

## Rank of a linear transformation.

Let  $G$  be a finite abelian group. Let  $N$  be the exponent of  $G$  and  $n$  be the order of  $G$ . A function  $f : G \rightarrow \mathbb{C} - \{0\}$  is called a character of  $G$  if it is a group homomorphism. If  $f$  is a character of a finite group, then each function value is a root of unity since all elements of a finite group have finite order. Notice that if  $g \in G$ , for any character  $\chi$  of  $G$  we have

$$\chi(g)^N = \chi(g^N) = \chi(e) = 1,$$

so the values of  $\chi$  lie among the  $N$ th roots of unity. Characters on finite abelian groups were first studied in number theory, since number theory is a source of many interesting finite abelian groups.

A finite abelian group of order  $n$  has exactly  $n$  distinct characters which are denoted by  $f_1, f_2, \dots, f_n$ .  $f_1$  is the trivial representation, that is,  $f_1(g) = 1$  for all  $g \in G$ . It is called the principal character of  $G$ ; the others are called nonprincipal characters, and  $f_i(g) \neq 1$  for some  $g \in G$ . The set of characters of  $G$  form an abelian group under multiplication called the character group.

Let  $\hat{G}$  denote the character group of  $G$ . Fix a primitive  $N$ -th root  $z$  of unity. Then, for each  $g \in G$  and  $\chi \in \hat{G}$ , there is a unique integer  $1 \leq r \leq N$  such that  $\chi(g) = z^r$ . We therefore obtain a pairing

$$\langle \cdot, \cdot \rangle : G \times \hat{G} \rightarrow \mathbb{Q}/\mathbb{Z}$$

defined by

$$\langle g, \chi \rangle = \left\{ \frac{r}{N} \right\}$$

where  $\{r/N\}$  denotes the fractional part of  $r/N$ . We may extend the previous pairing to

$$\langle \cdot, \cdot \rangle : \mathbb{Q}[G] \times \mathbb{Q}[\hat{G}] \rightarrow \mathbb{Q}$$

via linearity in the obvious way

$$\left\langle \sum_{g \in G} c_g \cdot g, \sum_{\chi \in \hat{G}} c_\chi \cdot \chi \right\rangle = \sum_{g \in G} \sum_{\chi \in \hat{G}} c_g \cdot c_\chi \langle g, \chi \rangle.$$

Here  $\mathbb{Q}[G]$  and  $\mathbb{Q}[\hat{G}]$  are group rings. In particular, both  $\mathbb{Q}[G]$  and  $\mathbb{Q}[\hat{G}]$  are  $\mathbb{Q}$ -vector spaces of dimension  $|G| = |\hat{G}|$ .

Let us now define a map  $f : \mathbb{Q}[\hat{G}] \rightarrow \mathbb{Q}[G]$  as follows:

$$f(a) = \sum_{g \in G} \langle g, a \rangle g, \quad \text{for any } a \in \mathbb{Q}[\hat{G}].$$

We may view  $f$  as a linear map between two vector spaces of dimension  $|G|$ . An important question in some applications in number theory is: what can we say about the kernel of  $f$ ?

The REU students will approach this question by studying the rank of  $f$  and determining the dimension of the kernel of  $f$ . We will work first with finite cyclic groups and, if time permits, we will also consider the general case.