Math 117 Homework No. 2

1. (a) Suppose that the sequence \( \{a_n\} \) converges to 0, and that the sequence \( \{b_n\} \) is bounded. 
Prove carefully that the sequence \( \{a_n b_n\} \) converges.

(b) Is (a) true if \( \{a_n\} \) converges to 1? In other words: Suppose \( \{a_n\} \) converges to 1 and the sequence \( \{b_n\} \) is bounded. Prove or disprove that \( \{a_n b_n\} \) converges.

2. (a) Prove carefully that if the sequence \( \{a_n\} \) is convergent with limit \( A \), and \( a_n \geq 0 \) for all \( n \), then \( A \geq 0 \).

(b) Give an example which shows that if in (a), we change the hypothesis to \( a_n > 0 \) for all \( n \), this does not imply \( A > 0 \).

(c) Using the result of part (a), show that if the sequence \( \{a_n\} \) is convergent with limit \( A \) and \( \lambda \leq a_n \leq \mu \) for all \( n \), then \( \lambda \leq A \leq \mu \).

3. (a) Suppose that \( \{a_n\} \) converges with limit \( A \). Fix some integer \( p > 1 \). Prove carefully that the sequence \( \{a_{p,n}\} \) (in other words the sequence formed by taking every \( p \)-th term of the original sequence) converges with limit \( A \).

(b) Is the converse of (a) true? In other words: Suppose \( \{a_n\} \) is a sequence with the property that for every fixed integer \( p > 1 \), the sequence \( \{a_{p,n}\} \) converges. Prove or disprove that \( \{a_n\} \) converges.