such that the order of $\mathcal{B}_H$ is less than the order of $\{\mathcal{R}_H(\lambda)v_1(\lambda), \ldots, \mathcal{R}_H(\lambda)v_p(\lambda)\}$. Therefore, by Lemma 3.3, we have order$(\mathcal{B}_H) <$ order$(\mathcal{B}_P) + p \cdot \kappa(P)$ (with $\kappa(P)$ as in Definition 3.1(b)). Since, by Lemma 3.3 again, $\mathcal{R}_H$ is an isomorphism, we have that $\widetilde{v}_i(\lambda) = \mathcal{R}_H(\lambda)\widehat{v}_i(\lambda)$, for $i = 1, \ldots, p$, where $\{\widehat{v}_1(\lambda), \ldots, \widehat{v}_p(\lambda)\}$ is a basis of $\mathcal{N}_r(P)$ consisting of polynomial vectors and whose order is equal to order$(\mathcal{B}_H) - p \cdot \kappa(P)$, which is less than the order of $\mathcal{B}_P$. But this is in contradiction with the fact that $\mathcal{B}_P$ is minimal. □

Though Theorem 3.4 is the one we need to get formulas for the eigenvectors and minimal bases of $H(\lambda)$, we include here for completeness the converse statement. The proof is an immediate consequence of Lemma 3.3 and the proof of Theorem 3.4, where we have seen that minimal bases of $H(\lambda)$ and $P(\lambda)$ are in one-to-one correspondence via the maps $\mathcal{R}_H$ and $\mathcal{L}_H$.

Theorem 3.5 m: *Let $H(\lambda)$ be a linearization of an $n \times n$ matrix polynomial $P(\lambda)$ of degree k, and let $\mathcal{R}_H(\lambda) \in \mathbb{C}[\lambda]^{nk\times n}$ and $\mathcal{L}_H(\lambda) \in \mathbb{C}[\lambda]^{nk\times n}$ be a right and a left eigencolumn of $H(\lambda)$, respectively.*

(a) If $P(\lambda)$ is regular: Let λ_0 be a finite eigenvalue of $P(\lambda)$ and x, y be, respectively, a right and a left eigenvector of $H(\lambda)$ associated with λ_0. Then there exist $v, w \in \mathbb{C}^n$ which are, respectively, a right and a left eigenvector of $P(\lambda)$ associated with λ_0 such that $x = \mathcal{R}_H(\lambda_0)v$ and $y = \mathcal{L}_H(\lambda_0)w$.
(b) If $P(\lambda)$ is singular: Let $x_1(\lambda), \ldots, x_p(\lambda)$ and $y_1(\lambda), \ldots, y_p(\lambda)$ be, respectively, a right and a left minimal basis of $H(\lambda)$. Then, there exist a right and a left minimal basis of $P(\lambda)$, $v_1(\lambda), \ldots, v_p(\lambda)$ and $w_1(\lambda), \ldots, w_p(\lambda)$, respectively, such that $x_i(\lambda) = \mathcal{R}_H(\lambda)v_i(\lambda)$ and $y_i(\lambda) = \mathcal{L}_H(\lambda)w_i(\lambda)$, for $i = 1, \ldots, p$.

By Theorem 3.4, an eigencolumn of a given linearization $H(\lambda)$ provides formulas for both eigenvectors and minimal bases of $H(\lambda)$. Moreover the eigencolumn relates the eigenvectors and minimal bases of $H(\lambda)$ with the eigenvectors and minimal bases of the polynomial $P(\lambda)$. Though $P(\lambda)$ in the statement of Theorem 3.4 is an arbitrary square matrix polynomial, we want to emphasize that the formula for eigenvectors makes only sense for regular polynomials, whereas the formulas for minimal bases are valid only for $P(\lambda)$ singular.

From the defining identity (2) of a linearization, we may get a right and a left eigencolumn for $H(\lambda)$. More precisely, let us consider $U(\lambda)$ and $V(\lambda)$ as block-partitioned matrices with $k \times k$ blocks of size $n \times n$ (each). Then, if U^L and V^R denote the last block-columns of $U(\lambda)^{\mathcal{B}}$ and $V(\lambda)$, respectively, we have that U^L and V^R fulfill property (c) in Definition 3.1 (see [6, Lemma 5.1]), and it is also trivial to see that they satisfy property (a). Actually, Lemma 3.2 tells us that every right and left eigencolumn are the last block-column of a certain unimodular matrix. The study of the structure of the block-column matrices V^R and U^L for the linearizations described in Section 2.3 is the main goal of this paper.

In the particular case where $\mathcal{R}_H(\lambda) = V^R$ and $\mathcal{L}_H(\lambda) = U^L$ as in the previous paragraph, with $H(\lambda)$ being a linearization within the families introduced in Section 2.3, we will see that both $\mathcal{R}_H(\lambda)$ and $\mathcal{L}_H(\lambda)$, when considered as block-partitioned column matrices consisting of k blocks of size $n \times n$, contain an identity block. We will also prove that they are degree-shifting maps between polynomial vectors, which is property (b). As a consequence, they will be right and left eigencolumns, respectively.

Summarizing the previous arguments, one way to obtain an expression for the right eigenvectors and right minimal bases of a given linearization $H(\lambda)$ of $P(\lambda)$ within the families of Section 2.3 is through the last block column V^R of the matrix $V(\lambda)$, with $V(\lambda)$ as in (2). Namely, if $v_1(\lambda), \ldots, v_p(\lambda)$ is a right minimal basis of $P(\lambda)$, the corresponding right minimal basis of $H(\lambda)$ is $V^R(\lambda)v_1(\lambda), \ldots, V^R(\lambda)v_p(\lambda)$. Similar expressions follow for the right eigenvectors via $V^R(\lambda_0)$, and also for the left minimal bases and left eigenvectors with U^L. In Section 4 we display formulas for $V^R(\lambda)$ and $U^L(\lambda)$ for linearizations within the families considered in Section 2.3.

4. Main results

By theorems 2.20 and 2.22, all pencils within the families considered in Section 2.3 are (strong) linearizations. The main goal of this paper is to derive formulas for the left and right eigenvectors and the left and right minimal bases of these linearizations. In particular, we want to relate the left and right eigenvectors and the left and right minimal bases of these linearizations with the ones of the polynomial $P(\lambda)$, as explained in Section 3.

In sections 4.1, 4.2 and 4.3 we display formulas for the left and right eigencolumns of the Fiedler pencils, the GF pencils and the FPR associated with type 1 tuples. From these formulas and Theorem 3.4 the corresponding formulas for eigenvectors and minimal bases will follow immediately. As we will see, all these eigencolums contain an identity block. These identity blocks allow us to go in the opposite direction and recover the eigenvectors and minimal bases of $P(\lambda)$ from the eigenvectors and minimal bases of the linearizations, as it was done in [4] for the GF pencils, and in [6] for Fiedler pencils. The proofs of all these formulas are addressed in Section 5.

From now on, when considering an ordered tuple $\mathbf{z}$ with ℓ entries, we will follow the convention of assigning the position 0 to the first entry in the tuple. Also, for each $0 \leq i \leq \ell$, $\mathbf{z}(i)$ will denote the number occupying the ith position in $\mathbf{z}$ and, for each $j \in \mathbf{z}$, $\mathbf{z}^{-1}(j)$ denotes the position of j in $\mathbf{z}$ (starting with 0). In other words, we see an index tuple $\mathbf{z}$ with ℓ elements, $j_1, \ldots, j_\ell$, as a bijection $\mathbf{z} : \{0, 1, \ldots, \ell-1\} \to \{j_1, \ldots, j_\ell\}$. We will also associate tuples of blocks to tuples of numbers. Then, according to the previous convention, when referring to "the position of a block" we understand that we start counting in 0 (the 0th position)

4.1. *Eigencolumns of Fiedler pencils*

The following theorem is a restatement of Lemma 5.3 in [6]. In our statement we implicitly use the fact that every permutation is equivalent to a permutation in column standard form.

Theorem 4.1 m: *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k, P_i be its ith Horner shift, for $i = 0, \ldots, k$, and let $\mathbf{z}$ be a permutation of $\{0, 1, \ldots, k-1\}$. Let $F_{\mathbf{z}}(\lambda) = \lambda M_{-k} - M_{\mathbf{z}}$ be the Fiedler pencil of $P(\lambda)$ associated with $\mathbf{z}$. Let $\mathbf{w} = (\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$ be the permutation of $\{0, 1, \ldots, k-1\}$ in column standard form equivalent to $\mathbf{z}$, with $\mathbf{b}_j = (t_{j-1} + 1 : t_j)$, for $j = 1, \ldots, s$.*

(a) A right eigencolumn for $F_{\mathbf{z}}(\lambda)$ is given by

$$\mathcal{R}_{\mathbf{z}}(P) := \left[B_0 \; B_1 \; \ldots \; B_{k-1} \right]^{\mathcal{B}}, \tag{8}$$

where, if $\mathbf{w}(i) \in \mathbf{b}_j$, for some $j = 1, \ldots, s$, then

$$B_i = \begin{cases} \lambda^{j-1} I, & \text{if } \; i = k - t_j - 1, \\ \lambda^{j-1} P_i, & \text{otherwise.} \end{cases} \tag{9}$$

Moreover, if $\mathbf{z}$ has $\mathfrak{c}_0$ consecutions at 0, then the $(k - \mathfrak{c}_0)$th block of $\mathcal{R}_{\mathbf{z}}(P)$ is equal to I_n.

(b) A left eigencolumn for $F_{\mathbf{z}}(\lambda)$ is given by $\mathcal{L}_{\mathbf{z}}(P) := \mathcal{R}_{\operatorname{rev}\mathbf{z}}(P^T)$. Moreover, if $\mathbf{z}$ has $\mathfrak{i}_0$ inversions at 0 then the $(k - \mathfrak{i}_0)$th block of $\mathcal{L}_{\mathbf{z}}(P)$ is equal to I_n.

Remark 1 m: We want to stress that $k - t_j - 1$ in (9) is the position in $\mathbf{z}$, starting with 0, of the first number of $\mathbf{b}_j$ (that is, $\mathbf{z}^{-1}(t_{j-1}+1) = k - t_j - 1$). Then, we may see the right eigencolumn of $F_{\mathbf{z}}(\lambda)$ as partitioned into s strings of blocks, each one corresponding to a string $\mathbf{b}_j$ in $\mathbf{z}$. More precisely, the string in $\mathcal{R}_{\mathbf{z}}(P)$ associated with $\mathbf{b}_j$ is of the form $\lambda^{j-1} \left[I \; P_{\mathbf{z}^{-1}(t_{j-1}+2)} \; \ldots \; P_{\mathbf{z}^{-1}(t_j)} \right]^{\mathcal{B}}$. Hence, the right eigencolumn $\mathcal{R}_{\mathbf{z}}(P)$ can be easily obtained from the column standard form of $\mathbf{z}$.

Remark 2 m: There is a duality between the formulas for the right and left eigencolumns of $P(\lambda)$ given in Theorem 4.1. More precisely, if the ith block, B_i, of $\mathcal{R}_{\mathbf{z}}$ in

(8), with $i \neq 0$, is of the form $\lambda^{j-1}P_i$, then the ith block, B'_i, of $\mathcal{L}_{\mathbf{z}}$ is $\lambda^{k-(j+i)}I$ and, similarly, if the ith block of $\mathcal{L}_{\mathbf{z}}$ is $\lambda^{j-1}P_i^T$, with $i \neq 0$, then the ith block of $\mathcal{R}_{\mathbf{z}}$ is $\lambda^{k-(j+i)}I$. Notice, finally, that $B_0 = \lambda^{s-1}I$ and $B'_0 = \lambda^{r-1}I$, with $s + r = k + 1$.

Example 4.2 Let $k = 13$ and $\mathbf{z} = (10 : 12, 9, 8, 6 : 7, 5, 2 : 4, 0 : 1)$. Note that $\mathbf{z}$ contains seven strings. Each string induces a string of blocks in $\mathcal{R}_{\mathbf{z}}$, where $F_{\mathbf{z}}(\lambda) = \lambda M_{-k} - M_{\mathbf{z}}$. The first entries of these strings correspond to the positions 0, 3, 4, 5, 7, 8 and 11, respectively. Then the right eigencolumn of $F_{\mathbf{z}}$ given by Theorem 4.1 is

$$\mathcal{R}_{\mathbf{z}} = \left[\lambda^6 I \;\; \lambda^6 P_1 \;\; \lambda^6 P_2 \middle| \lambda^5 I \middle| \lambda^4 I \middle| \lambda^3 I \;\; \lambda^3 P_6 \middle| \lambda^2 I \middle| \lambda I \;\; \lambda P_9 \;\; \lambda P_{10} \middle| I \;\; P_{12}\right]^{\mathcal{B}}.$$

For the left eigencolumn, we have $\operatorname{rev}\mathbf{z} \sim (12, 11, 7 : 10, 4 : 6, 3, 1 : 2, 0)$ in column standard form, so, from Theorem 4.1 we get

$$\mathcal{L}_{\mathbf{z}} = \left[\lambda^6 I \middle| \lambda^5 I \middle| \lambda^4 I \;\; \lambda^4 P_3^T \;\; \lambda^4 P_4^T \;\; \lambda^4 P_5^T \middle| \lambda^3 I \;\; \lambda^3 P_7^T \;\; \lambda^3 P_8^T \middle| \lambda^2 I \middle| \lambda I \;\; \lambda P_{11}^T \middle| I\right]^{\mathcal{B}}.$$

4.2. *Eigencolumns of GF pencils*

In this section we present an explicit expression for left and right eigencolumns of a GF pencil. Here we only address the case of PGF pencils and we postpone for Appendix A the case of non-proper GF pencils. This case deserves a special treatment and non-proper GF pencils do not seem to be relevant in applications (except in the particular case of the symmetric linearizations of even-degree regular matrix polynomials in [3]). It should be remarked that index tuples $\mathbf{q}$ and $\mathbf{m}$ in Definition 2.19 are both permutations and, so, they are equivalent to tuples in column standard form.

Theorem 4.3 m: *Let $P(\lambda)$ be an $n \times n$ matrix polynomial with degree k, let P_i, for $i = 0, 1, \ldots, k$ be its ith Horner shift, and let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a PGF pencil of $P(\lambda)$. Assume that $\mathbf{m}$ has $\mathfrak{c}_{-k}$ consecutions at $-k$, and write $\mathbf{m} \sim (\mathbf{m}_1, -k : -k+\mathfrak{c}_{-k})$ in column standard form. Let $\mathbf{z} = (\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$ be an index tuple in column standard form equivalent to $(-\operatorname{rev}\mathbf{m}_1, \mathbf{q})$.*

(a) A right eigencolumn $\mathcal{R}_K(P)$ for $K(\lambda)$ can be obtained as follows:
(a1) If $\mathfrak{c}_{-k} = 0$, then $\mathcal{R}_K(P) := \mathcal{R}_{\mathbf{z}}(P)$, with $\mathcal{R}_{\mathbf{z}}(P)$ as in (8).
(a2) If $\mathfrak{c}_{-k} > 0$, then

$$\mathcal{R}_K(P) := \left[\lambda^s (P_0 \;\; P_1 \; \ldots \; P_{\mathfrak{c}_{-k}-1}) \middle| B_{\mathfrak{c}_{-k}} \;\; B_{\mathfrak{c}_{-k}+1} \; \ldots \; B_{k-1}\right]^{\mathcal{B}}, \qquad (10)$$

where, if $\mathbf{z}(i) \in \mathbf{b}_j$, for some $j = 1, 2, \ldots, s$, then the block $B_{i+\mathfrak{c}_{-k}}$ is as in (9). Moreover, if $\mathbf{q}$ has $\mathfrak{c}_0$ consecutions at 0, then the $(k - \mathfrak{c}_0)$th block of $\mathcal{R}_K(P)$ is equal to I_n.

(b) A left eigencolumn for $K(\lambda)$ is given by $\mathcal{L}_K(P) := \mathcal{R}_{K^\sharp}(P^T)$, where $K^\sharp(\lambda) = \lambda M_{\operatorname{rev}\mathbf{m}}(P^T) - M_{\operatorname{rev}\mathbf{q}}(P^T)$. Moreover, if $\mathbf{q}$ has $\mathfrak{i}_0$ inversions at 0, then the $(k - \mathfrak{i}_0)$th block of $\mathcal{L}_K(P)$ is equal to I_n.

Remark 3 m: Notice that the B_i blocks in (10) follow the same rule as in (9). More precisely, the ith block B_i is of the form $\lambda^{j-1}I$ if $\mathbf{z}(i - \mathfrak{c}_{-k})$ is the first element in $\mathbf{b}_j$, and it is of the form $\lambda^{j-1}P_i$ if $\mathbf{z}(i - \mathfrak{c}_{-k}) \in \mathbf{b}_j$ but is not the first element of $\mathbf{b}_j$.

In the following, for simplicity and when there is no risk of confusion, we will drop the dependence on P in the eigencolumns $\mathcal{R}_K(P)$ and $\mathcal{L}_K(P)$.

Example 4.4 Let $k = 12$, $\mathbf{m} = (-4 : -3, -6, -12 : -10)$ and $\mathbf{q} = (7 : 9, 5, 0 : 2)$. Then, $\mathfrak{c}_{-k} = 2$. Note that $(-\operatorname{rev}\mathbf{m}_1, \mathbf{q}) = (6, 3 : 4, 7 : 9, 5, 0 : 2)$ is equivalent to $\mathbf{z} = (6 : 9, 3 : 5, 0 : 2)$ in column standard form. Thus, $s = 3$. Also,

$\operatorname{rev}\mathbf{m} \sim (-3,-4,-6,-10,-11,-12) = (\mathbf{m}_1',-12)$, in column standard form, and $\operatorname{rev}\mathbf{q} \sim (9,8,7,5,2,1,0)$. Then, $(-\operatorname{rev}\mathbf{m}_1', \operatorname{rev}\mathbf{q})$ is equivalent to $\mathbf{z}' = (11,10,9,8,6:7,4:5,3,2,1,0)$ in column standard form, so $s = 10$ in this case. If $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$, Theorem 4.3 gives

$$\mathcal{R}_K = \left[\lambda^3 P_0\ \lambda^3 P_1 \middle| \lambda^2 I\ \lambda^2 P_3\ \lambda^2 P_4\ \lambda^2 P_5 \middle| \lambda I\ \lambda P_7\ \lambda P_8 \middle| I\ P_{10}\ P_{11}\right]^{\mathcal{B}},$$

and

$$\mathcal{L}_K = \left[\lambda^9 I \middle| \lambda^8 I \middle| \lambda^7 I \middle| \lambda^6 I \middle| \lambda^5 I\ \lambda^5 P_5^T \middle| \lambda^4 I\ \lambda^4 P_7^T \middle| \lambda^3 I \middle| \lambda^2 I \middle| \lambda I \middle| I\right]^{\mathcal{B}}.$$

Example 4.5 Let $k = 12$, $\mathbf{m} = (-12 : -8)$, and $\mathbf{q} = (6 : 7, 5, 4, 0 : 3)$. In this case, $\mathfrak{c}_{-k} = 4$, $-\mathbf{m}_1$ is the empty tuple, and $\mathbf{z} = \mathbf{q}$. Therefore, $s = 4$. Similarly, $\operatorname{rev}\mathbf{m} = (-8,-9,-10,-11,-12) = (\mathbf{m}_1',-12)$, which is already in column standard form, $\operatorname{rev}\mathbf{q} = (3,2,1,0,4 : 5,7,6)$, so $(-\operatorname{rev}\mathbf{m}_1', \operatorname{rev}\mathbf{q})$ is equivalent to $\mathbf{z}' = (11,10,9,8,7,3:6,2,1,0)$ in column standard form, so $s = 9$ in this case. Then, if $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$, Theorem 4.3 gives

$$\mathcal{R}_K = \left[\lambda^4 P_0\ \lambda^4 P_1\ \lambda^4 P_2\ \lambda^4 P_3 \middle| \lambda^3 I\ \lambda^3 P_5 \middle| \lambda^2 I \middle| \lambda I \middle| I\ P_9\ P_{10}\ P_{11}\right]^{\mathcal{B}},$$

and

$$\mathcal{L}_K = \left[\lambda^8 I \middle| \lambda^7 I \middle| \lambda^6 I \middle| \lambda^5 I \middle| \lambda^4 I \middle| \lambda^3 I\ \lambda^3 P_6^T\ \lambda^3 P_7^T\ \lambda^3 P_8^T \middle| \lambda^2 I \middle| \lambda I \middle| I\right]^{\mathcal{B}}.$$

We want to stress that the palindromic linearizations introduced in [7] are, up to multiplication by certain nonsingular matrices, particular cases of PGF pencils. More precisely, the pencil $L_\tau(\lambda)$ in [7, Theorem 4.8] is a PGF pencil, and the palindromic linearization is $S_\tau \cdot R \cdot L_\tau(\lambda)$, with R and S_τ nonsingular matrices. Since multiplication on the left by nonsingular matrices does not affect the right eigencolumns, the formulas obtained in Theorem 4.3 are valid also for these palindromic linearizations.

4.3. *Eigencolumns of FPR*

We provide in this section formulas for the right (respectively, left) eigencolumns of FPR with $\mathbf{r}_m$ and $\mathbf{r}_q$ (resp. $\operatorname{rev}\mathbf{l}_m$ and $\operatorname{rev}\mathbf{l}_q$) in Definition 2.21 being type 1 tuples relative to $\mathbf{m}$ and $\mathbf{q}$ (resp. $\operatorname{rev}\mathbf{m}$ and $\operatorname{rev}\mathbf{q}$). This case seems to be the relevant one for applications. Indeed, all the symmetric families of linearizations considered in [21] correspond to this case. These families of symmetric linearizations are considered in Section 5.3.1. To derive formulas for the eigencolumns in the case where tuples are not type 1 seems to be quite involved and remains as an open problem.

Theorem 4.6 $\mathbf{m}$: *Let* $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$ *be a FPR of a matrix polynomial* $P(\lambda)$ *of degree* k.

(a) Assume that $\mathbf{r}_m$ *and* $\mathbf{r}_q$ *are type 1 tuples relative to* $\mathbf{m}$ *and* $\mathbf{q}$*, respectively. Let* $\mathfrak{s}(\mathbf{q},\mathbf{r}_q)$ *and* $\mathfrak{s}(\mathbf{m},\mathbf{r}_m)$ *be the simple tuple associated with* $(\mathbf{q},\mathbf{r}_q)$ *and* $(\mathbf{m},\mathbf{r}_m)$*, respectively. Then, a right eigencolumn of* $L(\lambda)$ *is given by* $\mathcal{R}_{\widetilde{K}}$*, where* $\widetilde{K}(\lambda) = \lambda M_{\mathfrak{s}(\mathbf{m},\mathbf{r}_m)} - M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)}$ *is a GF pencil. Moreover, if* $\mathfrak{s}(\mathbf{q},\mathbf{r}_q)$ *has* $\widetilde{\mathfrak{c}}_0$ *consecutions at* 0*, then the* $(k-\widetilde{\mathfrak{c}}_0)$*th block of* $\mathcal{R}_{\widetilde{K}}$ *is equal to* I_n.

(b) Assume that $\operatorname{rev}\mathbf{l}_m$ *and* $\operatorname{rev}\mathbf{l}_q$ *are type 1 tuples relative to* $\operatorname{rev}\mathbf{m}$ *and* $\operatorname{rev}\mathbf{q}$*, respectively. Let* $\mathfrak{s}(\operatorname{rev}\mathbf{q},\operatorname{rev}\mathbf{l}_q)$ *and* $\mathfrak{s}(\operatorname{rev}\mathbf{m},\operatorname{rev}\mathbf{l}_m)$ *be the simple tuple associated with* $(\operatorname{rev}\mathbf{q},\operatorname{rev}\mathbf{l}_q)$ *and* $(\operatorname{rev}\mathbf{m},\operatorname{rev}\mathbf{l}_m)$*, respectively. Then, a left eigencolumn of* $L(\lambda)$ *is given by* $\mathcal{R}_{\widehat{K}}$*, where* $\widehat{K}(\lambda) = \lambda M_{\mathfrak{s}(\operatorname{rev}\mathbf{m},\operatorname{rev}\mathbf{l}_m)}(P^T) - M_{\mathfrak{s}(\operatorname{rev}\mathbf{q},\operatorname{rev}\mathbf{l}_q)}(P^T)$ *is a GF*

pencil. Moreover, if $\mathfrak{s}(\mathrm{rev}\,\mathbf{q}, \mathrm{rev}\,\mathbf{r}_q)$ *has* $\widehat{\mathfrak{c}}_0$ *consecutions at* 0*, then the*$(k-\widehat{\mathfrak{c}}_0)$*th block of* $\mathcal{R}_{\widehat{K}}$ *is equal to* I_n.

Example 4.7 Let $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$ be the FPR associated with a matrix polynomial of degree $k = 12$, with $\mathbf{q} = (6, 1 : 5, 0)$, $\mathbf{r}_q = (1 : 4)$, $\mathbf{m} = (-7, -8, -12 : -9)$, $\mathbf{r}_m = (-12 : -10, -12 : -11)$, $\mathbf{l}_q = (0)$, $\mathbf{l}_m = (-8, -9)$. Then, $(\mathbf{q}, \mathbf{r}_q) = (6, 1 : 5, 0 : 4)$ and $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) = (6, 5, 0 : 4)$. Similarly, $(\mathbf{m}, \mathbf{r}_m) = (-7, -8, -12 : -9, -12 : -10, -12 : -11)$ and $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-7, -8, -9, -10, -12 : -11)$, so $\widetilde{\mathfrak{c}}_{-k} = 1$. Also, $(\mathrm{rev}\,\mathbf{q}, \mathrm{rev}\,\mathbf{l}_q) \sim (5 : 6, 4, 3, 2, 0 : 1, 0)$, $\mathfrak{s}(\mathrm{rev}\,\mathbf{q}, \mathrm{rev}\,\mathbf{l}_q) = (5 : 6, 4, 3, 2, 1, 0)$, $(\mathrm{rev}\,\mathbf{m}, \mathrm{rev}\,\mathbf{l}_m) \sim (-9 : -7, -10, -11, -12, -9 : -8)$, and $\mathfrak{s}(\mathrm{rev}\,\mathbf{m}, \mathrm{rev}\,\mathbf{l}_m) = (-7, -10 : -8, -11, -12)$, so $\widehat{\mathfrak{c}}_{-k} = 0$. Let $\widetilde{K}(\lambda) = \lambda M_{\mathfrak{s}(\mathbf{m},\mathbf{r}_m)} - M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)}$, and $\widehat{K}(\lambda) = \lambda M_{\mathfrak{s}(\mathrm{rev}\,\mathbf{m},\mathrm{rev}\,\mathbf{l}_m)} - M_{\mathfrak{s}(\mathrm{rev}\,\mathbf{q},\mathrm{rev}\,\mathbf{l}_q)}$. Following the notation in the statement of Theorem 4.3, we have $\widetilde{\mathbf{m}}_1 = (-7, -8, -9, -10)$ and then $\widetilde{\mathbf{z}} = (10, 9, 8, 7, 6, 5, 0 : 4)$. Similarly, $\widehat{\mathbf{m}}_1 = (-7, 10 : -8, -11)$ and $\widehat{\mathbf{z}} = (11, 8 : 10, 7, 5 : 6, 4, 3, 2, 1, 0)$. Hence

$$\mathcal{R}_L = \left[\, \lambda^7 P_0 \middle| \lambda^6 I \middle| \lambda^5 I \middle| \lambda^4 I \middle| \lambda^3 I \middle| \lambda^2 I \middle| \lambda I \middle| I \ P_8 \ P_9 \ P_{10} \ P_{11} \,\right]$$

and

$$\mathcal{L}_L = \left[\, \lambda^8 I \middle| \lambda^7 I \ \lambda^7 P_2^T \ \lambda^7 P_3^T \middle| \lambda^6 I \middle| \lambda^5 I \ \lambda^5 P_6^T \middle| \lambda^4 I \middle| \lambda^3 I \middle| \lambda^2 I \middle| \lambda I \middle| I \,\right].$$

5. Proof of the main results

In the following subsections we will prove theorems 4.1, 4.3 and 4.6. We will only prove the part regarding the right eigencolumns. The statements about the left ones can be obtained from the right one using the following observation. Given a index tuple $\mathbf{t}$, let us denote by $M_{\mathbf{t}}(P)$ the matrix in (7) associated with the polynomial $P(\lambda)$. Let $H(\lambda) = \lambda M_{\mathbf{a}}(P) - M_{\mathbf{b}}(P)$, where $\mathbf{a}$ and $\mathbf{b}$ are index tuples with indices from $\{0, 1, \dots, k, -0, -1, -2, \dots, -k\}$ (notice that this includes all three families of Fiedler pencils, GF pencils and FPR). Then $H(\lambda)^T = \lambda M_{\mathrm{rev}\,\mathbf{a}}(P^T) - M_{\mathrm{rev}\,\mathbf{b}}(P^T)$. Since the left eigencolumns of $H(\lambda)$ are the right eigencolumns of $H(\lambda)^T$, we can get formulas for the left eigencolumns by reversing the tuples of the coefficient matrices of $H(\lambda)$ and replacing the coefficients A_i by A_i^T in the formulas for the right eigencolumns.

5.1. *The case of Fiedler pencils*

Theorem 4.1 follows almost immediately from Lemma 5.3 in [6], where the authors derive formulas for the last block-column of $V(\lambda)$ and the last block-row of $U(\lambda)$ in (2) with $H(\lambda)$ being a Fiedler pencil. Our proof of Theorem 4.1 consists of relating our formulas (8) and (9) with the ones obtained in [6].

Proof of Theorem 4.1. First, let us recall the notion of *Consecution Inversion Structure Sequence (CISS)* of $\mathbf{z}$, introduced in [6, Def. 3.3]:

$$CISS(\mathbf{z}) = (c_1, i_1, c_2, i_2, \dots, c_\ell, i_\ell).$$

This means that $\mathbf{z}$ has c_1 consecutions at 0, then i_1 inversions at c_1, then c_2 consecutions at $c_1 + i_1$, then i_2 inversions at $c_1 + i_1 + c_2$, and so on. Notice that c_1 and i_ℓ in this list may be zero, but the remaining numbers are nonzero. Using this notation, and following Remark **1**, we may write

$$\mathcal{R}_{\mathbf{z}} = \left[\mathcal{I}_\ell \, \mathcal{C}_\ell \dots \mathcal{I}_1 \, \mathcal{C}_1\right]^{\mathcal{B}},$$

where, for $j = 1, \ldots, \ell$,

$$\mathcal{I}_j = \lambda^{i_1+\cdots+i_{j-1}+j} \begin{bmatrix} \lambda^{i_j-1} I \\ \vdots \\ \lambda I \\ I \end{bmatrix}^{\mathcal{B}} \quad \text{and} \quad \mathcal{C}_j = \lambda^{i_1+\cdots+i_{j-1}+j-1} \begin{bmatrix} I \\ P_{\alpha_1^j} \\ \vdots \\ P_{\alpha_{c_j}^j} \end{bmatrix}^{\mathcal{B}},$$

(where we set $i_0 := 0$) and

$$\alpha_i^j = k - (c_1 + i_1 + \cdots + c_{j-1} + i_{j-1} + c_j) + i - 1, \qquad \text{for} \quad i = 1, \ldots, c_j.$$

These are precisely formulas (5.3) in [6], which are the building blocks of formula (5.4) in [6], that corresponds to the right eigencolumn of the Fiedler pencil $F_{\mathbf{z}}$. The fact that $\mathcal{R}_{\mathbf{z}}$ contains an identity block follows immediately from this formula. This implies, in particular, that $\mathcal{R}_{\mathbf{z}}$ satisfies (a) in Definition 3.1. It is proved in [6, Theorem 5.7] that $\mathcal{R}_{\mathbf{z}}$ satisfies also part (b) of Definition 3.1. Hence $\mathcal{R}_{\mathbf{z}}$ is indeed a right eigencolumn. □

5.2. *The case of PGF pencils*

To prove Theorem 4.3 we use the following elementary observation. Let $\mathbf{B}$ be a block-column matrix consisting of k square blocks with n columns. When $\mathbf{B}$ is multiplied on the left by M_{k-1}, only the first and second blocks of $\mathbf{B}$ are modified. When multiplied by $M_{k-2}M_{k-1}$ only the first, second, and third blocks of $\mathbf{B}$ are modified. Thus, when multiplying $M_{(k-j:k-1)}\mathbf{B}$ the only blocks of $\mathbf{B}$ that can be altered are the blocks with indices from 1 to $j+1$.

Proof of Theorem 4.3. Let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a PGF pencil associated with a matrix polynomial $P(\lambda)$ such that $\mathbf{m}$ and $\mathbf{q}$ are index tuples in column standard form. We will focus on the right eigencolumn, because the arguments for the left one are similar using block transposition and the argument at the beginning of Section 5. We will obtain a right eigencolumn $\mathcal{R}_K$ of $K(\lambda)$ from strict equivalence with a right eigencolumn of a Fiedler pencil. This will ensure properties (a) and (b) in Definition 3.1, because multiplication by an invertible matrix preserves these properties. Moreover, our procedure will show that, if the Fiedler pencil is adequately chosen, then also (c) in Definition 3.1 is fulfilled. In the last part of the proof, we show that this strict equivalence preserves an identity block, proving the last part of the statement (note that the presence of this block also implies (a) in Definition 3.1).

Let us assume that $\mathbf{q}$ has $\mathfrak{c}_0$ consecutions at 0, and that $\mathbf{m}$ has $\mathfrak{c}_{-k}$ consecutions at $-k$. Then, there exists an index tuple $\mathbf{m}_1$ such that

$$K(\lambda) = \lambda\, M_{\mathbf{m}_1} M_{(-k:-k+\mathfrak{c}_{-k})} - M_{\mathbf{q}}. \tag{11}$$

Notice that the index tuple $(-\text{rev}\,\mathbf{m}_1, \mathbf{q})$ is a permutation of $\{0, 1, \ldots, k - \mathfrak{c}_{-k} - 1\}$. Let $\mathbf{z} = (\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$ be an index tuple in column standard form equivalent to $(-\text{rev}\,\mathbf{m}_1, \mathbf{q})$ and $\widetilde{\mathbf{z}}$ be an index tuple in column standard form equivalent to $(-\text{rev}\,\mathbf{m}_1, \mathbf{q}, k - \mathfrak{c}_{-k} : k-1)$. We construct the following Fiedler pencil associated with $P(\lambda)$:

$$F_{\widetilde{\mathbf{z}}}(\lambda) = M_{-\text{rev}\,\mathbf{m}_1} K(\lambda) M_{(k-\mathfrak{c}_{-k}:k-1)} = \lambda M_{-k} - M_{(-\text{rev}\,\mathbf{m}_1, \mathbf{q}, k-\mathfrak{c}_{-k}:k-1)}, \tag{12}$$

where $M_{(k-\mathfrak{c}_{-k}:k-1)} = I$ if $\mathfrak{c}_{-k} = 0$. We know that there exist $U(\lambda)$ and $V(\lambda)$ unimodular such that

$$U(\lambda)F_{\widetilde{\mathbf{z}}}(\lambda)V(\lambda) = \begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix},$$

which can be rewritten as

$$(U(\lambda)M_{-\operatorname{rev}\mathbf{m}_1})K(\lambda)(M_{(k-\mathfrak{c}_{-k}:k-1)}V(\lambda)) = \begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

The right eigencolumn $\mathcal{R}_K$ of $K(\lambda)$ will be given by $M_{(k-\mathfrak{c}_{-k}:k-1)}\mathcal{R}_{\widetilde{\mathbf{z}}}$, which is the last block-column of $M_{(k-\mathfrak{c}_{-k}:k-1)}V(\lambda)$. Recall that the explicit expression for $\mathcal{R}_{\widetilde{\mathbf{z}}}$ is given in Theorem 4.1. Thus, if $\mathfrak{c}_{-k} = 0$, then $\mathcal{R}_K = \mathcal{R}_{\widetilde{\mathbf{z}}} = \mathcal{R}_{\mathbf{z}}$, and this proves part (a1) in the statement.

Now assume that $\mathfrak{c}_{-k} \neq 0$. Let $\mathbf{b}_s = (w : k - \mathfrak{c}_{-k} - 1)$, for some $w > 0$. Then $\widetilde{\mathbf{z}}$ is equivalent to $(w : k-1, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$. Now, by Theorem 4.1,

$$\mathcal{R}_{\widetilde{\mathbf{z}}} = \left[\lambda^{s-1}(I\ P_1 \ldots P_{k-1-w})\middle| B_{k-w}\ B_{k-w+1} \ldots B_{k-1}\right]^{\mathcal{B}}, \tag{13}$$

where B_i, for $i = k-w, \ldots, k-1$, are as in the statement. Now, multiplying $\mathcal{R}_{\widetilde{\mathbf{z}}}$ on the left by $M_{(k-\mathfrak{c}_{-k}:k-1)}$ only affects the first $\mathfrak{c}_{-k}+1$ blocks of $\mathcal{R}_{\widetilde{\mathbf{z}}}$. Since $(w : k-1)$ contains at least $\mathfrak{c}_{-k}+1$ elements, only some of the first $k-w$ blocks in (13) will be modified.

It is easy to check by direct multiplication that $M_{(k-\mathfrak{c}_{-k}:k-1)}\mathcal{R}_{\widetilde{\mathbf{z}}}$ is equal to

$$\left[\lambda^{s}(P_0\ P_1 \ldots P_{\mathfrak{c}_{-k}-1})\ \lambda^{s-1}(I\ P_{\mathfrak{c}_{-k}+1} \ldots P_{k-1-w})\ B_{k-w} \ldots B_{k-1}\right]^{\mathcal{B}},$$

and this proves (a2).

Finally, for the claim on the identity block, we first assume that $k - \mathfrak{c}_{-k} \neq \mathfrak{c}_0 + 1$, and then $\mathfrak{c}_0 + 1 \in \mathbf{m}_1$ or $\mathfrak{c}_0 + 1 \in \mathbf{q}$. This implies that $s \geq 2$. From Theorem 4.1, the $(k-\mathfrak{c}_0)$th block of $\mathcal{R}_{\widetilde{\mathbf{z}}}$ (given by (13)) is equal to I_n and, since multiplying on the left by $M_{(k-\mathfrak{c}_{-k}:k-1)}$ does not affect this block, the identity block remains in $\mathcal{R}_K$. If $k - \mathfrak{c}_{-k} = \mathfrak{c}_0 + 1$, then $s = 1$ and, by the previous arguments, $\mathcal{R}_K = [\mathbf{B}_1\ \mathbf{B}_2]^{\mathcal{B}}$, where the first block of of $\mathbf{B}_2$ is equal to I_n. This is, precisely, the $(k-\mathfrak{c}_0)$th block of $\mathcal{R}_K$. □

The following corollary deals with the case where $\mathbf{q}$ in Theorem 4.3 is a permutation of $\{0, 1, \ldots, h\}$, for some $0 \leq h \leq k-1$ and, therefore, $\mathbf{m}$ is a permutation of $\{-k, -k+1, \ldots, -h-1\}$. It will be used in the proof of Lemma 5.2, which is used in turn to prove Theorem 4.6.

Corollary 5.1m: *Let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a PGF pencil such that $\mathbf{q}$ is a permutation of $\{0, 1, \ldots, h\}$, for some $0 \leq h \leq k-1$. Let $\mathbf{q} = (\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$ and $\mathbf{m} \setminus \{-k\} = (\mathbf{b}'_r, \mathbf{b}'_{r-1}, \ldots, \mathbf{b}'_1)$ be in column standard form, with $\mathbf{b}_i = (t_{i-1}+1 : t_i)$ and $\mathbf{b}'_j = (-t'_{j-1}+1 : -t'_j)$, for $i = 1, \ldots, s$ and $j = 1, \ldots, r$. Set $\mathbf{z} = (-\operatorname{rev}\mathbf{b}'_1, \ldots, -\operatorname{rev}\mathbf{b}'_r, \mathbf{b}_s, \ldots, \mathbf{b}_1)$. Then a right eigencolumn for $K(\lambda)$ is of the form*

$$\mathcal{R}_K = \left[\mathbf{B}_1\middle|\mathbf{B}_2\right]^{\mathcal{B}} = \left[B_0\ B_1 \ldots B_{k-h-2}\middle|B_{k-h-1} \ldots B_{k-1}\right]^{\mathcal{B}},$$

where

- *for $1 \leq i \leq k-h-2$,*

$$B_i = \begin{cases} \lambda^{r+s-j} I, & \text{if } \mathbf{z}(i) \in -\operatorname{rev}\mathbf{b}'_j \text{ and } i = k - t'_{j-1} \\ \lambda^{r+s-j} P_i, & \text{if } \mathbf{z}(i) \in -\operatorname{rev}\mathbf{b}'_j \text{ and } i \neq k - t'_{j-1}; \end{cases}$$

- *for* $k-h-1 \leq i \leq k-1$,

$$B_i = \begin{cases} \lambda^{j-1} I\,, & \textit{if } \mathbf{z}(i) \in \mathbf{b}_j \textit{ and } i = k - t_j - 1 \\ \lambda^{j-1} P_i\,, & \textit{if } \mathbf{z}(i) \in \mathbf{b}_j \textit{ and } i \neq k - t_j - 1\,; \end{cases}$$

and

$$B_0 = \begin{cases} \lambda^{r+s-1} I\,, & \textit{if } (-k, -k+1) \textit{ form an inversion in } \mathbf{m} \\ \lambda^{r+s-1} A_k\,, & \textit{otherwise}. \end{cases}$$

Proof: The result is an immediate consequence of Theorem 4.3 (see also Remark **3**). Just notice that the column standard form of the tuple $(-\mathrm{rev}\,\mathbf{b}'_1, \ldots, -\mathrm{rev}\,\mathbf{b}'_r, \mathbf{b}_s, \ldots, \mathbf{b}_1)$ is itself, so applying Theorem 4.3 we get the blocks B_i in the statement. □

Remark 1 $\mathbf{m}$: The right eigencolumn obtained in Corollary 5.1 can be seen as partitioned into $r+s$ strings of blocks

$$\left[\mathbf{B}'_1 \ldots \mathbf{B}'_{r-1}\, \mathbf{B}'_r \middle| \mathbf{B}_s\, \mathbf{B}_{s-1} \ldots \mathbf{B}_1\right]^{\mathcal{B}},$$

where the string $\mathbf{B}_i$ corresponds to the string $\mathbf{b}_i$ of $\mathbf{q}$. More precisely, the string $\mathbf{B}_i$ is of the form $\lambda^j \left[I\; P_{\mathbf{z}^{-1}(t_{j-1}+2)} \cdots P_{\mathbf{z}^{-1}(t_j)}\right]^{\mathcal{B}}$, where j is the number of strings to the right of $\mathbf{B}_i$. Similarly, each $\mathbf{B}'_i$ corresponds to the string $\mathbf{b}'_i$ in $\mathbf{m}$ and is of the form $\lambda^j \left[I\; P_{\mathbf{z}^{-1}(t'_{j-1}+2)} \cdots P_{\mathbf{z}^{-1}(t'_j)}\right]^{\mathcal{B}}$, with j being the number of strings to the right of $\mathbf{B}'_i$. Also, the first block of $\mathcal{R}_K$ is $\lambda^{r+s-1} I$ or $\lambda^{r+s-1} A_k$, depending on whether there is a consecution at $-k$ or not. Hence, the right eigencolumn $\mathcal{R}_K$ can be easily obtained from the column standard form of both $\mathbf{m} \setminus \{-k\}$ and $\mathbf{q}$.

5.3. *The case of FPR*

Let $P(\lambda)$ be a matrix polynomial of degree k as in (1) and let $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$ be a FPR for $P(\lambda)$. Here we assume that A_0 (resp. A_k) is nonsingular if 0 (resp. $-k$) is an index in $\mathbf{l}_q$, $\mathbf{r}_q$, or both (resp. in $\mathbf{l}_m$, $\mathbf{r}_m$, or both). In order to find an explicit expression for a right eigencolumn of $L(\lambda)$, first notice that $K(\lambda) = M_{-\mathrm{rev}\,\mathbf{l}_q} M_{-\mathrm{rev}\,\mathbf{l}_m} L(\lambda) M_{-\mathrm{rev}\,\mathbf{r}_m} M_{-\mathrm{rev}\,\mathbf{r}_q}$ is a PGF pencil. Therefore,we can get a right eigencolumn of $L(\lambda)$ by computing $M_{-\mathrm{rev}\,\mathbf{r}_m} M_{-\mathrm{rev}\,\mathbf{r}_q} \mathcal{R}_K$, with $\mathcal{R}_K$ as in Theorem 4.3. Hence, we can assume without loss of generality that $M_{\mathbf{l}_q}$ and $M_{\mathbf{l}_m}$ are both the identity matrix for this purpose.

Note that, if $-k$ has $\mathfrak{c}_{-k}$ consecutions and we write $\mathbf{m} = (\mathbf{m}_1, -k : -k + \mathfrak{c}_{-k})$, then $K(\lambda) = \lambda M_{\mathbf{m}_1} M_{(-k:-k+\mathfrak{c}_{-k})} - M_{\mathbf{q}}$, where $\mathbf{q}$ is a permutation of $\{0, 1, \ldots, h\}$, for some $h < k$ (actually, $h < k - \mathfrak{c}_{-k}$). Let $\mathbf{z}$ be an index tuple in column standard form equivalent to $(-\mathrm{rev}\,\mathbf{m}_1, \mathbf{q})$. Theorem 4.3 provides explicit formulas for $\mathcal{R}_K$ depending on $\mathbf{z}$. Notice that $\mathbf{m}$ (and, as a consequence, also $\mathbf{m}_1$) contains indices from $\{-k, -k+1, \ldots, -h-1\}$. Also, $\mathbf{r}_q$ contains indices from $\{0, 1, \ldots, h-1\}$ and $\mathbf{r}_m$ contains indices from $\{-k, -k+1, \ldots, -h-2\}$ (by definition of FPR). All these observations, together with Corollary 5.1, imply that $\mathcal{R}_K = \left[\mathbf{B}_1 \middle| \mathbf{B}_2\right]^{\mathcal{B}}$, where $\mathbf{B}_1$ consists of $k-h-1$ blocks and depends only on $\mathbf{m}$, and $\mathbf{B}_2$ consists of $h+1$ blocks and depends only on $\mathbf{q}$. Now, multiplying on the left by $M_{-\mathrm{rev}\,\mathbf{r}_q}$ only affects some blocks in $\mathbf{B}_2$, while multiplying on the left by $M_{-\mathrm{rev}\,\mathbf{r}_m}$ only affects some blocks in $\mathbf{B}_1$.

Here we will study thoroughly the product $M_{-\mathrm{rev}\,\mathbf{r}_q} \mathcal{R}_K$. A similar procedure can be applied for $M_{-\mathrm{rev}\,\mathbf{r}_m} \cdot (M_{-\mathrm{rev}\,\mathbf{r}_q} \mathcal{R}_K)$. More precisely, based on the observations above, the multiplication by $M_{-\mathrm{rev}\,\mathbf{r}_m}$ only affects some of the first $k-h-1$ blocks of $\mathcal{R}_K$

that were not modified when multiplying by $M_{-\operatorname{rev}\mathbf{r}_q}$, so the product by $M_{-\operatorname{rev}\mathbf{r}_m}$ and $M_{-\operatorname{rev}\mathbf{r}_q}$ will never overlap. Then, these multiplications can be addressed independently.

Lemma 5.2m: *Let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a PGF pencil such that $\mathbf{q}$ is a permutation of $\{0, 1, \ldots, h\}$, for some $0 \leq h \leq k-1$. Let $\mathbf{q} = (\mathbf{b}_s, \mathbf{b}_{s-1}, \ldots, \mathbf{b}_1)$ be in column standard form, with $\mathbf{b}_i = (t_{i-1}+1 : t_i)$, for $i = 1, \ldots, s$. Let $\mathbf{r}_q = (h_1 : h_2)$, with $0 \leq h_1 \leq h_2 \leq h-1$ such that $(\mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Then*

$$M_{-\operatorname{rev}\mathbf{r}_q}\mathcal{R}_K = \left[\, B_0\ B_1\ \ldots\ B_{k-1} \right]^{\mathcal{B}},$$

where, for $i \notin \{k-h_2-1, \ldots, k-h_1\}$, B_i are as in Corollary 5.1 and

(a) if $t_{d-1}+1 < h_1 \leq h_2 < t_d$, for some $d \geq 1$, then

$$B_i = \begin{cases} \lambda^{d-1}P_{k-h_1}, & i = k-h_2-1, \\ \lambda^{d-1}(P_{i-1} + A_{k-i}P_{k-h_1}), & i = k-h_2, k-h_2+1, \ldots, k-h_1; \end{cases}$$

(b) if $0 = h_1 \leq h_2 < t_1$, then

$$B_i = \begin{cases} -A_0^{-1}P_{k-1}, & i = k-h_2-1, \\ P_{i-1} - A_{k-i}A_0^{-1}P_{k-1}, & i = k-h_2, k-h_2+1, \ldots, k-1; \end{cases}$$

(c) If $t_{d-1}+1 = h_1 \leq h_2 < t_d$, for some $d > 1$, then

$$B_i = \begin{cases} \lambda^{d-2}I, & i = k-h_2-1 \\ \lambda^{d-2}P_i & i = k-h_2, \ldots, k-h_1. \end{cases}$$

Proof: The proof can be carried out by keeping track of the blocks of $\mathcal{R}_K$ after multiplying on the left by $M_{-\operatorname{rev}\mathbf{r}_q} = M_{(-h_2:-h_1)}$. It is straightforward to see that, for $h_1 > 0$,

$$M_{(-h_2:-h_1)} = \left[\begin{array}{c|cc|c} I_{n(k-h_2-1)} & & & \\ \hline & 0 & I & \\ & I & A_{h_2} & \\ & & \ddots\ \ \vdots & \\ & & I\ \ A_{h_1} & \\ \hline & & & I_{n(h_1-1)} \end{array}\right],$$

whereas, if $h_1 = 0$,

$$M_{(-h_2:0)} = \left[\begin{array}{c|cc} I_{n(k-h_2-1)} & & \\ \hline & 0 & A_0^{-1} \\ & I & A_{h_2}A_0^{-1} \\ & \ddots & \vdots \\ & & I\ \ A_1A_0^{-1} \end{array}\right]$$

(see [21, p. 325]). In particular, if we denote by $\widetilde{B}_0, \widetilde{B}_1, \ldots, \widetilde{B}_{k-1}$ the blocks of $\mathcal{R}_K$, only the blocks $\widetilde{B}_i$ with $i = k-h_2-1, k-h_2, \ldots, k-h_1$ are modified by this multiplication. Now the result follows from Theorem 4.3 by direct multiplication. In case (c), for $i = k-h_2-1, k-h_2, \ldots, k-h_1$, we have used that $\lambda P_{i+1} + A_{k-i} = P_i$. □

We have the counterpart of Lemma 5.2 for strings of negative elements.

Lemma 5.3m: *Let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a PGF pencil such that $\mathbf{m}$ is a permutation of $\{-k, -k+1, \ldots, -h-2\}$, for some $0 \le h \le k-1$. Let $\mathbf{m} \setminus \{k\} = (\mathbf{b}'_r, \mathbf{b}'_{r-1}, \ldots, \mathbf{b}'_1)$, with $\mathbf{b}'_i = (-t'_{i-1}+1 : -t'_i)$, for $i = 1, \ldots, r$, be in column standard form. Let $\mathbf{r}_m = (-h'_1 : -h'_2)$, with $-k \le -h'_1 < -h'_2 \le -h-2$, be such that $(\mathbf{m}, \mathbf{r}_m)$ satisfies the SIP. Then*

$$M_{-\mathrm{rev}\,\mathbf{r}_m} \mathcal{R}_K = \begin{bmatrix} B_0 \; B_1 \; \ldots \; B_{k-1} \end{bmatrix}^{\mathcal{B}},$$

where, for $i \notin \{k-h'_1-1, \ldots, k-h'_2\}$, B_i are as in Corollary 5.1 and

(a) if $-t'_{d-1}+1 < -h'_1 \le -h'_2 < -t'_d$, for some $d \ge 1$, then

$$B_i = \begin{cases} \lambda^{r+s-d}(P_{i+1} - A_{k-i}P_{k-h'_1}), & i = k-h'_1-1, k-h'_1+1, \ldots, k-h'_2-1, \\ \lambda^{r+s-d}P_{k-h'_1}, & i = k-h'_2; \end{cases}$$

(b) if $-k = -h'_1 \le -h'_2 < -t'_1$, then

$$B_i = \begin{cases} \lambda^{r+s-1}I, & i = 0, \\ \lambda^{r+s-1}P_i, & i = 1, \ldots, k-h'_2-1, \\ \lambda^{r+s-2}I, & i = k-h'_2; \end{cases}$$

(c) if $-t'_{d-1}+1 = -h'_1 \le -h'_2 < -t'_d$, for some $d > 1$, then

$$B_i = \begin{cases} \lambda^{r+s+1-d}P_i, & i = k-h'_1-1, \ldots, k-h'_2-1, \\ \lambda^{r+s-d}I, & i = k-h'_2. \end{cases}$$

Proof: As in the proof of Lemma 5.2, we have to keep track of the blocks of $\mathcal{R}_K$ after multiplying on the left by $M_{-\mathrm{rev}\,\mathbf{r}_m} = M_{(h'_2:h'_1)}$. For this, we make use of [21, p. 332]

$$M_{(h'_2:h'_1)} = \left[\begin{array}{c|ccc|c} I_{n(k-h'_1-1)} & & & & \\ \hline & -A_{h'_1} & I & & \\ & \vdots & & \ddots & \\ & -A_{h'_2} & & I & \\ & I & 0 & \ldots\ 0 & \\ \hline & & & & I_{n(h'_2-1)} \end{array}\right],$$

valid for $h'_1 < k$, and

$$M_{(h'_2:k)} = \left[\begin{array}{cccc|c} -A_1A_k^{-1} & I & & & \\ \vdots & & \ddots & & \\ -A_{h'_2-1}A_k^{-1} & & & I & \\ -A_{h'_2}A_k^{-1} & 0 & \ldots & 0 & \\ \hline & & & & I_{n(h'_2-1)} \end{array}\right]$$

The proof is analogous to that of Lemma 5.2. For claim (c), we use the identity $\lambda^{r+s-d}(-A_{k-j}+P_j) = \lambda^{r+s+1-d}P_{j-1}$. □

Example 5.4 Let $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q}$ be the FPR of a matrix polynomial $P(\lambda)$ of degree $k = 15$ with $\mathbf{q} = (8, 4 : 7, 0 : 3)$, $\mathbf{m} = (-11 : -9, -12, -15 : -13)$, and $\mathbf{r}_q = (5 : 6)$. Let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$. Notice that $\mathbf{r}_q$ consists of only one string. This string corresponds to Case (a) in Lemma 5.2. According to Lemma 5.2, only blocks in positions 8th, 9th, and 10th of $\mathcal{R}_K$ (starting with 0) are modified when multiplying on the left by $M_{-\mathrm{rev}\,\mathbf{r}_q}$. More precisely, we have

$$\mathcal{R}_K = \begin{bmatrix} \lambda^5 P_0 \\ \lambda^5 P_1 \\ \hline \lambda^4 I \\ \hline \lambda^3 I \\ \lambda^3 P_4 \\ \lambda^3 P_5 \\ \hline \lambda^2 I \\ \hline \lambda I \\ \lambda P_8 \\ \lambda P_9 \\ \lambda P_{10} \\ \hline I \\ P_{12} \\ P_{13} \\ P_{14} \end{bmatrix}, \qquad M_{-\mathrm{rev}\, r_q}\mathcal{R}_K = \begin{bmatrix} \lambda^5 P_0 \\ \lambda^5 P_1 \\ \hline \lambda^4 I \\ \hline \lambda^3 I \\ \lambda^3 P_4 \\ \lambda^3 P_5 \\ \hline \lambda^2 I \\ \hline \lambda I \\ \lambda \mathbf{P_{10}} \\ \lambda(\mathbf{P_8} + \mathbf{A_6 P_{10}}) \\ \lambda(\mathbf{P_9} + \mathbf{A_5 P_{10}}) \\ \hline I \\ P_{12} \\ P_{13} \\ P_{14} \end{bmatrix}.$$

This case has been only included for illustrative purposes, though it will not be addressed in this paper because $\mathbf{r}_q$ is not a type 1 string relative to $\mathbf{q}$. However, if we take $\mathbf{r}_q = (4 : 6)$, then $\mathbf{r}_q$ is a type 1 string relative to $\mathbf{q}$ and $M_{-\mathrm{rev}\,\mathbf{r}_q}\mathcal{R}_K$ is equal to

$$\left[\ \lambda^5 P_0\ \ \lambda^5 P_1 \mid \lambda^4 I \mid \lambda^3 I\ \ \lambda^3 P_4\ \ \lambda^3 P_5 \middle|\ \lambda^2 I \middle|\ \lambda I \middle|\ I\ \ P_9\ \ P_{10}\ \ P_{11}\ \ P_{12}\ \ P_{13}\ \ P_{14}\ \right]^{\mathcal{B}}.$$

Notice that the formula for the right eigencolumn of Case (c) in both lemmas 5.2 and 5.3 corresponds to a PGF pencil. More precisely, it corresponds to the PGF pencil $\lambda M_{\mathfrak{s}(\mathbf{m},\mathbf{r}_m)} - M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)}$, where $\mathfrak{s}(\mathbf{m},\mathbf{r}_m)$ and $\mathfrak{s}(\mathbf{q},\mathbf{r}_q)$ are the simple tuples associated with $(\mathbf{m},\mathbf{r}_m)$ and $(\mathbf{q},\mathbf{r}_q)$, respectively. This observation is key in the proof of Theorem 4.6.

Proof of Theorem 4.6. Set $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$. We will consider separately the cases where: (a) neither $\mathbf{r}_m$ nor $\mathbf{r}_q$ contain 0; and (b) at least one of $\mathbf{r}_m$ and $\mathbf{r}_q$ contains 0.

(a) Since $\mathbf{l}_m$ and $\mathbf{l}_q$ do not affect the right eigencolumn of $\mathcal{R}_K$, and $\mathbf{r}_m, \mathbf{r}_q$ modify blocks with different indices (in particular, $\mathbf{r}_m$ modifies only blocks with indices from 0 to $k-h-2$ and $\mathbf{r}_q$ modifies blocks with indices from $k-h-1$ to k), we may concentrate only on $\mathbf{r}_q$, and we may assume that $\mathbf{r}_m = \emptyset$. The proof for the blocks modified by $\mathbf{r}_m$ when $\mathbf{r}_m \neq \emptyset$ can be carried out with similar arguments using Lemma 5.3. We will first prove the result for $\mathbf{r}_q$ consisting of just one string. The result for more than one string will follow recursively from this case.

Let $\mathbf{r}_q = (t_{d-1}+1 : h_2)$, for some $d = 2, \ldots, s$ and with $h_2 < t_d$ (notice that the case $d = 1$ is excluded because $\mathbf{r}_q$ does not contain 0). Let $\mathfrak{s}(\mathbf{q},\mathbf{r}_q)$ be the simple tuple associated with $(\mathbf{q},\mathbf{r}_q)$ and $\widetilde{K}(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)}$. Now, from Case (c) in Lemma 5.2, we have

$$M_{-\mathrm{rev}\,\mathbf{r}_q}\mathcal{R}_K = \mathcal{R}_{\widetilde{K}},$$

and then the result follows. Notice that the last equality directly implies the claim on the identity matrix.

Now, if $\mathbf{r}_q$ contains more than one string, we can iterate the previous argument. More precisely, let $\mathbf{r}_q \sim (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_g)$ in column standard form, with $\mathbf{c}_1, \ldots, \mathbf{c}_g$ strings. Let us denote $\mathfrak{s}_0 := \mathbf{q}$, and by $\mathfrak{s}_i$ the simple tuple associated with $(\mathfrak{s}_{i-1}, \mathbf{c}_i)$, for $i = 1, \ldots, g$.

Then

$$M_{-\mathrm{rev}\,\mathbf{r}_q}\mathcal{R}_K = M_{-\mathrm{rev}\,(\mathbf{c}_2,\ldots,\mathbf{c}_g)}M_{-\mathrm{rev}\,\mathbf{c}_1}\mathcal{R}_K = M_{-\mathrm{rev}\,(\mathbf{c}_2,\ldots,\mathbf{c}_g)}\mathcal{R}_{K_1} = M_{-\mathrm{rev}\,(\mathbf{c}_3,\ldots,\mathbf{c}_g)}M_{-\mathrm{rev}\,\mathbf{c}_2}\mathcal{R}_{K_1} = M_{-\mathrm{rev}\,(\mathbf{c}_3,\ldots,\mathbf{c}_g)}\mathcal{R}_{K_2} = \ldots = M_{-\mathrm{rev}\,\mathbf{c}_g}\mathcal{R}_{K_{g-1}} = \mathcal{R}_{K_g} = \mathcal{R}_{\widetilde{K}},$$

where $K_1 = \lambda M_{\mathbf{m}} - M_{\mathfrak{s}_1}$, $K_2 = \lambda M_{\mathbf{m}} - M_{\mathfrak{s}_2}, \ldots, K_{g-1} = \lambda M_{\mathbf{m}} - M_{\mathfrak{s}_{g-1}}$, $K_g = \lambda M_{\mathbf{m}} - M_{\mathfrak{s}_g} = \widetilde{K}$.

(b) Now let us consider the case where at least one of $\mathbf{r}_m$ or $\mathbf{r}_q$ contains zero. Again, we will focus on $\mathbf{r}_q$, because the arguments for $\mathbf{r}_m$ are similar. We will assume again that $\mathbf{r}_m = \emptyset$. Let us first consider the case where $\mathbf{r}_q$ consists of just one string, $\mathbf{r}_q = (0 : h_2)$, with $h_2 < t_1$. Notice that, since $0 \in \mathbf{r}_q$, A_0 must be nonsingular. Using the identity

$$\lambda P_j + A_{k-j-1} = P_{j+1}, \qquad j = 1, \ldots, k-1, \tag{14}$$

and the fact that $P_k = P$, we get

$$-\lambda A_0^{-1} P_{k-1} = I - A_0^{-1} P$$

and

$$\lambda(P_j - A_{k-j-1}A_0^{-1}P_{k-1}) = -A_{k-j-1}\left(I + \lambda A_0^{-1}P_{k-1}\right) + P_{j+1} = P_{j+1} - A_{k-j-1}A_0^{-1}P.$$

Then we have

$$\lambda \begin{bmatrix} -A_0^{-1}P_{k-1} \\ P_{k-h_2-1} - A_{h_2}A_0^{-1}P_{k-1} \\ P_{k-h_2} - A_{h_2+1}A_0^{-1}P_{k-1} \\ \vdots \\ P_{k-2} - A_{h_2-1}A_0^{-1}P_{k-1} \end{bmatrix} = \begin{bmatrix} I \\ P_{k-h_2} \\ P_{k-h_2+1} \\ \vdots \\ P_{k-1} \end{bmatrix} - \begin{bmatrix} A_0^{-1} \\ A_{h_2}A_0^{-1} \\ \vdots \\ A_1 A_0^{-1} \end{bmatrix} P(\lambda),$$

hence

$$\lambda M_{-\mathrm{rev}\,\mathbf{r}_q}\mathcal{R}_K = \mathcal{R}_{\widetilde{K}} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_0^{-1} \\ A_{h_2}A_0^{-1} \\ A_{h_2-1}A_0^{-1} \\ \vdots \\ A_1 A_0^{-1} \end{bmatrix} P(\lambda)$$

or, equivalently,

$$\mathcal{R}_{\widetilde{K}} = \lambda M_{-\mathrm{rev}\,\mathbf{r}_q}\mathcal{R}_K + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_0^{-1} \\ A_{h_2}A_0^{-1} \\ A_{h_2-1}A_0^{-1} \\ \vdots \\ A_1A_0^{-1} \end{bmatrix} P(\lambda). \tag{15}$$

We have seen so far that $\mathcal{R}_{\widetilde{K}}$ in the statement fulfills (c) in Definition 3.1. The fact that it fulfills also (a) and (b) is an immediate consequence of Theorem 4.3, because these properties hold for the right eigencolumns $\mathcal{R}_K$ obtained in this theorem, and multiplication by constant invertible matrices preserves these two properties. For (b), we need to use also (15), and notice that, if $v(\lambda) \in \mathcal{N}_r(P)$, then $\mathcal{R}_{\widetilde{K}}v(\lambda) = \lambda M_{-\mathrm{rev}\,\mathbf{r}_q}\mathcal{R}_K v(\lambda)$.

In the case where $\mathbf{r}_m \neq \emptyset$ and $\mathbf{r}_m$ contains $-k$, which is the corresponding case to the one addressed in (b) for $\mathbf{r}_q$, the result is an immediate consequence of Lemma 5.3 (b). □

Example 5.5 Let $L(\lambda) = \lambda M_{\mathbf{l}_m}M_{\mathbf{l}_q}M_{\mathbf{m}}M_{\mathbf{r}_q} - M_{\mathbf{l}_m}M_{\mathbf{l}_q}M_{\mathbf{q}}M_{\mathbf{r}_q}$ be the FPR of a matrix polynomial $P(\lambda)$ of degree $k = 15$ with $\mathbf{q} = (8, 4 : 7, 0 : 3)$, $\mathbf{m} = (-11 : -9, -12, -15 : -13)$, and $\mathbf{r}_q = (4 : 6)$, $\mathbf{r}_m = \emptyset$. Then, the simple tuple associated with $(\mathbf{q}, \mathbf{r}_q)$ is $\widetilde{\mathbf{q}} = (8, 7, 0 : 6)$. Therefore, a right eigencolumn of $L(\lambda)$ is given by $\mathcal{R}_{\widetilde{K}}$, where $\widetilde{K}(\lambda) = \lambda M_{\mathbf{m}} - M_{\widetilde{\mathbf{q}}}$. In this case, $(-\mathrm{rev}\,\mathbf{m}_1, \mathbf{q}) = (12, 9 : 11, 8, 7, 0 : 6)$, thus

$$\mathcal{R}_{\widetilde{K}} = \left[\lambda^5 P_0\ \lambda^5 P_1 | \lambda^4 I | \lambda^3 I\ \lambda^3 P_4\ \lambda^3 P_5 | \lambda^2 I | \lambda I | I\ P_9\ P_{10}\ P_{11}\ P_{12}\ P_{13}\ P_{14}\right]^{\mathcal{B}},$$

as we have already seen in Example 5.4.

Now set $\mathbf{r}_q = \emptyset$, $\mathbf{r}_m = (-15 : -14)$. Then, the simple tuple associated with $(\mathbf{m}, \mathbf{r}_m)$ is $\widetilde{\mathbf{m}} = (-11 : -9, -12, -13, -15 : -14)$. Therefore, a right eigencolumn of $L(\lambda)$ is given by $\mathcal{R}_{\widetilde{K}}$, where $\widetilde{K}(\lambda) = \lambda M_{\widetilde{\mathbf{m}}} - M_{\mathbf{q}}$. In this case, $(-\mathrm{rev}\,\mathbf{m}_1, \mathbf{q}) = (13, 12, 9 : 11, 8, 4 : 7, 0 : 3)$, thus

$$\mathcal{R}_{\widetilde{K}} = \left[\lambda^6 P_0 | \lambda^5 I | \lambda^4 I | \lambda^3 I\ \lambda^3 P_4\ \lambda^3 P_5 | \lambda^2 I | \lambda I\ \lambda P_8\ \lambda P_9\ \lambda P_{10} | I\ P_{12}\ P_{13}\ P_{14}\right]^{\mathcal{B}}.$$

Example 5.6 Let $K(\lambda) = \lambda M_{-5}M_{-4}M_{-3}M_{-8}M_{-7}M_{-6} - M_2M_0M_1$ be the PGF pencil associated with a matrix polynomial $P(\lambda)$ with degree $k = 8$. We have $\mathbf{m} = (-5 : -3, -8 : -6)$ and $\mathbf{q} = (2, 0 : 1)$ in column standard form. By direct computation we get

$$K(\lambda) = \begin{bmatrix} -I & 0 & \lambda A_8 & 0 & 0 & 0 & 0 & 0 \\ \lambda I & -I & \lambda A_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & \lambda I & 0 & 0 \\ 0 & \lambda I & \lambda A_6 & -I & 0 & \lambda A_5 & 0 & 0 \\ 0 & 0 & 0 & \lambda I & -I & \lambda A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda I & \lambda A_3 + A_2 & A_1 & -I \\ 0 & 0 & 0 & 0 & 0 & -I & \lambda I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_0 & \lambda I \end{bmatrix}$$

and, from Theorem 4.3,

$$\mathcal{R}_K = \left[\lambda^3 A_8\ \lambda^3 P_1 | \lambda^2 I\ \lambda^2 P_3\ \lambda^2 P_4 | \lambda I | I\ P_7\right]^{\mathcal{B}}.$$

It is straightforward to see that $K(\lambda)\mathcal{R}_K = \left[0\,0\,0\,0\,0\,0\,0\,P(\lambda)\right]^{\mathcal{B}}$, so $\mathcal{R}_K$ is indeed a right eigencolumn for $K(\lambda)$. Now, set $\mathbf{r}_m = (-5:-4)$ and $\mathbf{r}_q = (0)$. We have that both $(\mathbf{m}, \mathbf{r}_m)$ and $(\mathbf{q}, \mathbf{r}_q)$ satisfy the SIP and also that both $\mathbf{r}_m$ and $\mathbf{r}_q$ are of type 1 relative to $\mathbf{m}$ and $\mathbf{q}$, respectively. Moreover, a simple computation gives

$$\mathcal{R}_L := M_{-\mathrm{rev}\,\mathbf{r}_m} M_{-\mathrm{rev}\,\mathbf{r}_q} \mathcal{R}_K = \left[\lambda^3 A_8\ \lambda^3 P_1\ \lambda^3 P_2\ \lambda^3 P_3 \middle| \lambda^2 I \middle| \lambda I \middle| I\ -A_0^{-1} P_7\right]^{\mathcal{B}},$$

which is in accordance with lemmas 5.2 and 5.3. It is also immediate to see that the FPR defined as $L(\lambda) := K(\lambda) M_{\mathbf{r}_m} M_{\mathbf{r}_q}$ is

$$L(\lambda) = \begin{bmatrix} -I & 0 & 0 & 0 & \lambda A_8 & 0 & 0 & 0 \\ \lambda I & -I & 0 & 0 & \lambda A_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & \lambda I & 0 & 0 \\ 0 & \lambda I & -I & 0 & \lambda A_6 - A_5 & \lambda A_5 & 0 & 0 \\ 0 & 0 & \lambda I & -I & \lambda A_5 - A_4 & \lambda A_4 & 0 & 0 \\ 0 & 0 & 0 & \lambda I & \lambda A_4 & \lambda A_3 + A_2 & A_1 & A_0 \\ 0 & 0 & 0 & 0 & 0 & -I & \lambda I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_0 & -\lambda A_0 \end{bmatrix},$$

and that $L(\lambda)\mathcal{R}_L = \left[0\,0\,0\,0\,0\,0\,0\,P(\lambda)\right]^{\mathcal{B}}$, so $\mathcal{R}_L$ is a right eigencolumn for $L(\lambda)$. However, Theorem 4.6 gives the following right eigencolumn for $L(\lambda)$:

$$\mathcal{R}_{\widetilde{K}} := \left[\lambda^4 A_8\ \lambda^4 P_1\ \lambda^4 P_2\ \lambda^4 P_3 \middle| \lambda^3 I \middle| \lambda^2 I \middle| \lambda I \middle| I\right]^{\mathcal{B}},$$

which corresponds to the PGF pencil $\widetilde{K}(\lambda) = \lambda M_{\mathfrak{s}(\mathbf{m},\mathbf{r}_m)} - M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)}$, where $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-3, -8 : -4)$ and $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) = (2, 1, 0)$ are the simple tuples associated with $(\mathbf{m}, \mathbf{r}_m)$ and $(\mathbf{q}, \mathbf{r}_q)$, in column standard form. It is straightforward to check that $L(\lambda)\mathcal{R}_{\widetilde{K}} = \left[0\,0\,0\,0\,0\,P(\lambda)\,0\,0\right]^{\mathcal{B}}$, so $\mathcal{R}_{\widetilde{K}}$ is indeed a right eigencolumn for $L(\lambda)$.

The case of tuples which are not of type 1 will not be addressed in this work. When both $\mathbf{r}_m$ and $\mathbf{r}_q$ contain at most one string, say the ith one, not being of type 1 relative to $\mathfrak{s}_{i-1}$, we may use lemmas 5.2 and 5.3 to determine the blocks in $M_{-\mathrm{rev}\,\mathbf{r}_q} M_{-\mathrm{rev}\,\mathbf{r}_m} \mathcal{R}_K$ (we are using the notation of Definition 2.27). However, if there are more than one in $\mathbf{r}_m$ or $\mathbf{r}_q$ not being of type 1, then the problem of keeping track of the blocks which are moved after successive multiplications by the corresponding M_j matrices becomes an involved task, and remains as an open problem.

5.3.1. Symmetric pencils with repetition

Here we consider two different families of symmetric linearizations that belong to the Fiedler families in Section 2.3.

Let us begin with the symmetric linearizations considered in [14] and [15], and recently analyzed in [21] in the context of Fiedler pencils. These linearizations are FPR. In particular, for a given $0 \le h \le k-1$, we set $L^S_{k,h}(\lambda) := \lambda M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$, with $\mathbf{q} = (0 : h)$, $\mathbf{m} = (-k : -h-1)$, $\mathbf{r}_q = (0 : h-1, 0 : h-2, \ldots, 0 : 1, 0)$, and $\mathbf{r}_m = (-k : -h-2, -k : -h-3, \ldots, -k : -k+1, -k)$ (see [21, Cor. 2]). Notice that, with the notation introduced in Section 2.3, we have $\mathbf{l}_q = \mathbf{l}_m = \emptyset$ for all these pencils.

Notice that both $\mathbf{r}_q$ and $\mathbf{r}_m$ are of type 1 relative to $\mathbf{q}$ and $\mathbf{m}$, respectively. Moreover, with the notation of Theorem 4.6, we have $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) = (h, h-1, h-2, \ldots, 1, 0)$ and $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-h-1, -h-2, \ldots, -k)$. Therefore, a right eigencolumn for $L^S_{k,h}(\lambda)$ is

$$\mathcal{R}_{L^S_{k,h}} = \left[\lambda^{k-1} I\ \lambda^{k-2} I\ \lambda^{k-3} I \ldots \lambda I\ I\right]^{\mathcal{B}}.$$

Note that this eigencolumn does not depend on h. By the symmetry of the construction, this right eigencolumn is a also a left eigencolumn of $L^S_{k,h}(\lambda)$. As an example of these pencils, let us consider the case $k = 4$ and $h = 2$. We have

$$L^S_{4,2}(\lambda) = \lambda M_{(-4:-3)} M_{(0:1,0)} M_{(-4)} - M_{(0:2)} M_{(0:1,0)} M_{(-4)} =$$

$$\begin{bmatrix} -A_4 & \lambda A_4 & 0 & 0 \\ \lambda A_4 & \lambda A_3 + A_2 & A_1 & A_0 \\ 0 & A_1 & -\lambda A_1 + A_0 & -\lambda A_0 \\ 0 & A_0 & -\lambda A_0 & 0 \end{bmatrix}.$$

Notice that $L^S_{4,2}\mathcal{R}_{L^S_{4,2}} = \begin{bmatrix} 0 \; P(\lambda) \; 0 \; 0 \end{bmatrix}^{\mathcal{B}}$, and that $(L^S_{4,2})^T\mathcal{R}_{L^S_{4,2}} = \begin{bmatrix} 0 \; P(\lambda)^T \; 0 \; 0 \end{bmatrix}^{\mathcal{B}}$, so $\mathcal{R}_{L^S_{4,2}}$ is indeed a right and a left eigencolumn of $L^S_{4,2}$.

We want to emphasize that, as mentioned in [21, p. 336], the pencils $L^S_{k,h}(\lambda)$ are a basis for the vector space $\mathbb{DL}(P)$ introduced in [16]. This is an immediate consequence of the following three facts:

(i) Every $L^S_{k,h}(\lambda)$ belongs to $\mathbb{DL}(P)$ [15, p. 225].
(ii) The dimension of the vector space spanned by $L^S_{k,0}(\lambda), \ldots, L^S_{k,k-1}(\lambda)$ is k (provided that $A_k \neq 0$) [15, Lemma 10].
(iii) The dimension of the vector space $\mathbb{DL}(P)$ is k [16, Cor. 5.4].

Next we consider a recent construction of symmetric linearizations introduced by Vologiannidis and Antoniou in [21, p. 338]. Let $0 \leq h \leq k-1$ and consider the cases:

(a) h is odd: Set $\mathbf{q} = (\mathbf{q}_{odd}, \mathbf{q}_{even})$ and $\mathbf{m} = (\mathbf{m}_{odd}, \mathbf{m}_{even})$, where $\mathbf{q}_{odd} = (1, 3, \ldots, h)$, $\mathbf{q}_{even} = (0, 2, \ldots, h-1)$, $\mathbf{m}_{odd} = (-h-2, -h-4, \ldots)$, and $\mathbf{m}_{even} = (-h-1, -h-3, \ldots)$. Also, $\mathbf{l}_q = \mathbf{q}_{even}$, $\mathbf{r}_q = \emptyset$, $\mathbf{l}_m = \emptyset$, $\mathbf{r}_m = \mathbf{m}_{odd}$.

   Notice that the column standard form of $\mathbf{q}$ and $\mathbf{m}$ is $(h, h-2 : h-1, h-4 : h-3, \ldots, 1 : 2, 0)$ and $(-h-2 : -h-1, -h-4 : -h-3, \ldots)$, respectively. Thus, $\mathbf{r}_m$ is of type 1 relative to $\mathbf{m}$. Moreover, with the notation of Theorem 4.6, we have $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-h-1, -h-3 : -h-2, -h-5 : -h-4, \ldots, -k)$ if k is odd, and $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-h-1, -h-3 : -h-2, -h-5 : -h-4, \ldots, -k : -k+1)$ if k is even. However, $\operatorname{rev} \mathbf{l}_q$ is not of type 1 relative to $\operatorname{rev} \mathbf{q}$. Nonetheless, by the symmetry of the construction, a right eigencolumn is also a left eigencolumn for these linearizations (replacing A_i by A_i^T).
(b) h is even: Set $\mathbf{q} = (\mathbf{q}_{odd}, \mathbf{q}_{even})$ and $\mathbf{m} = (\mathbf{m}_{odd}, \mathbf{m}_{even})$, where now $\mathbf{q}_{odd} = (1, 3, \ldots, h-1)$, $\mathbf{q}_{even} = (0, 2, \ldots, h)$, $\mathbf{m}_{odd} = (-h-1, -h-3, ...)$, $\mathbf{m}_{even} = (-h-2, -h-4, \ldots)$. Also, $\mathbf{l}_q = \emptyset$, $\mathbf{r}_q = \mathbf{q}_{odd}$, $\mathbf{l}_m = \mathbf{m}_{even}$, $\mathbf{r}_m = \emptyset$.

   As in the previous case, $\mathbf{r}_q$ is of type 1 relative to $\mathbf{q}$.

Example 5.7 Let $k = 6$ and $h = 3$. Then $\mathbf{q} = (\mathbf{q}_{even}, \mathbf{q}_{odd}) = ((1, 3), (0, 2))$ and $\mathbf{m} = (\mathbf{m}_{even}, \mathbf{m}_{odd}) = ((-5), (-4, -6))$, $\mathbf{r}_m = (-5)$, $\mathbf{l}_q = (0 : 2)$ and $\mathbf{r}_q = \emptyset = \mathbf{l}_m$. Then

$$L(\lambda) = \lambda M_{(0,2)} M_{(-5:-4,-6)} M_{-5} - M_{(0,2)} M_{(3,1:2,0)} M_{-5} =$$

$$\begin{bmatrix} 0 & -I & \lambda I & 0 & 0 & 0 \\ -I & \lambda A_6 - A_5 & \lambda A_5 & 0 & 0 & 0 \\ \lambda I & \lambda A_5 & \lambda A_4 + A_3 & A_2 & -I & 0 \\ 0 & 0 & A_2 & -\lambda A_2 + A_1 & \lambda I & A_0 \\ 0 & 0 & -I & \lambda I & 0 & 0 \\ 0 & 0 & 0 & A_0 & 0 & -\lambda A_0 \end{bmatrix}.$$

Notice that $L(\lambda)$ is, indeed, block-symmetric.

The simple tuple associated with $(\mathbf{m}, \mathbf{r}_m)$ in column standard form is $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-4, -6 : -5)$, and the simple tuple associated with $(\mathbf{q}, \mathbf{r}_q)$ in column standard form is $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) = (3, 1 : 2, 0)$. Then, following the notation of Theorem 4.6, $\widetilde{\mathbf{m}}_1 = (-4)$ and $\widetilde{\mathbf{z}} = (4, 3, 1 : 2, 0)$ is the tuple in column standard form similar to $(-\widetilde{\mathbf{m}}_1, \mathfrak{s}(\mathbf{q}, \mathbf{r}_q))$. Hence, by Theorem 4.6, a right eigencolumn for $L(\lambda)$ is given by

$$\mathcal{R}_L = \left[\lambda^4 A_6 | \lambda^3 I | \lambda^2 I | \lambda I \ \lambda P_4 | I\right]^{\mathcal{B}}.$$

It is straightforward to check that $L(\lambda)\mathcal{R}_L = \left[0\ 0\ 0\ 0\ P(\lambda)\ 0\right]^{\mathcal{B}}$, so $\mathcal{R}_L$ is indeed a right eigencolumn of $L(\lambda)$. Since $L(\lambda)$ is block-symmetric, we have that

$$\mathcal{R}_L(P^T) = \left[\lambda^4 A_6^T | \lambda^3 I | \lambda^2 I | \lambda I \ \lambda P_4^T | I\right]^{\mathcal{B}}$$

is a left eigencolumn of $L(\lambda)$.

6. Conclusions and future work

We have obtained explicit formulas for the left and right eigencolumns of the following families of linearizations of square matrix polynomials: (a) the Fiedler pencils; (b) the GF pencils; and (c) the FPR with type 1 tuples. We have also analyzed two particular families of symmetric linearizations that belong to the last family. It remains, as an open problem, to obtain formulas for eigenvectors and minimal bases of FPR containing tuples which are not of type 1. The formulas for the eigencolumns give rise directly to formulas for the left and right eigenvectors and minimal bases for these linearizations, and relate these eigenvectors and minimal bases with the eigenvectors and minimal bases of the polynomial. The formulas for the left and right eigenvectors may be useful in the comparison of the conditioning of eigenvalues of matrix polynomials through linearizations. We think that this is now one of the most challenging questions regarding the PEP solved by linearizations. There are several previous pioneer works where the conditioning of eigenvalues of linearizations and the conditioning of eigenvalues of the polynomial have been compared [12, 13]. The present paper may be useful for the continuation of these works. In particular, to compare the conditioning of eigenvalues in the Fiedler families (including the Fiedler pencils, the GF pencils and the FPR) with the conditioning of eigenvalues in the matrix polynomial.

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Appendix A. Eigencolumns of GF pencils that are not proper

Theorem A.1 $\mathbf{m}$: *Let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a GF pencil of a regular matrix polynomial $P(\lambda)$ of degree k. Then a right eigencolumn $\mathcal{R}_K$ for $K(\lambda)$ is given by the following formulas.*

(a) *Assume $0, k \in \mathbf{q}$. Let $\mathbf{q}' = \mathbf{q} \setminus \{k\}$ and $\mathbf{z}$ be a permutation of $\{0, 1, \dots, k-1\}$ in column standard form equivalent to $(-\mathrm{rev}\, \mathbf{m}, \mathbf{q}')$. We distinguish two cases:*

(a1) *If $k-1$ is to the left of k in $(-\mathrm{rev}\, \mathbf{m}, \mathbf{q})$, then*

$$\mathcal{R}_K = \begin{bmatrix} A_k \\ \mathcal{R}_{\mathbf{z}}(2:k) \end{bmatrix},$$

with $\mathcal{R}_{\mathbf{z}}$ as in (4.1).

(a2) *If $k-1$ is to the right of k in $(-\mathrm{rev}\, \mathbf{m}, \mathbf{q})$, then*

$$\mathcal{R}_K = \mathcal{R}_{\mathbf{z}}\,.$$

(b) *Assume $-0, -k \in \mathbf{m}$. Let $(-\mathfrak{c}_{-0} : -0, \mathbf{m}')$ be the tuple in column standard form equivalent to $\mathbf{m}$.*

(b1) *If $\mathfrak{c}_{-0} = k$, then*

$$\mathcal{R}_K = \begin{bmatrix} \lambda I \ \lambda P_1 \dots \lambda P_{k-2} \ A_0 \end{bmatrix}^{\mathcal{B}}.$$

(b2) *If $\mathfrak{c}_{-0} < k$, then*

$$\mathcal{R}_K = \mathcal{R}_{\widetilde{K}},$$

where $\widetilde{K}(\lambda) = \lambda M_{\mathbf{m}'} - M_{(0:\mathfrak{c}_{-0})} M_{\mathbf{q}}$ is a PGF pencil.

(c) *Assume $-0 \in \mathbf{m}$ and $k \in \mathbf{q}$. Set $(-\mathfrak{c}_{-0} : -0, \mathbf{m}')$ and $(t : k, \mathbf{q}')$ for the tuples in column standard form equivalent to $\mathbf{m}$ and $\mathbf{q}$, respectively. We distinguish the following two cases:*

(c1) *If $t > \mathfrak{c}_{-0} + 1$, then*

$$\mathcal{R}_K = \mathcal{R}_{\widetilde{K}},$$

where $\widetilde{K}(\lambda) = \lambda M_{(-k:-t)} M_{\mathbf{m}'} - M_{(0:\mathfrak{c}_{-0})} M_{\mathbf{q}'}$ is a PGF pencil.

(c2) *If $t = \mathfrak{c}_{-0} + 1$, then*

$$\mathcal{R}_K = \begin{bmatrix} A_k \ P_1 \dots P_{k-1} \end{bmatrix}^{\mathcal{B}}.$$

Proof: (a1) In the conditions of the statement, we have that $(-\mathrm{rev}\, \mathbf{m}, \mathbf{q})$ is equivalent to $(-\mathrm{rev}\, \mathbf{m}, \mathbf{q}', k)$, so $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}'} M_k$, and then $F_\sigma(\lambda) := M_{-\mathrm{rev}\, \mathbf{m}} K(\lambda) M_{-k} = \lambda M_{-k} - M_{-\mathrm{rev}\, \mathbf{m}} M_{\mathbf{q}'}$ is a Fiedler pencil. Now the claim is a consequence of Theorem 4.1 applied to $F_\sigma(\lambda)$.

(a2) In this case we have that $(-\text{rev}\,\mathbf{m}, \mathbf{q})$ is equivalent to $(k, -\text{rev}\,\mathbf{m}, \mathbf{q}')$, so $F_\sigma(\lambda) := M_{-k}M_{-\text{rev}\,\mathbf{m}}K(\lambda) = \lambda M_{-k} - M_{-\text{rev}\,\mathbf{m}}M_{\mathbf{q}'}$ is also a Fiedler pencil, and the result is again a consequence of Theorem 4.1 applied to $F_\sigma(\lambda)$.

(b1) In this case we have

$$K(\lambda) = \lambda M_{-k}M_{-k+1}\cdots M_{-1}M_{-0} - I,$$

so $K(\lambda)M_0 = \lambda M_{-k}M_{-k+1}\cdots M_1 - M_0$ is a PGF pencil, and the result is an immediate consequence of Theorem 4.3 applied to this pencil.

(b2) Notice that, in this case, $K(\lambda) = \lambda M_{(-\mathfrak{c}_{-0}:-0)}M_{\mathbf{m}'} - M_{\mathbf{q}}$, so $\widetilde{K}(\lambda) = M_{(0:\mathfrak{c}_{-0})}K(\lambda)$ is a PGF pencil, and the result follows.

(c1) Now we have $K(\lambda) = \lambda M_{(-\mathfrak{c}_{-0}:-0)}M_{\mathbf{m}'} - M_{(t:k)}M_{\mathbf{q}'}$, so $\widetilde{K}(\lambda) = M_{(0:\mathfrak{c}_{-0})}M_{(-k:-t)}K(\lambda)$ is a PGF pencil, and the result follows.

(c2) In this case, we have $K(\lambda) = \lambda M_{(-\mathfrak{c}_{-0}:-0)} - M_{(\mathfrak{c}_{-0}+1:k)}$, so $M_{(0:\mathfrak{c}_{-0})}K(\lambda)M_{-k} = C_1(\lambda)$ is the first companion form. Hence, the claim is a consequence of Theorem 4.1. □

For the left eigencolumn, similar results can be stated using the reversal of all tuples appearing in Theorem A.1 and the polynomial P^T.

Appendix B. The infinite eigenvalue

A matrix polynomial $P(\lambda)$ is said to have an *infinite eigenvalue* if $\text{rev}\,P(\lambda)$ has an eigenvalue 0. Moreover, the left and right eigenspaces of the infinite eigenvalue of $P(\lambda)$ are the left and right eigenspaces of the zero eigenvalue of $\text{rev}\,P(\lambda)$, respectively.

In this appendix we will provide formulas for the left and right eigenvectors associated with the infinite eigenvalue in the following cases: (a) Fiedler pencils; (b) PGF pencils; and (c) FRP with type 1 tuples. Hence, the results we will state here are complementary to the ones in theorems 4.1, 4.3 and 4.6, respectively, for finite eigenvalues.

The key in deriving formulas for the left and right eigenvectors associated with the infinite eigenvalue relies in the following fact: Given a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, with $A_k \neq 0$, then v (respectively w) is a right (resp. left) eigenvector of $P(\lambda)$ associated with the infinite eigenvalue if and only if $A_k v = 0$ (resp. $A_k^T w = 0$), that is: left and right eigenvectors of a matrix polynomial associated with the infinite eigenvalue are vectors belonging to the left and right nullspace, respectively, of its leading coefficient. In all three statements below, $P(\lambda)$ is assumed to be a regular matrix polynomial as in (1), and the eigenvectors of linearizations are partitioned into k blocks with length n.

Theorem B.1 m: *Let $F_\sigma(\lambda)$ be a Fiedler pencil of $P(\lambda)$. Then:*

(a) A right eigenvector associated with the infinite eigenvalue of $P(\lambda)$ is of the form $\begin{bmatrix} v\ 0 \dots 0 \end{bmatrix}^{\mathcal{B}} \in \mathbb{C}^{nk\times n}$, where $v \neq 0$ is such that $A_k v = 0$.

(b) A left eigenvector associated with the infinite eigenvalue of $P(\lambda)$ is of the form $\begin{bmatrix} w\ 0 \dots 0 \end{bmatrix}^{\mathcal{B}} \in \mathbb{C}^{nk\times n}$, where $w \neq 0$ is such that $A_k^T w = 0$.

Proof: The result is an immediate consequence of the observation in the paragraph just before the statement and the fact that the leading coefficient of every Fiedler pencil is $M_{-k} = \text{diag}\,(A_k, I_{n(k-1)})$. □

Theorem B.2 m: *Let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a PGF pencil associated with $P(\lambda)$, and $\mathfrak{c}_{-k}$, $\mathfrak{i}_{-k}$ be, respectively, the number of consecutions and inversions of $\mathbf{m}$ at $-k$.*

(i) Let $v \neq 0$ be such that $A_k v = 0$. Then $\begin{bmatrix} v_1 \dots v_{\mathfrak{c}_{-k}}\ v\ 0 \dots 0 \end{bmatrix}^{\mathcal{B}}$, where $v_i = -A_{k-i}v$, for $i = 1, \dots, \mathfrak{c}_{-k}$, is a right eigenvector of $K(\lambda)$ associated with the infinite eigenvalue.

(ii) Let $w \neq 0$ *be such that* $A_k^T w = 0$*. Then* $\begin{bmatrix} w_1 \dots w_{\mathfrak{i}_k}\, w\, 0 \dots 0 \end{bmatrix}^{\mathcal{B}}$*, where* $w_i = -A_{k-i}^T w$*, for* $i = 1, \dots, \mathfrak{i}_{-k}$*, is a left eigenvector of* $K(\lambda)$ *associated with the infinite eigenvalue.*

Proof: The result for the right eigenvectors is an immediate consequence of the fact that, if we write $\mathbf{m} = (-\mathrm{rev}\,\mathbf{m}_1, -k : -k + \mathfrak{c}_{-k})$, then $M_{\mathbf{m}} x = 0$ if and only if $M_{(-k:-k+\mathfrak{c}_{-k})} x = 0$, and

$$M_{(-k:-k+\mathfrak{c}_{-k})} = \left[\begin{array}{cc|c} 0 & A_k & \\ I & A_{k-1} & \\ \ddots & \vdots & \\ & I\ A_{k-\mathfrak{c}_{-k}} & \\ \hline & & I_{n(k-\mathfrak{c}_{-k}-1)} \end{array}\right].$$

The result for the left eigenvectors is a consequence of *(i)* applied to $K(\lambda)^T$. □

Theorem B.3 m: *Let* $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$ *be a FPR of a matrix polynomial* $P(\lambda)$*. Assume* $\mathbf{r}_m$*,* $\mathbf{r}_q$*,* $\mathrm{rev}\,\mathbf{l}_m$ *and* $\mathrm{rev}\,\mathbf{l}_q$ *are of type 1 relative to* $\mathbf{m}$*,* $\mathbf{q}$*,* $\mathrm{rev}\,\mathbf{m}$ *and* $\mathrm{rev}\,\mathbf{q}$*, respectively. Let* $\mathfrak{c}_{-k}$ *be the number of consecutions of* $-k$ *in the simple tuple associated with* $(\mathbf{m}, \mathbf{r}_m)$ *and* $\mathfrak{i}_{-k}$ *be the number of inversions of* $-k$ *in the simple tuple associated with* $(\mathbf{l}_m, \mathbf{m})$*.*

(i) Let $v \neq 0$ *be such that* $A_k v = 0$*. Then* $\begin{bmatrix} v_1 \dots v_{\mathfrak{c}_{-k}}\ v\ 0 \dots 0 \end{bmatrix}^{\mathcal{B}}$*, where* $v_i = -A_{k-i} v$*, for* $i = 1, \dots, \mathfrak{c}_{-k}$*, is a right eigenvector of* $L(\lambda)$ *associated with the infinite eigenvalue.*

(ii) Let $w \neq 0$ *be such that* $A_k^T w = 0$*. Then* $\begin{bmatrix} w_1 \dots w_{\mathfrak{i}_{-k}}\ w\ 0 \dots 0 \end{bmatrix}^{\mathcal{B}}$*, where* $w_i = -A_{k-i}^T w$*, for* $i = 1, \dots, \mathfrak{i}_{-k}$*, is a left eigenvector of* $L(\lambda)$ *associated with the infinite eigenvalue.*

Proof: The proof can be carried out in a similar way as the proof of Theorem B.2. □

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