Continuous symmetrized Sobolev inner products of order N (I)

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Abstract

Given a symmetric Sobolev inner product of order N, the corresponding sequence of monic orthogonal polynomials $\{Q_n\}$ satisfies that $Q_{2n}(x) = P_n(x^2)$, $Q_{2n+1}(x) = xR_n(x^2)$ for certain sequences of monic polynomials $\{P_n\}$ and $\{R_n\}$. In this paper, we deduce the integral representation of the inner products such that $\{P_n\}$ and $\{R_n\}$ are the corresponding sequences of orthogonal polynomials. Moreover, we state a relation between both inner products which extends the classical result for symmetric linear functionals.

Key words: Sobolev inner product, orthogonal polynomials, symmetrization process.

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1 Introduction

It is well known (see e.g. [8, p. 43]) that Hermite polynomials can be obtained from Laguerre polynomials using a symmetrization process. More precisely,

$$H_{2n}(x) = L_n^{(-1/2)}(x^2), \quad H_{2n+1}(x) = xL_n^{(1/2)}(x^2)\,, \quad n \geq 0,$$

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where $\{L_n^{(\alpha)}\}$ and $\{H_n\}$ denote the classical Laguerre and Hermite monic polynomials, respectively.

An extension of this result due to Chihara [8, p. 41] states that, if a linear functional $\mathbf{U}$ defined in the linear space $\mathbb{P}$ of polynomials with real coefficients is *symmetric*, i.e.,

$$\mathbf{U}(x^{2n+1}) = 0\,, \quad n \geq 0\,,$$

and *quasi-definite*, i.e., there exists a sequence of monic polynomials $\{Q_n\}$ such that

(1) $\deg(Q_n) = n$,
(2) $\mathbf{U}(Q_n Q_m) = K_n \delta_{n,m}\,, \quad K_n \neq 0$,

then

$$Q_{2n}(x) = P_n(x^2)\,, \quad Q_{2n+1}(x) = xR_n(x^2)\,, \tag{1.1}$$

for certain sequences of polynomials $\{P_n\}$ and $\{R_n\}$. Chihara proved that $\{P_n\}$ is the sequence of monic polynomials orthogonal with respect to the linear functional $\mathbf{L}$ given by

$$\mathbf{L}(x^n) = \mathbf{U}(x^{2n})\,.$$

Furthermore, $\{R_n\}$ is the sequence of monic polynomials orthogonal with respect to the linear functional $\mathbf{L}^*$ given by

$$\mathbf{L}^*(p) = \mathbf{L}(xp) = \mathbf{x}\mathbf{L}(p)\,, \quad p \in \mathbb{P}\,, \tag{1.2}$$

namely, $\{R_n\}$ denotes the sequence of monic kernel polynomials with parameter 0 associated with $\{P_n\}$.

On the other hand, taking into account the three-term recurrence relation satisfied by $\{Q_n\}$,

$$xQ_n(x) = Q_{n+1}(x) + \gamma_n Q_{n-1}(x)\,, \quad \gamma_n \neq 0\,,$$

Chihara was able to deduce the parameters of the three-term recurrence relations that $\{P_n\}$ and $\{R_n\}$ satisfy, in terms of the parameters $\{\gamma_n\}$. In [1], an extension of this problem is considered when mass points are added to an absolutely continuous and symmetric measure.

Let us consider the more general case of a Sobolev inner product of order N defined in the linear space $\mathbb{P} \times \mathbb{P}$,

$$\langle p, q \rangle_s = \int_{\mathbb{R}} p\, q\, d\mu_0 + \sum_{i=1}^{N} \lambda_i \int_{\mathbb{R}} p^{(i)}\, q^{(i)}\, d\mu_i\,, \tag{1.3}$$

where $p^{(i)}$ denotes the ith derivative of p, and λ_i are positive real numbers [9]. We assume that μ_0, μ_1,..., μ_N are nondiscrete positive Borel measures supported on a subset of the real line, such that the corresponding sequences of moments are finite,

$$c_n^{(i)} = \int_{\mathbb{R}} x^n d\mu_i < \infty\,, \qquad i = 0, 1, ..., N\,, \qquad n \geq 0\,.$$

This kind of inner products, as well as their corresponding sequences of orthogonal polynomials, have been exhaustively studied during the last ten years. An extension of (1.3) can be given in terms of a matrix of measures $d\Omega$ in such a way that

$$\langle p, q\rangle = \int_{\mathbb{R}} [\, p,\, p',\, \ldots,\, p^{(N)}\,]\, d\Omega[\, q,\, q',\, \ldots,\, q^{(N)}\,]^t\,, \tag{1.4}$$

where $d\Omega$ is a square matrix of size $N+1$ with real measures as entries. These nondiagonal Sobolev inner products were introduced by J. Blankenagel in his Doctoral Dissertation [2], but the problem of stating a general theory for the corresponding sequences of orthogonal polynomials remains open.

The Sobolev product in (1.3) is said to be *symmetrized* if $\langle x^n, x^m\rangle_s = 0$ when $n+m$ is an odd number. It can be easily shown that this condition holds if and only if μ_0, μ_1,..., μ_N are supported on a subset of the real line which is symmetric with respect to the origin and the measures themselves are also symmetric, i.e.

$$c_{2n+1}^{(i)} = 0\,, \quad i = 0, 1, ..., N\,, \quad n \geq 0\,.$$

The concept of symmetrized Sobolev inner product constitutes an extension of the definition of symmetric linear functional. Given a quasi-definite symmetrized Sobolev inner product of order N, let $\{Q_n\}$ be the corresponding sequence of monic orthogonal polynomials. Then, it is easy to prove that there exist two sequences of polynomials $\{P_n\}$ and $\{R_n\}$ such that (1.1) holds. The analog of Chihara's symmetrization problem for Sobolev products would then be to find, for the sequences $\{P_n\}$ and $\{R_n\}$, the explicit expressions of the bilinear functionals with respect to which they are orthogonal, recurrence relations with a finite number of terms, and explicit algebraic relations between them.

This problem has already been solved for $N = 1$ (see [4]). The aim of our contribution is to extend some of the results in [4] for any $N \geq 1$. More precisely, we prove that the sequences $\{P_n\}$ and $\{R_n\}$ are orthogonal with respect to nondiagonal Sobolev inner products of the form (1.4), which we denote, respectively, by $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$. In Theorem 3.1, we state an explicit relation between both inner products, that extends the well-known result (1.2) for the linear symmetrization problem. Moreover, we explicitly construct the corresponding matrices of measures $d\Omega_1$ and $d\Omega_2$, and we prove that they are congruent, that is, we find a nonsingular matrix A such that $d\Omega_2 = A\, d\Omega_1\, A^t$ (see Corollary 3.4 below). The derivation of recurrence relations for the sequences $\{P_n\}$ and $\{R_n\}$ as well as explicit algebraic relations between them is left for a separate paper [6]. Finally, we apply our results to Freud-Sobolev orthogonal polynomials [7]. In particular we prove that, in this case, the matrices of measures $d\Omega_1$ and $d\Omega_2$ are diagonal, which means that the sequences $\{P_n\}$ and $\{R_n\}$ are orthogonal with respect to standard Sobolev inner products.

2 Orthogonality properties for $\{P_n\}$ and $\{R_n\}$

In this section we prove that the sequences $\{P_n\}$ and $\{R_n\}$ are orthogonal with respect to nondiagonal Sobolev inner products, and we construct the corresponding matrices of measures.

The orthogonality of the sequence $\{Q_n\}$ yields, for $n \neq m$,

$$0 = \langle Q_{2n}, Q_{2m} \rangle_s = \int_{\mathbb{R}} P_n(x^2) P_m(x^2)\, d\mu_0 + \sum_{i=1}^{N} \lambda_i \int_{\mathbb{R}} \frac{d^i P_n(x^2)}{dx^i} \frac{d^i P_m(x^2)}{dx^i}\, d\mu_i \,. \tag{2.1}$$

Using Faà di Bruno's formula for the ith derivative of the composition of two functions, we get

$$\frac{d^i P_n(x^2)}{dx^i} = i! \sum_{k=\left[\frac{i+1}{2}\right]}^{\min\{i,n\}} \frac{(2x)^{2k-i} P_n^{(k)}(x^2)}{(2k-i)!(i-k)!} \,, \tag{2.2}$$

where $p^{(l)}(z) := \frac{d^l}{dz^l} p(z)$ (see Appendix A for a detailed derivation). From now on, and for the sake of simplicity, we consider that the upper limit of the previous sum is i, although this implies that some of its terms can be zero.

Plugging (2.2) into (2.1), we get

$$\begin{aligned} 0 = & \int_{\mathbb{R}} P_n(x^2) P_m(x^2) d\mu_0 \\ & + \sum_{i=1}^{N} \lambda_i \sum_{k=\left[\frac{i+1}{2}\right]}^{i} \sum_{s=\left[\frac{i+1}{2}\right]}^{i} \tilde{\beta}_{i,k,s} \int_{\mathbb{R}} P_n^{(k)}(x^2) P_m^{(s)}(x^2) x^{2k+2s-2i} d\mu_i \,, \end{aligned}$$

where

$$\tilde{\beta}_{i,k,s} := \frac{2^{2k+2s-2i}\, (i!)^2}{(2k-i)!(i-k)!(2s-i)!(i-s)!} \,.$$

Introducing the change of variable $t = x^2$, we obtain

$$\begin{aligned} 0 = & \, 2 \int_0^{\infty} P_n(t) P_m(t) d\hat{\mu}_0 \\ & + \sum_{i=1}^{N} \lambda_i \sum_{k=\left[\frac{i+1}{2}\right]}^{i} \sum_{s=\left[\frac{i+1}{2}\right]}^{i} \beta_{i,k,s} \int_0^{\infty} P_n^{(k)}(t) P_m^{(s)}(t) t^{k+s-i} d\hat{\mu}_i \,, \end{aligned}$$

where $d\hat{\mu}_j := d\mu_j(t^{1/2})$, $j = 0, ..., N$, and

$$\beta_{i,k,s} = 2\tilde{\beta}_{i,k,s} = \frac{2^{2k+2s-2i+1}\,(i!)^2}{(2k-i)!(i-k)!(2s-i)!(i-s)!}\,. \tag{2.3}$$

This means that $\{P_n\}$ is a sequence of monic polynomials orthogonal with respect to the non-diagonal Sobolev inner product

$$\begin{aligned}\langle p, q\rangle_1 = 2\int_0^\infty p(t)q(t)d\hat{\mu}_0 \\ + \sum_{i=1}^{N} \lambda_i \sum_{k=\left[\frac{i+1}{2}\right]}^{i} \sum_{s=\left[\frac{i+1}{2}\right]}^{i} \beta_{i,k,s} \int_0^\infty p^{(k)}(t)q^{(s)}(t)t^{k+s-i}d\hat{\mu}_i\,.\end{aligned} \tag{2.4}$$

The inner product $\langle\cdot,\cdot\rangle_1$ can also be expressed in terms of a matrix of measures $d\Omega_1$ of size $N+1$ as

$$\langle p, q\rangle_1 = \int_0^\infty [p,\ p',\ \dots,\ p^{(N)}]\, d\Omega_1[q,\ q',\ \dots,\ q^{(N)}]^t\,, \tag{2.5}$$

where

$$d\Omega_1(f+1, c+1) = \sum_{i=\max\{f,c\}}^{\min\{2\min\{f,c\},N\}} \lambda_i\ \beta_{i,f,c}\ x^{f+c-i}d\hat{\mu}_i\,, \quad 0 \le f, c \le N\,. \tag{2.6}$$

In this expression, $\lambda_0 = 1$ and $d\Omega_1(f+1, c+1) = 0$ if $\min\{2\min\{f, c\}, N\} < \max\{f, c\}$.

It is also possible to find the inner product such that the corresponding sequence of monic orthogonal polynomials is $\{R_n\}$. For $n \neq m$,

$$\begin{aligned}0 &= \langle Q_{2n+1}, Q_{2m+1}\rangle_s \\ &= \int_{\mathbb{R}} x^2 R_n(x^2)R_m(x^2)d\mu_0\ +\ \sum_{i=1}^{N} \lambda_i \int_{\mathbb{R}} Q^{(i)}_{2n+1}(x)Q^{(i)}_{2m+1}(x)d\mu_i\,.\end{aligned} \tag{2.7}$$

For the ith derivative of the polynomial $Q_{2n+1}(x) = xR_n(x^2)$, Leibniz's rule gives

$$\frac{d^iQ_{2n+1}(x)}{dx^i} = x\frac{d^iR_n(x^2)}{dx^i} + i\frac{d^{i-1}R_n(x^2)}{dx^{i-1}}\,.$$

The derivatives in the right-hand side can be evaluated in explicit form by means of (2.2). A straightforward calculation then leads to

$$\frac{d^iQ_{2n+1}(x)}{dx^i} = \frac{d^i[xR_n(x^2)]}{dx^i} = (i+1)! \sum_{k=\left[\frac{i}{2}\right]}^{i} R^{(k)}_n(x^2)\frac{2^{2k-i}x^{2k-i+1}}{(2k-i+1)!(i-k)!}\,. \tag{2.8}$$

Substituting this expression into Eq. (2.7), we have

$$0 = \int_{\mathbb{R}} x^2 R_n(x^2) R_m(x^2) d\mu_0 + \sum_{i=1}^{N} \lambda_i \sum_{k=\left[\frac{i}{2}\right]}^{i} \sum_{s=\left[\frac{i}{2}\right]}^{i} \tilde{\gamma}_{i,k,s} \int_{\mathbb{R}} R_n^{(k)}(x^2) R_m^{(s)}(x^2) x^{2k+2s-2i+2} d\mu_i \,,$$

where

$$\tilde{\gamma}_{i,k,s} := \frac{2^{2k+2s-2i}\,[(i+1)!]^2}{(2k-i+1)!(i-k)!(2s-i+1)!(i-s)!} \,.$$

The change of variable $t = x^2$ yields

$$0 = 2\int_0^{\infty} t R_n(t) R_m(t) d\hat{\mu}_0 + \sum_{i=1}^{N} \lambda_i \sum_{k=\left[\frac{i}{2}\right]}^{i} \sum_{s=\left[\frac{i}{2}\right]}^{i} \gamma_{i,k,s} \int_0^{\infty} R_n^{(k)}(t) R_m^{(s)}(t) t^{k+s-i+1} d\hat{\mu}_i \,,$$

where

$$\gamma_{i,k,s} = 2\tilde{\gamma}_{i,k,s} = \frac{2^{2k+2s-2i+1}\,[(i+1)!]^2}{(2k-i+1)!(i-k)!(2s-i+1)!(i-s)!} \,. \tag{2.9}$$

Therefore, $\{R_n\}$ is the sequence of monic polynomials orthogonal with respect to the non-diagonal Sobolev inner product

$$\langle p, q\rangle_2 = 2\int_0^{\infty} t p(t) q(t) d\hat{\mu}_0 + \sum_{i=1}^{N} \lambda_i \sum_{k=\left[\frac{i}{2}\right]}^{i} \sum_{s=\left[\frac{i}{2}\right]}^{i} \gamma_{i,k,s} \int_0^{\infty} p^{(k)}(t) q^{(s)}(t) t^{k+s-i+1} d\hat{\mu}_i \,. \tag{2.10}$$

The inner product $\langle \cdot, \cdot \rangle_2$ can also be given in terms of a non-diagonal matrix of measures $d\Omega_2$ of size $N+1$ as

$$\langle p, q\rangle_2 = \int_0^{\infty} [p,\, p',\, \ldots,\, p^{(N)}]\, d\Omega_2 [q,\, q',\, \ldots,\, q^{(N)}]^t \,, \tag{2.11}$$

where

$$d\Omega_2(f+1, c+1) = \sum_{i=\max\{f,c\}}^{\min\{2\min\{f,c\},N\}} \lambda_i\, \gamma_{i,f,c}\, t^{f+c-i+1}\, d\hat{\mu}_i \,, \quad 0 \le f, c \le N \,, \tag{2.12}$$

with $\lambda_0 = 1$, and $d\Omega_2(f+1, c+1) = 0$ if $\min\{2\min\{f,c\}, N\} < \max\{f,c\}$.

3 Relation between the inner products associated with $\{P_n\}$ and $\{R_n\}$

It is natural to ask whether in the bilinear case a result similar to (1.2) can be obtained for the functionals such that $\{P_n\}$ and $\{R_n\}$ are the corresponding sequences of orthogonal polynomials. The answer is in the affirmative. The inner products given in (2.4) and (2.10) are not independent, and the following theorem gives the relation between them.

Theorem 3.1 *If $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are the inner products defined in (2.4) and (2.10), respectively, then the following relation holds:*

$$\langle p, q \rangle_2 = \langle x^{1/2} p, x^{1/2} q \rangle_1 \,. \tag{3.1}$$

Remark 3.2 *Notice that the inner product (2.4) is defined in the linear space of polynomials with real coefficients. If we extend its domain to the set of continuously differentiable real functions, then Theorem 3.1 makes sense.*

PROOF. Taking into account (2.4),

$$\langle x^{1/2} p, x^{1/2} q \rangle_1 = 2 \int_0^\infty x p(x) q(x) d\hat{\mu}_0$$
$$+ \sum_{i=1}^{N} \lambda_i \sum_{k=\left[\frac{i+1}{2}\right]}^{i} \sum_{s=\left[\frac{i+1}{2}\right]}^{i} \beta_{i,k,s} \int_0^\infty [x^{1/2} p(x)]^{(k)} [x^{1/2} q(x)]^{(s)} x^{k+s-i} d\hat{\mu}_i \,. \tag{3.2}$$

The derivative $[x^{1/2}p(x)]^{(k)}$ can be expressed as a combination of $p, p', ..., p^{(k)}$. Using Leibniz's rule and the explicit expression of the $(k-i)$th derivative of $x^{1/2}$, we get

$$[x^{1/2} p(x)]^{(k)} = \sum_{i=0}^{k} m_{k,i} \, x^{i-k+1/2} \, p^{(i)}(x) \,,$$

where

$$m_{k,i} = \binom{k}{i} (-1)^{k-i} \left(-\frac{1}{2}\right)_{k-i} \,, \tag{3.3}$$

and $(x)_n$ denotes the Pochhammer symbol defined as

$$(x)_n = x(x+1) \cdots (x+n-1) \,, \quad (x)_0 = 1 \,.$$

Substituting the previous expression into (3.2), we obtain

$$\langle x^{1/2} p, x^{1/2} q \rangle_1 = 2 \int_0^\infty x p(x) q(x) d\hat{\mu}_0$$
$$+ \sum_{i=1}^{N} \lambda_i \sum_{k=0}^{i} \sum_{s=0}^{i} \delta_{i,k,s} \int_0^\infty p^{(k)}(x) q^{(s)}(x) x^{k+s-i+1} d\hat{\mu}_i \,, \tag{3.4}$$

where

$$\delta_{i,k,s} = \sum_{t=\max\left\{k,\left[\frac{i+1}{2}\right]\right\}}^{i} m_{t,k} \left[\sum_{l=\max\left\{s,\left[\frac{i+1}{2}\right]\right\}}^{i} \beta_{i,t,l}\, m_{l,s} \right] . \tag{3.5}$$

On the other hand, notice from (2.9) that $\gamma_{i,k,s} = 0$ when $k < \left[\frac{i}{2}\right]$ or $s < \left[\frac{i}{2}\right]$ since, in that case, either $2k - i + 1 < 0$ or $2s - i + 1 < 0$. Then, (2.10) can also be written as

$$\begin{aligned} \langle p, q \rangle_2 = 2 \int_0^\infty x p(x) q(x) d\hat{\mu}_0 \\ + \sum_{i=1}^{N} \lambda_i \sum_{k=0}^{i} \sum_{s=0}^{i} \gamma_{i,k,s} \int_0^\infty x^{k+s-i+1} p^{(k)}(x) q^{(s)}(x) d\hat{\mu}_i \, . \end{aligned} \tag{3.6}$$

Comparison of Eqs. (3.4) and (3.6) reveals that Theorem 3.1 is equivalent to

$$\delta_{i,k,s} = \gamma_{i,k,s} \quad \forall\, i, k, s \, . \tag{3.7}$$

To prove (3.7), we represent the sum inside the brackets in (3.5) as a hypergeometric function. Recall that the *generalized hypergeometric function* ${}_pF_q$ is defined as

$${}_pF_q\left(\begin{matrix} a_1,\, a_2,\, \ldots\, ,\, a_p \\ b_1,\, b_2,\, \ldots\, ,\, b_q \end{matrix} \middle|\, x \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k k!}\, x^k \, ,$$

and it is said to be *balanced* if

$$\sum_{l=1}^{p} a_l = \sum_{l=1}^{q} b_l \; - 1 \, .$$

Taking into account the well-known properties

$$x! = \Gamma(x+1) \, , \tag{3.8}$$

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = (-1)^n \frac{\Gamma(1-x)}{\Gamma(1-x-n)} \, , \tag{3.9}$$

$$\Gamma(2x) = 2^{2x-1} \frac{\Gamma(x+\frac{1}{2})\Gamma(x)}{\Gamma(\frac{1}{2})} \, , \tag{3.10}$$

Equations (2.3) and (3.3) can be written as

$$\beta_{i,t,l} = \frac{2^{-2i+1} (-1)^{t+l} (-i)_t (-i)_l}{[\Gamma(1-i)]^2 \left(1 - \frac{i}{2}\right)_t \left(\frac{1}{2} - \frac{i}{2}\right)_t \left(1 - \frac{i}{2}\right)_l \left(\frac{1}{2} - \frac{i}{2}\right)_l} \, ,$$

$$m_{l,s} = \frac{(-1)^l (1)_l \left(-\frac{1}{2} - s\right)_l}{s!\, \Gamma(1-s) \left(\frac{3}{2}\right)_s (1-s)_l} \, .$$

These expressions enable us to represent the sum inside the brackets in (3.5) as a balanced ${}_4F_3$ hypergeometric function of unit argument,

$$\sum_{l=\max\left\{s,\left[\frac{i+1}{2}\right]\right\}}^{i} \beta_{i,t,l}\, m_{l,s} = \sum_{l=0}^{\infty} \beta_{i,t,l}\, m_{l,s}$$

$$= \frac{2^{-2i+1}(-1)^t(-i)_t}{s!\,\Gamma(1-s)\left(\frac{3}{2}\right)_s [\Gamma(1-i)]^2 \left(1-\frac{i}{2}\right)_t \left(\frac{1}{2}-\frac{i}{2}\right)_t} \sum_{l=0}^{\infty} \frac{(-i)_l(1)_l\left(-\frac{1}{2}-s\right)_l}{\left(1-\frac{i}{2}\right)_l\left(\frac{1}{2}-\frac{i}{2}\right)_l(1-s)_l}$$

$$= \frac{2^{-2i+1}(-1)^t(-i)_t}{s!\,\Gamma(1-s)\left(\frac{3}{2}\right)_s [\Gamma(1-i)]^2 \left(1-\frac{i}{2}\right)_t \left(\frac{1}{2}-\frac{i}{2}\right)_t}\; {}_4F_3\left(\left.\begin{matrix} -i,\, 1,\, 1,\, -\frac{1}{2}-s \\ 1-\frac{i}{2},\, \frac{1}{2}-\frac{i}{2},\, 1-s \end{matrix}\right| 1\right).$$

Taking advantage of the transformation formula [11],

$$ {}_4F_3\left(\left.\begin{matrix} -n,\, b,\, c,\, d \\ \frac{b-n}{2},\, \frac{b-n+1}{2},\, c+d+\frac{1}{2} \end{matrix}\right| 1\right)$$

$$= \frac{(2d-b+1)_n}{(1-b)_n}\; {}_3F_2\left(\left.\begin{matrix} -n,\, 2d,\, \frac{1}{2}+d-c \\ c+d+\frac{1}{2},\, 2d-b+1 \end{matrix}\right| 1\right), \qquad n \in \mathbb{N},$$

and noting from (3.9) that $(0)_n = (-1)^n/\Gamma(1-n)$, we find that

$$ {}_4F_3\left(\left.\begin{matrix} -i,\, 1,\, 1,\, -\frac{1}{2}-s \\ \frac{1}{2}-\frac{i}{2},\, 1-\frac{i}{2},\, 1-s \end{matrix}\right| 1\right) = \frac{(-1-2s)_i\,\Gamma(1-i)}{(-1)^i}\; {}_2F_1\left(\left.\begin{matrix} -i,\, -1-s \\ 1-s \end{matrix}\right| 1\right).$$

The ${}_2F_1$ hypergeometric function can be evaluated in closed form by means of the well-known Gauss summation formula,

$$ {}_2F_1\left(\left.\begin{matrix} a,\, b \\ c \end{matrix}\right| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,, \qquad \mathrm{Re}(c-a-b) > 0\,,$$

which leads to

$$ {}_4F_3\left(\left.\begin{matrix} -i,\, 1,\, 1,\, -\frac{1}{2}-s \\ \frac{1}{2}-\frac{i}{2},\, 1-\frac{i}{2},\, 1-s \end{matrix}\right| 1\right) = \frac{(-1-2s)_i\,\Gamma(1-i)\Gamma(1-s)\Gamma(i+2)}{(-1)^i\,\Gamma(1-s+i)}\,. \tag{3.11}$$

Thus we conclude that

$$\sum_{l=\max\left\{s,\left[\frac{i+1}{2}\right]\right\}}^{i} \beta_{i,t,l}\, m_{l,s} = \frac{2^{-2i+1}(-1)^t(-i)_t(-1-2s)_i\,\Gamma(i+2)}{s!\left(\frac{3}{2}\right)_s \Gamma(1-i)\left(1-\frac{i}{2}\right)_t\left(\frac{1}{2}-\frac{i}{2}\right)_t(-1)^i\,\Gamma(1-s+i)}\,.$$

Plugging this result into (3.5), this double sum can be expressed as

$$\delta_{i,k,s} = \frac{(-1)^i \, 2^{-2i+1} \, (-1-2s)_i \, \Gamma(i+2)}{k! \, s! \left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_s \Gamma(1-k)\Gamma(1-i)\Gamma(1-s+i)} \times {}_4F_3\left(\left. \begin{array}{c} -i, \, 1, \, 1, \, -\frac{1}{2}-k \\ 1-\frac{i}{2}, \, \frac{1}{2}-\frac{i}{2}, \, 1-k \end{array} \right| 1 \right) ,$$

which using (3.11) simplifies to

$$\delta_{i,k,s} = \frac{2^{-2i+1} \, [(i+1)!]^2 \, (-1-2s)_i \, (-1-2k)_i}{(i-s)! \, (i-k)! \, k! \, s! \left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_s} .$$

From (3.9) and (3.10), we can easily prove that

$$(-1-2t)_i = \frac{(-1)^i \, (1+2t)!}{(1+2t-i)!} = \frac{(-1)^i \, 2^{1+2t} \, \Gamma\left(\frac{3}{2}+t\right)\Gamma(1+t)}{(1+2t-i)! \, \Gamma\left(\frac{1}{2}\right)} .$$

Thus, we have

$$\delta_{i,k,s} = \frac{2^{2k+2s-2i+1} \, [(i+1)!]^2}{(i-s)!(i-k)!(1+2s-i)!(1+2k-i)!} \cdot \frac{4\,\Gamma\left(\frac{3}{2}+s\right)\Gamma\left(\frac{3}{2}+k\right)}{\left[\Gamma\left(\frac{1}{2}\right)\right]^2 \left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_s} ,$$

and Equation (3.7) follows. □

Remark 3.3 *When considering general symmetrized and symmetric bilinear functionals, the functional associated with the sequence* $\{R_n\}$ *is not uniquely determined by that associated with* $\{P_n\}$ *[5]. However, Theorem 3.1 shows that, in the case of symmetrized Sobolev inner products defined in terms of absolutely continuous measures, the inner product* $\langle \cdot, \cdot \rangle_2$ *can be deduced directly from* $\langle \cdot, \cdot \rangle_1$*. Therefore, in this case,* $\{R_n\}$ *is said to be the sequence of* generalized kernel polynomials associated with $\{P_n\}$.

The result given in Theorem 3.1 can also be expressed as a relation between the matrices of measures that define the inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$. In fact, we can prove that both matrices are congruent.

Corollary 3.4 *Consider the matrices of measures given by (2.6) and (2.12). Then,*

$$d\Omega_2 = A \, d\Omega_1 \, A^t ,$$

where A *is an upper triangular matrix whose* j*th column* $A(:,j)$ *is given by*

$$A(:,j) = \left[m_{j,0} \, x^{1/2-j}, \, m_{j,1} \, x^{3/2-j}, \, \ldots, \, m_{j,j} \, x^{1/2}, \, 0, \, \ldots, \, 0 \right]^t ,$$

and $m_{j,i}$ *is defined in (3.3).*

PROOF. It suffices to take into account (2.6), (2.12) and

$$[\, x^{1/2}p,\ (x^{1/2}p)^{(1)},\, \ldots,\, (x^{1/2}p)^{(k)}\,] = [\, p,\, p^{(1)},\, \ldots,\, p^{(k)}\,]A\,. \qquad \square$$

4 Example: Freud-Sobolev orthogonal polynomials

In this section we apply the previous results to the case when $d\mu_i = d\mu = e^{-x^4}dx$ for $i = 0, 1, \ldots, N$, i.e., we consider the inner product

$$\langle p, q\rangle_s = \int_{\mathbb{R}} p\, q\, e^{-x^4} dx \;+\; \sum_{i=1}^{N} \lambda_i \int_{\mathbb{R}} p^{(i)}\, q^{(i)}\, e^{-x^4} dx\,. \tag{4.1}$$

The polynomials $\{Q_n\}$ orthogonal with respect to (4.1) are a particular case of the so-called Freud-Sobolev polynomials [7]. Applying to this example the main results in Section 2, we deduce that the corresponding sequences $\{P_n\}$ and $\{R_n\}$ are orthogonal with respect to the inner products (2.4) and (2.10), respectively, where, for $i = 0, 1, \ldots, N$,

$$d\hat{\mu}_i = d\hat{\mu} = \hat{\omega}(x)dx\,, \quad \hat{\omega}(x) = \frac{1}{2}x^{-1/2}e^{-x^2}\,. \tag{4.2}$$

Notice that the weight function $\hat{\omega}$ is semiclassical, since it satisfies the Pearson equation $(\phi\hat{\omega})' = \psi\hat{\omega}$ with $\phi(x) = 2x$, $\psi(x) = 1 - 4x^2$, and boundary conditions $\phi(x)p(x)\hat{\omega}(x)|_0^\infty = 0$ for any polynomial p. Taking advantage of general results for semiclassical linear functionals obtained in [9], we get

$$\phi^N(x)D^N(\hat{\omega}(x)) = \psi(x, N)\hat{\omega}(x)\,, \tag{4.3}$$

where

$$\psi(x, 0) = 1\,,$$
$$\psi(x, N) = 2x\psi'(x, N-1) + \psi(x, N-1)[1 - 2N - 4x^2]\,, \quad N \geq 1\,,$$

and

$$\deg(\psi(x, N)) \leq 2N\,. \tag{4.4}$$

In general, the inner products (2.4) and (2.10) are non-diagonal. But we will prove that, in the case of Freud-Sobolev polynomials, they are standard Sobolev products.

Proposition 4.1 *In the particular case (4.2), the inner products given by (2.4) and (2.10) can both be expressed in terms of diagonal matrices of measures of size* $N + 1$*.*

PROOF. The matrix of measures associated with $\{P_n\}$ will be diagonal if the inner product

$$\langle p,q\rangle_1 = 2\int_0^\infty p(x)q(x)d\hat{\mu} + \sum_{i=1}^N \lambda_i \sum_{k=\left[\frac{i+1}{2}\right]}^{i} \sum_{s=\left[\frac{i+1}{2}\right]}^{i} \beta_{i,k,s}\int_0^\infty p^{(k)}(x)q^{(s)}(x)x^{k+s-i}d\hat{\mu}\ , \tag{4.5}$$

can be expressed in such a way that it contains only terms in which p and q are affected by the same derivative, i.e., $p^{(k)}q^{(k)}$. For a fixed i, $1 \le i \le N$, we consider the change of indices

$$l = 2i-k-s\,, \quad r = i-k = s-i+l\,.$$

The corresponding term of the sum in (4.5) then becomes

$$\sum_{k=\left[\frac{i+1}{2}\right]}^{i} \sum_{s=\left[\frac{i+1}{2}\right]}^{i} \beta_{i,k,s}\int_0^\infty p^{(k)}(x)q^{(s)}(x)x^{k+s-i}d\hat{\mu} = \sum_{l=0}^{2i-2\left[\frac{i+1}{2}\right]} \sum_{r=\tilde{r}}^{\tilde{l}} \beta_{i,i-r,i-l+r}\int_0^\infty p^{(i-r)}(x)q^{(i-l+r)}(x)x^{i-l}d\hat{\mu}\ , \tag{4.6}$$

where

$$\tilde{r} = \max\left\{0, l-i+\left[\frac{i+1}{2}\right]\right\}, \quad \tilde{l} = \min\left\{i-\left[\frac{i+1}{2}\right], l\right\}.$$

Notice that

$$\begin{aligned} \tilde{r} = 0 \quad &\Longrightarrow \quad \tilde{l} = l\ , \\ \tilde{r} = l-i+\left[\frac{i+1}{2}\right] \quad &\Longrightarrow \quad \tilde{l} = i-\left[\frac{i+1}{2}\right]. \end{aligned} \tag{4.7}$$

For a fixed l, consider the sum

$$\sum_{r=\tilde{r}}^{\tilde{l}} \beta_{i,i-r,i-l+r}\int_0^\infty p^{(i-r)}(x)q^{(i-l+r)}(x)x^{i-l}d\hat{\mu} = \beta_{i,i-\tilde{r},i-l+\tilde{r}} \sum_{r=\tilde{r}}^{\tilde{l}} \frac{\beta_{i,i-r,i-l+r}}{\beta_{i,i-\tilde{r},i-l+\tilde{r}}}\int_0^\infty p^{(i-r)}(x)q^{(i-l+r)}(x)x^{i-l}d\hat{\mu}\ . \tag{4.8}$$

Now we add and subtract from (4.8) the term

$$\beta_{i,i-\tilde{r},i-l+\tilde{r}}\int_0^\infty [p^{(i-l+\tilde{r})}(x)q^{(i-l+\tilde{r})}(x)]^{(l-2\tilde{r})}x^{i-l}d\hat{\mu}\ ,$$

which using Leibniz's rule can also be expressed as

$$\beta_{i,i-\tilde{r},i-l+\tilde{r}} \sum_{r=\tilde{r}}^{l-\tilde{r}} \binom{l-2\tilde{r}}{r-\tilde{r}} \int_0^\infty p^{(i-r)}(x) q^{(r+i-l)}(x) x^{i-l} d\hat{\mu} \, .$$

We see from (4.7) that $\tilde{r}+\tilde{l}=l$, hence the upper limit of the previous sum is $l-\tilde{r}=\tilde{l}$. Then, we obtain

$$\begin{aligned} &\beta_{i,i-\tilde{r},i-l+\tilde{r}} \sum_{r=\tilde{r}}^{\tilde{l}} \beta^{(1)}_{i,i-r,i-l+r} \int_0^\infty p^{(i-r)}(x) q^{(i-l+r)}(x) x^{i-l} d\hat{\mu} \\ &+ \ \beta_{i,i-\tilde{r},i-l+\tilde{r}} \int_0^\infty [p^{(i-l+\tilde{r})}(x) q^{(i-l+\tilde{r})}(x)]^{(l-2\tilde{r})} x^{i-l} d\hat{\mu} \, , \end{aligned} \tag{4.9}$$

where

$$\beta^{(1)}_{i,i-r,i-l+r} = \frac{\beta_{i,i-r,i-l+r}}{\beta_{i,i-\tilde{r},i-l+\tilde{r}}} - \binom{l-2\tilde{r}}{r-\tilde{r}} \, .$$

Since $\tilde{r}+\tilde{l}=l$, and $\beta_{i,k,s}=\beta_{i,s,k}$, the following property holds,

$$\beta_{i,i-\tilde{r},i-l+\tilde{r}} = \beta_{i,i-\tilde{l},i-l+\tilde{l}} \, . \tag{4.10}$$

Notice that $\beta^{(1)}_{i,i-\tilde{r},i-l+\tilde{r}} = \beta^{(1)}_{i,i-\tilde{l},i-l+\tilde{l}} = 0$. Then, (4.9) simplifies to

$$\begin{aligned} &\beta_{i,i-\tilde{r},i-l+\tilde{r}} \sum_{r=\tilde{r}+1}^{\tilde{l}-1} \beta^{(1)}_{i,i-r,i-l+r} \int_0^\infty p^{(i-r)}(x) q^{(i-l+r)}(x) x^{i-l} d\hat{\mu} \\ &+ \ \beta_{i,i-\tilde{r},i-l+\tilde{r}} \int_0^\infty [p^{(i-l+\tilde{r})}(x) q^{(i-l+\tilde{r})}(x)]^{(l-2\tilde{r})} x^{i-l} d\hat{\mu} \, . \end{aligned} \tag{4.11}$$

If $i-l=0$, then $\tilde{r}=\left[\frac{i+1}{2}\right]$ and $l-2\tilde{r}=0$, so it suffices to apply the previous procedure to the first integral in (4.11) to obtain the result. If $i-l\neq 0$, we apply integration by parts to the second integral in (4.11) and obtain

$$-\int_0^\infty p^{(i-l+\tilde{r})}(x) q^{(i-l+\tilde{r})}(x) [x^{i-l}\hat{\omega}(x)]^{(l-2\tilde{r})} dx \, , \tag{4.12}$$

since $x^{i-l}\hat{\omega}(x)\Big|_0^\infty = 0$. Then,

$$\begin{aligned} [x^{i-l}\hat{\omega}(x)]^{(l-2\tilde{r})} &= \sum_{j=0}^{l-2\tilde{r}} \binom{l-2\tilde{r}}{j} (x^{i-l})^{(l-2\tilde{r}-j)} (\hat{\omega}(x))^{(j)} \\ &= \sum_{j=0}^{l-2\tilde{r}} \binom{l-2\tilde{r}}{j} (i-2l+2\tilde{r}+j-1)_{l-2\tilde{r}-j} \, x^{i-2l+2\tilde{r}+j} (\hat{\omega}(x))^{(j)} \, . \end{aligned} \tag{4.13}$$

Taking into account (4.3), we find that

$$\begin{aligned}
&[x^{i-l}\hat{\omega}(x)]^{(l-2\tilde{r})} \\
&= \sum_{j=0}^{l-2\tilde{r}} \binom{l-2\tilde{r}}{j} (i-2l+2\tilde{r}+j-1)_{l-2\tilde{r}-j}\, x^{i-2l+2\tilde{r}} \frac{1}{2^j}\, \psi(x,j)\, \hat{\omega}(x) \\
&\equiv \hat{\omega}(x) R(x) \ ,
\end{aligned}$$

where $\deg(R(x)) \leq i-2\tilde{r}$ due to Eq. (4.4). Finally, we plug the previous result in (4.12) to get

$$-\int_0^\infty p^{(i-l+\tilde{r})}(x) q^{(i-l+\tilde{r})}(x) R(x) d\hat{\mu} \ ,$$

which completes the proof. A similar reasoning enables us to prove that the matrix of measures associated with $\{R_n\}$ is also diagonal. □

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A Appendix: Derivation of Eq. (2.2) from Faà di Bruno's formula

Faà di Bruno's formula for the ith derivative of the composition $(f \circ g)(x) = f(g(x))$ [3,10] states that

$$\frac{d^i f(g(x))}{dx^i} = i! \sum_{k=0}^{i} f^{(k)}(g(x)) \sum_{k_1,k_2,\ldots,k_i} \prod_{j=1}^{i} \frac{[g^{(j)}(x)]^{k_j}}{(j!)^{k_j}\, k_j!} \ , \tag{A.1}$$

where the inner summation is extended over all partitions satisfying

$$k_1 + k_2 + \ldots + k_i = k \ , \qquad k_1 + 2k_2 + \ldots + ik_i = i \ . \tag{A.2}$$

In particular, when $g(x) = x^2$, $[g^{(j)}(x)]^{k_j}$ vanishes for $j \geq 3$ unless $k_j = 0$, in which case it equals unity. Conditions (A.2) then read

$$k_1 + k_2 = k \ , \qquad k_1 + 2k_2 = i \ , \qquad k_3 = \ldots = k_i = 0 \ ,$$

and the first two of these equations uniquely determine the values of k_1 and k_2 in terms of k and i,

$$k_1 = 2k - i \ , \qquad k_2 = i - k \ .$$

Application of Faà di Bruno's formula (A.1) then yields

$$\frac{d^i f(x^2)}{dx^i} = i! \sum_{k=\left[\frac{i+1}{2}\right]}^{i} f^{(k)}(x^2) \frac{(2x)^{2k-i}}{(2k-i)!(i-k)!} \ , \tag{A.3}$$

where the lower bound in the sum over k, with $[x]$ denoting the integer part of x, follows from the fact that $k_1 = 2k - i$ is a nonnegative integer. When $f(x) = P_n(x)$ is a polynomial of degree n in x, $f^{(k)}(x^2)$ vanishes for $k > n$, so that the upper bound i can be replaced by $\min\{i, n\}$. Thus we obtain (2.2). □

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