Real Functions and Measure Theory
Budapest Semesters in Mathematics
Spring Semester, 2002
Analysis Practice Problems

1. Let τ_s denote the topology of $\mathbb{R}$ generated by the half-closed intervals $[a, b) : a, b \in \mathbb{R}$ (tau_s is called the *Sorgenfrey topology*). Show that $(\mathbb{R}, \tau_s)$ and $\mathbb{R}$ with the usual topology are not homeomorphic spaces, but the Borel σ-algebras are identical.

2. Let $\mathcal{A}$ denote the σ-algebra of the countable and co-countable subsets of $\mathbb{R}$. Decide if $\sin x$ is measurable with respect to $\mathcal{A}$.

3. Let $\mathcal{B}$ denote the σ-algebra in $\mathbb{R}$ generated by the open intervals. Describe the σ-algebra generated by the
   (a) closed intervals;
   (b) left half-open intervals;
   (c) like in (a) and (b), but with rational endpoints only.

4. Describe the σ-algebra $\mathcal{A}$ in $\mathbb{R}$ generated by the subsets of rational numbers. Can you find real functions f and g such that f is $\mathcal{B}$-measurable by not $\mathcal{A}$ measurable and g the other way around?

5. What is the smallest σ-algebra $\mathcal{A}$ on $\mathbb{R}$ such that every monotone real function is measurable?

6. What kind of measurable space is $(X, \mathcal{A})$ if
   (i) every finite valued measurable function $f : X \to \mathbb{R}$ is simple?

   (ii) the vector space of simple measurable functions is finite dimensional?

7. A sequence (a_n) is eventually monotone if there is N such that $(a_n)_{n \geq N}$ is monotone. Let $(X, \mathcal{A})$ be a measurable space, and let f_n be a sequence of measurable functions on X. Show that the set of points where f_n is eventually monotone is a measurable set.

8. Let $(X, \mathcal{A}, \mu)$ be a probability space, A_n are measurable subsets such that $\mu(A_n) \geq c$ for every n. Prove that the set H of points that belong to infintely many of the sets A_n is measurable and $\mu(H) \geq c$.

9. Let f be a Borel measurable real function on $\mathbb{R}$. Prove that the set of points where it takes an extremal value is always Borel measurable. Show that the measurability of f can be dropped if strict extremal values are considered only.

10. Construct a sequence of simple Borel functions such that it converges increasingly o $f(x) = \left|\frac{1}{x}\right|$.

11. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. For $A, B \in \mathcal{A}$, define $A \sim B$ if and only if $\mu(A \triangle B) = 0$. Prove that $\sim$ is an equivalence relation. For any two equivalence classes C_1, C_2, define

$$d(C_1, C_2) = \mu(A \triangle B),$$

where $A \in C_1, B \in C_2$. Show that the definition makes sense, and d is a metric on the set of equivalence classes.

12. Prove that the metric d of the previous exercise is complete.

13. What is the smallest σ-algebra $\mathcal{A}$ on $\mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0 \end{cases}$$

is measurable with respect to $\mathcal{A}$? Describe the measurable functions.

14. Decide if the pointwise limit of an increasing sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ is always continuous / lower semicontinuous / upper semicontinuous.

15. A sequence $\{a_n\}$ is eventually increasing if there is an N such that $\{a_n\}_{n \geq N}$ is increasing. Let $(X, \mathcal{A})$ be a measurable space, and let f_n be a sequence of measurable functions on X. Show that the set of points where f_n is eventually monotone is a measurable set. Show by example that the Lebesgue's Monotone Convergence Theorem does not remain true if we replace the condition "increasing" by "eventually increasing."

16. Let $(X, \mathcal{A}, \mu)$ be a probability space. Suppose A_n are measurable subsets such that $\mu(A_n) > \frac{3}{4}$ for every n. Let H denote the set of points $x \in X$ such that x is contained by exactly two of the sets $A_1, A_2, \ldots$. Show that H is measurable, and $\mu(H) \leq \frac{1}{4}$.

17. Prove that the Cantor set has Lebesgue measure 0.

18. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space (that is $X = \cup X_n, X_n \in \mathcal{A}, \mu(X_n) < \infty$), and let f be a measurable real function on X. Prove that $\mu(f^{-1}(c)) = 0$ for all but countably many $c \in \mathbb{R}$.

19. Let μ be a Borel measure on $\mathbb{R}$ such that $\mu((a,b)) = b - a$ for every rational interval (a,b). Show that μ is the Lebesgue measure (restricted on the Borel sets).

20. Prove that any finite Borel measure μ in $\mathbb{R}^n$ is regular. (Hint: use Riesz's representation theorem on $\Lambda f = \int f d\mu$.)

21. Let μ be a finite Borel measure on $\mathbb{R}$, and put $f(x) = \mu((-\infty, x))$ (the distribution function of μ. Prove that f is an increasing function and express $\mu((a,b)), \mu([a,b]), \mu([a,b)), \mu((a,b]), \mu(\{a\})$ in terms of f. Show that f is continuous in a given point x if and only if $\mu(\{x\}) = 0$.

22. Let μ be a finite Borel measure on $\mathbb{R}$ such that every value taken by μ is a rational number. Show that μ can take at most finitely many different values.

23. Let $(X, \mathcal{A}, \mu)$ be a measure space. $H \in \mathcal{A}$ is said to be an atom (w.r.t. μ) if $\mu(H) > 0$ and for every $A \subset H, A \in \mathcal{A}$, either $\mu(A) = 0$ or $\mu(H \setminus A) = 0$. Prove that if f is a measurable function $f : X \to \mathcal{R}$, and H is an atom, then f is constant almost everywhere on H.

24. Let μ be a finite Borel measure on $\mathbb{R}$. Show that apart from a set of measure zero, any atom is a singleton. Describe the atoms in terms of the distribution function of μ.

25. Let $(X, \mathcal{A}, \mu)$ be an atom-free measure space, and $\mu(X) = 1$. Prove that μ takes every value in $[0, 1]$.

26. Let Λ be a positive linear functional on $C([0, 1])$. Suppose $f_n \in C([0, 1])$ converges to $f \in C([0, 1])$ (i) uniformly; (ii) pointwise with $\max_x |f_n(x)| \leq 1$. Show that $\lim \Lambda(f_n) = \Lambda f$ in either case.

27. Let $\{r_n\}$ be a sequence containing all rational numbers, and denote $\Lambda f = \sum_{n=1}^{\infty} \frac{f(r_n)}{2^n}$ for $f \in C_c(\mathbb{R})$. Prove that Λ is a positive linear functional on $C_c(\mathbb{R})$, and describe the corresponding representing measure space $(\mathbb{R}, \mathcal{A}, \mu)$. Give a formula expressing $\mu([0, 1])$ and compute $\mu(\mathbb{R})$.

28. Let f denote the Cantor function. Compute $\int_0^1 f(x)dx$ and $\int_{[0,1]} f' d\lambda$.

29. Verify that $\Lambda f = \int_0^1 f(x)e^x dx$ is a positive linear functional on $C([0, 1])$. Describe the σ-algebra of locally regular sets and find the measure (according to Riesz's theorem) that represents Λ.

30. Let (X, τ) be the space of Ex. 14 and for $f \in C_c(X)$, define

$$\Lambda f = \sum_x \int_{\mathbb{R}} f(x, y)dy.$$

Show that λ is a positive linear functional and find the representing measure λ.

31. Find

$$\lim_{n\to\infty} \int_0^{2\pi} (\sin x)^n dx.$$

32. Let

$$\lambda(H) = m(H) - \int_H \frac{5}{1 + x^2} dm.$$

Characterize the total, positive, and negative variations of λ and find the Hahn decomposition for λ.

33. Show that

$$\lambda f = \sum_{n=0}^{\infty} \int_{2n}^{2n+1} f(x)dx \quad (f \in C_c(\mathbb{R}))$$

is a positive linear functional on $C_c(\mathbb{R})$. Let $(\mathbb{R}, \mathcal{A}, \mu)$ be the corresponding Riesz representation. Express $\mu(A)$ in terms of the Lebesgue measure for any $A \in \mathcal{A}$. Is μ a translation invariant measure?

34. Let μ be a measure on $(X, \mathcal{A})$, $\mu(X) < \infty$. Let $A \in \mathcal{A}, B \in \mathcal{A}$ be given, and define $\lambda(H) = \mu(H \cap A) - \mu(H \cap B)$ for every $H \in M$. Show that λ is a signed measure and find its Hahn decomposition. Prove that $\lambda \ll \mu$, and find the Radon-Nikodym derivative.

35. Let μ be a regular Borel measure on a compact Hausdorff space X with $\mu(X) = 1$. Prove that there exists a compact set $K \subset X$ such that $\mu(K) = 1$ and $\mu(V) > 0$ for every open set V with $V \cap K \neq \emptyset$.

36. Does there exist a signed measure ϑ on the measurable space $(\mathbb{Z}, \mathcal{P}(\mathbb{Z})$ such that

(i) $\vartheta(\{n\}) = 1$ for every even, and $\vartheta(\{n\}) = -1$ for every odd integer n;

(i) $\vartheta(\{n\}) = 1$ for every even, and $\vartheta(\{n\}) = -2^{-|n|}$ for every odd integer n?

37. Let $\vartheta(H) = \lambda(H) - \int_H \frac{2}{1+x^2} d\lambda$. Characterize the variations $|\vartheta|, \pi, \nu$ of ϑ and find the Hahn decomposition of ϑ.

38. Let δ denote the measure of unit mass in $\{0\}$, that is $\delta(H) = 1$ if $0 \in H$, and $\delta(H) = 0$ if $0 \notin H$, where $H \subset \mathbb{R}$ is arbitrary. Find the Lebesgue decomposition of δ w.r.t. λ, and that of λ w.r.t. δ.

39. Solve the last two exercises for $\vartheta = \delta - \lambda$, and generalize them if δ is supported on a finite, or a countable subset of $\mathbb{R}$.

40. Let $r \in \mathbb{R}$, $f \in L_1(\lambda)$, $\vartheta(H) = \int_H f d\lambda$, and denote $H + r = \{y : y = x + r, x \in H\}$. Prove that for the translated measure $\vartheta_r(H) = \vartheta(H + r)$ we have $\vartheta_r \ll \lambda$ and find the corresponding Radon-Nikodym derivative. Decide which one of the statements:

    (i) $\vartheta \perp \vartheta_r$

    (ii) $\vartheta \ll \vartheta_r$

    (iii) $\vartheta_r \ll \vartheta$

    holds true (absolute continuity w.r.t. a signed measure ϑ means a.c. w.r.t. its total variation $|\vartheta|$).

41. Let $I_1^n, I_2^n, \ldots, I_{2^n}^n$ denote the pairwise disjoint closed intervals of length $\frac{1}{3^n}$ we used in the nt'th step of the definition of Cantor's middle third set C. Prove that there exists a Borel measure γ on $[0, 1]$ such that $\gamma(I_k^n) = \frac{1}{2^n}$ for every $n = 1, 2, \ldots$ and for every $k = 1, \ldots, 2^n$. Show that γ is supported on C.

42. Find the variations and the Hahn decomposition for $\vartheta = \gamma - \lambda$ on $([0, 1], \mathcal{B})$. Decide which one of the statements

    (i) $\gamma \ll \lambda$

    (ii) $\lambda \ll \gamma$

    (iii) $\gamma \perp \lambda$

    holds true.

43. Let $\mu_1, \ldots, mu_n, \varphi_1, \ldots, \varphi_k$ be positive measures on $(X, \mathcal{M})$. Show that $(\mu_1 + \cdots + \mu_n) \perp (\varphi_1 + \cdots + \varphi_k)$ if and only if $\mu_j \perp \varphi_l$ for every $a \leq j \leq n$, $1 \leq l \leq k$.

44. Let μ be a positive measure on $(X, \mathcal{M})$. Characterize the measurable functions f such that $\mu \ll |\vartheta|$, where $\vartheta(H) = \int_H f d\mu$.

45. Let μ be a positive measure on $(X, \mathcal{A})$, let $H \in \mathcal{A}$ be fixed and define $\vartheta(E) = \mu(E \cap H)$ for every $E \in \mathcal{A}$. Show that $\vartheta \ll \mu$ and find the Radon-Nikodym derivative.

46. Let μ_1, μ_2 be finite positive measures on $(X, \mathcal{A})$ and define $\varphi = \mu_1 - \mu_2$. Show that assuming $\mu_1 \perp \mu_2$ we have $\pi_\varphi = \mu_1$, $\nu_\vartheta = \mu_2$.