UCSB Mathematics Qualifying Exam

May 23, 2005

Answer 6 of 7. This exam is three hours.

1. Prove that every simply connected closed 3-manifold is homeomorphic to the 3-sphere.

2. Define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Prove that if s_0 is a zero of $\zeta(s)$ that is not a negative even integer, then $\Re(s_0) = \frac{1}{2}$.

3. Prove that on a projective non singular algebraic variety over $\mathbb{C}$, any Hodge class is a rational linear combination of classes $\bar{Z}$ of algebraic cycles. (*Hint: By Chow's theorem, algebraic cycles are the same as closed analytic subspaces on complex projective varieties.*)

4. Let G be any compact simple gauge group. Prove that quantum Yang-Mills theory on $\mathbb{R}^4$ exists and has mass gap $\Delta > 0$.

5. Let Δ be the discriminant of the cubic and define

$$\begin{aligned} N_p &= \#\{\text{solutions of } y^2 = x^3 + ax + b \mod p\} \\ a_p &= p - N_p \\ L(C,s) &= \prod_{p \nmid 2\Delta} (1 - a_p p^{-1} + p^{1-2s})^{-1}. \end{aligned}$$

Prove that the Taylor expansion of $L(C,s)$ at $s = 1$ has the form

$$L(C,s) = c(s-1)^r + \text{ higher order terms}$$

with $c \neq 0$ and $r = \text{rank}(C(\mathbb{Q}))$. (*Hint: recall that if C is an elliptic curver over $\mathbb{Q}$ then $C(\mathbb{Q})$ is isomorphic to the direct sum of $\mathbb{Z}^r$ and a finite group.*)

6. The Navier-Stokes equations are given by

$$\frac{\partial}{\partial t} u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \mu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \qquad \text{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0,$$

for $x \in \mathbb{R}^n$ and $t \geq 0$, and initial conditons $u(x,0) = u^o(x)$. Take $\nu > 0$ and $n = 3$. Let $u^o(x)$ be any smooth divergence-free vector field satisfying $|\partial_x^\alpha u^o(x)| \leq C_{\alpha K}(1+|x|)^{-K}$ for any α, K. Suppose $f(x,t) = 0$. Prove that there exists smooth functions $p(x,t)$, $u_i(x,t)$ on $C^\infty(\mathbb{R}^3 \times [0,\infty))$ that satisfy the Navier-Stokes equations, with the extra condition that

$$\int_{\mathbb{R}^n} |u(x,t)|^2 \, dx < C, \quad \text{for all } t \geq 0.$$

7. Prove that $P \neq NP$.