# Subgroup separability and virtual retractions of groups

D. D. Long,* & A. W. Reid†

August 25, 2004

## 1  Introduction

W begin by recalling that if $\Gamma$ is a group, and $H$ a subgroup of $\Gamma$, then $\Gamma$ is called $H$-**separable** if for every $g \in \Gamma \setminus H$, there is a subgroup $K$ of finite index in $\Gamma$ such that $H \subset K$ but $g \notin K$. The group $\Gamma$ is called **subgroup separable** (or LERF) if $\Gamma$ is $H$-separable for all finitely generated subgroups $H$. As is well-known LERF is a powerful property in the setting of low-dimensional topology which has attracted a good deal of attention (see [3] and [34] for example), however, it is a property established either positively or negatively for very few classes of groups. Much recent work has suggested that the correct condition to impose on the subgroup is not finite generation but geometrical finiteness (or quasi-convexity in the case of a negatively curved group) and this article takes this theme further by exploring a new related condition which is even more relevant for topological applications and which is reminiscent of old and very classical topological considerations, namely map extension properties and ANR's. We begin with a simple definition that underpins much of what follows.

**Definition 1.1** *Let $\Gamma$ be a group and $H$ a subgroup. Then a homomorphism $\theta : H \longrightarrow A$ **extends over the finite index subgroup** $V \leq \Gamma$ if*

- *$H \leq V$*

- *There is a homomorphism $\Theta : V \longrightarrow A$ which is the homomorphism $\theta$ when restricted to $H$.*

One of the motivations in our setting for making this definition is the following straightforward theorem proved in §3.

**Theorem 1.2** *Suppose that $\Gamma$ is LERF, and $H$ a finitely generated subgroup. Suppose $\theta : H \to A$ is a homomorphism onto a finite group. Then there is subgroup of finite index $V$ of $\Gamma$ containing $H$ and a homomorphism $\Theta : V \to A$ with $\Theta|_H = \theta$.*

In other words, if the ambient group is LERF, then every homomorphism of a finitely generated subgroup onto a finite group can be extended to some subgroup of finite index in $\Gamma$. We shall say that the homomorphism $\theta$ **virtually extends** and that $\Gamma$ has the **local extension property** for homomorphisms onto finite groups. We shall show in Theorem 1.3 that, somewhat surprisingly, this extension property comes close to characterising LERF, we refer the reader to §2 for the definitions.

**Theorem 1.3** *Let $\Gamma$ be a word hyperbolic group which has the local extension property for homomorphisms of quasi-convex subgroups onto $\mathbf{Z}/2$.*
    *Then $\Gamma$ is almost separable on its quasi-convex subgroups.*

It turns out that these extension properties give elegant expressions of several well established notions, for example residual finiteness:

**Theorem 1.4** *$\Gamma$ is residually finite if and only if $\Gamma$ has the property that for each of its cyclic groups there is a virtual extension of at least one of the maps onto a nontrivial (cyclic) group.*

An important generalisation of the extension property is the following:

**Definition 1.5** *Let $\Gamma$ be a group and $H$ a subgroup. Then $\Gamma$ **virtually retracts** $H$ if there is a finite index subgroup $V$ of $\Gamma$ with*

- *$H \leq V$*

- *There is a homomorphism $\theta : V \longrightarrow H$ which is the identity when restricted to $H$.*

The finite index subgroup $V$ will be called a **retractor**. One should regard a virtual retraction as an extension of the identity homomorphism $H \longrightarrow H$, to some finite index subgroup $V$ of $\Gamma$ and clearly, given any such retraction, we can extend any homomorphism $H \rightarrow A$ over the finite index subgroup $V$.

Retractions are presumably somewhat rare, but one can ask for their existence in more restricted circumstances, for example, we might require that $H$ be a finite subgroup, or an infinite cyclic subgroup or a geometrically finite (or quasiconvex) subgroup in some more geometric setting.

Motivated by this, we define a group $\Gamma$ to have Property $LR$ (it admits **L**ocal **R**etractions) if for every finitely generated subgroup $H < \Gamma$, there is a finite index subgroup which retracts to it. If we wish to retrict to some class of subgroups, we refer to $\Gamma$ being $LR$ over finite, cyclic, freely indecomposable etc subgroups.

Despite their rarity one can prove that retractions exist for several important classes of of examples; indeed, in the course of proving that finitely generated free groups are LERF, M. Hall showed that if $F$ is free and and $H$ a finitely generated subgroup of $F$ then there is a finite index subgroup of $F$ containing $H$ as a free factor, that is $F$ has Property $LR$ (see §5 for a detailed discussion). We show.

**Theorem 1.6** *Suppose that $\Gamma$ is a finitely generated group which virtually embeds into an all right hyperbolic Coxeter subgroup of $Isom(\mathbf{H}^n)$.*
*Then $\Gamma$ virtually retracts to its geometrically finite subgroups.*

That is to say, $\Gamma$ is $LR$ over geometrically finite subgroups. Our interest comes from the geometric applications for which we have the especially important conjecture:

**Conjecture 1.7** *Suppose that $\Gamma$ is the fundamental group of a closed hyperbolic 3-manifold.*
*Then $\Gamma$ virtually retracts to any of its cyclic subgroups.*

There is some exploration of this question and its implications in §2.3. We note that although this question is significantly stronger than the traditional virtually Haken conjecture, phrased in these terms it places the question closer in spirit to questions about extensions of cyclic groups as in the notion of residual finiteness as in the statement of Theorem 1.4. We note that it neither implies, nor is implied by LERF.

Questions regarding subgroup separability often interact with difficult issues concerning Property T and the congruence subgroup property and an advantage of the property of being $LR$ over cyclic groups is that one can often sidestep such issues and apply geometric methods. For example, despite the fact little or nothing is known about subgroup separability in this context we are able to show using harmonic methods:

2

**Theorem 1.8** *Let $\Gamma$ be a cocompact torsion-free lattice in $\mathrm{PU}(n,1)$. Let $\Delta < \Gamma$ be isomorphic to a lattice in $\mathrm{PO}_0(m,1)$ where $m \geq 3$. Then $\Gamma$ does not virtually retract to $\Delta$.*

Indeed, despite its powerful geometric consequences, the question of whether a group is $LR$ over its cyclic groups is stable in a way that subgroup separability is not; for example it is well known that LERF is not preserved by Cartesian products of groups: in contrast, the property of being $LR$ over $\mathbf{Z}$ is (see Theorem 2.13).

The paper is organized as follows. In §2 we provide a quick review of separability properties, and in §3 prove Theorem 1.2. In addition, and perhaps of note, we provide a partial converse to Theorem 1.2, as well as give a characterization of residually finite in the language of extensions. We go on to discuss retractions in detail and discuss how the theory of retractions can be used to see geometry that, as yet, is undetected by LERF. In §3 we give an application of the theory of extensions to 3-manifold topology, and in particular we show.

**Corollary 1.9** *Let $M = \mathbf{H}^3/\Gamma$ be a non-compact orientable arithmetic hyperbolic 3-manifold and $H$ a non-elementary subgroup of $\Gamma$ of infinite index generated by two parabolic elements.*

*Then there is a finite index subgroup $\Gamma_H < \Gamma$, a homomorphism $\phi : \Gamma_H \to \mathbf{Z}$ for which $H$ is contained in $\ker \phi$.*

In the final section we raise some questions that are closely connected to our work.

## 2 Residual properties of groups and extensions

### 2.1 Background on subgroup separability

Let $\Gamma$ be a group and $H$ a subgroup. It will be convenient to record the following equivalent definitions of $\Gamma$ being $H$-separable. Define the subgroup $H^*$ of $\Gamma$ to be

$$H^* = \bigcap \{ K \ : \ H < K, [\Gamma : K] < \infty \}.$$

Then $\Gamma$ is $H$-separable if and only $H^* = H$. Closely related to this is that $\Gamma$ is $H$-separable if and only if $H$ is closed in the profinite topology on $\Gamma$. We recall that in these terms, a group $\Gamma$ is *residually finite* if and only the trivial subgroup is separable in $\Gamma$.

When $\Gamma$ is a group of hyperbolic isometries, a useful geometric version of LERF that will be important for us, is GFERF. Recall that if $\Gamma < \mathrm{Isom}(\mathbf{H}^n)$ then $\Gamma$ is called GFERF if it is $H$-separable for every finitely generated geometrically finite subgroup $H$.

**Convention:** All geometrically finite groups considered will be finitely generated, so that our definition of geometrical finiteness is $\mathrm{vol}(N_\eta(C(\Gamma))) < \infty$ where $C(\Gamma)$ is the convex core of $\mathbf{H}^n/\Gamma$.

In dimension 2, since there are no finitely generated geometrically infinite subgroups, it follows that LERF coincides with GFERF. In dimension 3, Agol's recent proof of tameness [2], shows that if a finite co-volume Kleinian group is GFERF, it is also LERF; since an affirmative solution to the tameness conjecture implies that every finitely generated geometrically infinite subgroup of a finite covolume Kleinian group is virtually a fiber in a fibration over the circle (see [11] Corollary 8.3). Such groups are known to be separable by work of Bonahon and Thurston (see [6] and [36]).

When $\Gamma$ is word hyperbolic group (in particular it is finitely generated) we can replace GFERF by QCERF. That is, $\Gamma$ is called QCERF if it is $H$-separable for every quasi-convex subgroup $H$. Note that if $H$ is a quasi-convex subgroup of a word hyperbolic group it is finitely generated (see [8] page 460).

### 2.2 Extensions

The following simple observation expressing a basic connection between extensions and LERF seems to have gone unnoticed previously:

**Theorem 2.1** *Suppose that $\Gamma$ is LERF, and $H$ a finitely generated subgroup. Suppose $\theta : H \to A$ is a homomorphism onto a finite group. Then there is subgroup of finite index $V$ of $\Gamma$ containing $H$ and a homomorphism $\Theta : V \to A$ with $\Theta|_H = \theta$.*

**Proof:** We can assume that $H$ has infinite index in $\Gamma$. Let $K = \ker \theta$, a finite index subgroup of $H$, and hence finitely generated. Since $\Gamma$ is LERF, there is a finite index subgroup $K' < \Gamma$ such that $K' \cap H = K$. Define $\Delta = \bigcap \{ hK'h^{-1} : h \in H \}$. Note that since $K$ is normal in $H$, $K < \Delta$ and moreover, since $K$ has finite index in $H$, the above intersection consists of only a finite number of conjugates of $K'$. Hence $\Delta$ has finite index in $K'$ (and also $\Gamma$). Let $V$ denote the group generated by $H$ and $\Delta$. It is easy to check that, by construction, $\Delta$ is a normal subgroup of $V$, so that $V = H.\Delta$. The canonical projection $\Theta : V \to V/\Delta$ defines the required extension. $\square$

A corollary of this is the following.

**Corollary 2.2** *Suppose that $\Gamma$ is LERF and contains a free non-abelian subgroup. Then given any finite group $G$, there is a finite index subgroup $\Gamma_G$ of $\Gamma$ and a surjection $\Gamma_G \to G$.* $\square$

**Remarks:**

**1.** This corollary implies the well-known fact that lattices in semi-simple Lie groups that have the Congruence Subgroup Property are not LERF.

**2.** There are many non-LERF groups that share the property stated in Corollary 2.2; the most well known being the group $F_2 \times F_2$.

Somewhat surprisingly, there is almost a converse to Theorem 2.1 in the negatively curved setting. To this end, we recall (the terminology is that of [27], see also [31]) that a subgroup $H$ is *almost separable* if $[H^* : H] < \infty$. For many geometrical purposes, almost separability suffices, since the subgroup $H^*$ is visibly separable.

**Theorem 2.3** *Let $\Gamma$ be a word hyperbolic group which has the local extension property for homomorphisms of quasi-convex subgroups onto $\mathbf{Z}/2$.*

*Then $\Gamma$ is almost separable on its quasi-convex subgroups.*

**Proof:** Let $H$ be any quasi-convex subgroup of $\Gamma$ whose limit set is a proper subset of the Gromov boundary at infinity of $\Gamma$, denoted by $\partial G$. We will show that $H$ is *engulfed*; that is, it is contained in some proper subgroup of finite index in $\Gamma$. The result of [31] (extending the earlier work of [27]) will then show that if $H$ is any quasi-convex subgroup of $\Gamma$ then $[H^* : H]$ is finite.

Using the techniques of [27] (as extended in [31]) we may find an infinite cyclic subgroup $C$ in $\Gamma$ so that the group $\Delta = \langle H, C \rangle = H * C$ and has its limit set being a proper subset of $\partial G$ and in particular is quasi-convex.

Choose a homomorphism $\theta : \Delta \to \mathbf{Z}/2$ which kills $H$. By assumption, $\Gamma$ has the local extension property for homomorphisms onto $\mathbf{Z}/2$, so $\theta$ may be extended to $\Theta : V \longrightarrow \mathbf{Z}/2$.

We deduce that our given $H < ker(\Theta) < \Gamma$ is engulfed as required. $\square$

We next prove that there is a simple characterization of residually finite in the language of extensions.

**Theorem 2.4** *$\Gamma$ is residually finite if and only if $\Gamma$ has the property that for each of its cyclic groups there is an extension of at least one of the maps onto a nontrivial cyclic group.*

**Proof:** If $\Gamma$ has the stated extension property, then given a nonidentity $g \in \Gamma$ we can extend some homomorphism of $\langle g \rangle \longrightarrow \mathbf{Z}/k$ to $V \longrightarrow \mathbf{Z}/k$, where $V$ has finite index in $\Gamma$. Then the kernel of this map has finite index in $\Gamma$ and excludes $g$ so that $\Gamma$ is residually finite. Conversely, suppose that $\Gamma$ is residually finite, and let some nonidentity $g \in \Gamma$ be given. Choose some normal subgroup of finite index in $\Gamma$, $N$ say, which excludes $g$.

Then the map

$$N.\langle g \rangle \longrightarrow N.\langle g \rangle / N \cong \langle g \rangle / \langle g^k \rangle$$

for some $k$. This extends the map $\langle g \rangle \to \mathbf{Z}/k$, this map being nontrivial by choice of $N$. $\square$

## 2.3   Virtual retractions

We now discuss virtual retractions. These appear to be interesting for a variety of reasons that we discuss further below. Despite their apparent rarity, one can prove that virtual retractions exist for an important class of examples (see [3] and [34]). The result 2.6 motivated the definition, and our interest in retractions. We first recall some definitions.

**Definition 2.5** *Let $\Gamma$ be a group and $H$ a subgroup. Then $\Gamma$ **virtually retracts** $H$ if there is a finite index subgroup $V$ of $\Gamma$ with*

- $H \leq V$

- There is a homomorphism $\theta : V \longrightarrow H$ which is the identity when restricted to $H$.

Now let $\mathcal{G}$ be a finite graph (without loops and without multiple edges) with vertex set $V(\mathcal{G}) = \{s_1, s_2, \ldots, s_n\}$, and edge set $E(\mathcal{G})$. A labelling of $\mathcal{G}$ is a function $\mathbf{m} : E(\mathcal{G}) \to \{2, 3, \ldots\}$. If $e \in E(\mathcal{G})$ has endpoints $s_i$ and $s_j$, we write $\mathbf{m}(e) = m_{ij}$, and by convention we set $m_{ii} = 1$. Then the **Coxeter group** $C(\mathcal{G}, \mathbf{m})$ associated to the labelled graph $(\mathcal{G}, \mathbf{m})$ is the group with presentation

$$< s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 > .$$

Thus there is a relation for every pair $\{s_i, s_j\}$ such that there is an edge $e \in E(\mathcal{G})$ with endpoints $s_i$ and $s_j$, together with the relations $s_i^2 = 1$, $1 \leq i \leq n$. A Coxeter group is called **an all right** Coxeter group if all finite labels are 2.

**Theorem 2.6** *Suppose that $\Gamma$ is a finitely generated group which embeds into an all right hyperbolic Coxeter subgroup of $Isom(\mathbf{H}^n)$.*

*Then $\Gamma$ virtually retracts to any of its geometrically finite subgroups.*

**Proof:** Let $P$ be an all right polyhedron in $\mathbf{H}^n$, $G(P)$ the Coxeter group generated by reflections in the faces of $P$ and $H$ a geometrically finite subgroup of $G(P)$. As in the proof of Theorem 3.1 of [3] (see also [34]), the geometric convex hull of $H$ can be expanded to a tiling hull $Y(H)$ coming from the all right Coxeter group. Briefly, if $C(H)$ denotes the convex hull of the limit set of $H$, then $Y(H)$ is defined to be the smallest convex union of copies of $P$ which contains $C(H)$ in its interior. It is easy see that $Y(H)$ is $H$-invariant, and the key step in the separability argument of [3] and [34] is to show that the quotient $Y(H)/H$ continues to have finite volume.

The group generated by reflections in the sides of $Y(H)$, denoted by $K(H)$, is discrete with fundamental polyhedron $Y(H)$. By the $H$-invariance, it follows that $K(H)$ is normalized by $H$ and $K \cap H = 1$. Define $V(H) = K(H) \cdot H$. Then $V(H)$ visibly has the properties of a retractor, the only issue being whether $[G(P) : V(H)]$ is finite. This follows from the fact that $Y(H)/H$ has finite volume (see [3] and [34]). Thus $V(H)$ is a retractor for $H$ in $G(P)$. If $H$ is a subgroup of $\Gamma$, then $\Gamma \cap V(H)$ is a retractor for $H$ in $\Gamma$. $\square$

**Remarks:1.** Examples of groups $\Gamma$ as in Theorem 2.6 are plentiful (see [3] and [30]). All right Coxeter subgroups as in Theorem 2.6 exist in dimensions $2 \leq n \leq 8$.

**2.** Since Hall's result that finitely generated free groups are LERF can be reproved using these techniques (see [34]), it is perhaps not too surprising that a stronger statement than GFERF can be made under the hypothesis of Theorem 2.6. However, until now, the full strength of the construction does not appear to have been noticed.

**3.** An interesting feature of this method, is that given the embedding into an all right Coxeter group acting on $\mathbf{H}^n$, to every geometrically finite subgroup one can canonically assign a subgroup of finite index, namely the retractor subgroup.

**4.** Recently, Haglund [23] has extended the methods of [3] and [34] to the setting of quasi-convex subgroups of abstract all right Coxeter groups. More precisely, he shown that if $C(\mathcal{G}, \mathbf{m})$ is any all right Coxeter group, then the quasi-convex subgroups are separable (there is no assumption made about negative curvature). Furthermore, the method of proof in [23] closely resembles that of [3] and [34], and one can deduce from it (see in particular Lemma 5.1, Theorem 4.5 and Corollary 4.6), the following extension of Theorem 2.6.

**Theorem 2.7** *Suppose that $\Gamma$ is a finitely generated group which embeds into an all right hyperbolic Coxeter group. Then every quasi-convex subgroup $H$ of $\Gamma$ admits a virtual retraction.* □

Note that $F_2 \times F_2$ is a subgroup of finite index in an all right Coxeter group, and so although it is not LERF, the geometrically well-behaved subgroups are separable by [23], and there are virtual retractions onto quasi-convex subgroups.

An important corollary of 2.6 from the topological viewpoint is the following.

**Corollary 2.8** *Suppose that $M = \mathbf{H}^3/\Gamma$ is a finite volume hyperbolic 3-manifold for which $\Gamma$ embeds into an all right hyperbolic Coxeter subgroup of $Isom(\mathbf{H}^3)$.*

*Then if $\gamma \in \Gamma$ is any non-trivial element, there is a finite cover $M_\gamma$ of $M$ for which $\gamma$ represents a generator of an infinite cyclic factor of $H_1(M_\gamma; \mathbf{Z})$.*

**Proof:** Apply Theorem 2.6 to the infinite cyclic subgroup generated by $\gamma$. □

As far as the authors are aware these are the only examples of hyperbolic 3-manifolds for which it is known that every non-trivial element virtually represents a non-trivial element in first homology.

This corollary motivates consideration of an important special case which we single out:

**Definition 2.9** *A group is $LR$ over its cyclic subgroups, if given any cyclic subgroup, $C$, there is a virtual retraction $V \longrightarrow C$.*

It is a well known conjecture that a hyperbolic 3-manifold group is $LR$ over *some* infinite cyclic subgroup. We conjecture that actually a hyperbolic 3-manifold group is $LR$ over *all* its cyclic subgroups. This seems to be the correct conjecture as this stronger property behaves rather well. We note that it follows immediately that if a group is $LR$ over its cyclic subgroups, then it clearly satisfies Theorem 2.4, so that it is residually finite.

We now discuss some properties of $LR$ (and $LR$ over cyclic subgroups) that are reminiscent of how LERF behaves. In fact, Property $LR$ behaves better in certain circumstances, as we now discuss. In the following, to avoid the cumbersome clutter of multiple repetitions of the phrase "(and $LR$ over cyclic subgroups)" we shall suppress it and only point out occasions when there is any difference of note.

As with LERF (see [34]), Property $LR$ is well behaved for subgroups. That is to say, if $\Gamma$ has Property LR and $K \leq \Gamma$, then $K$ has Property $LR$. For if $H < K$ is a finitely generated subgroup and $V$ a subgroup of finite index in $\Gamma$ which is a retractor for $H$, then $V \cap K$ is a retractor for $H$ in $K$.

For finite supergroups there is also a similar result to [34] with some mild hypotheses:

**Theorem 2.10** *Suppose that $\Gamma$ is a finitely generated linear group and $K$ a subgroup of $\Gamma$ of finite index. If $K$ has Property LR, then $\Gamma$ has Property LR.*

**Proof:** We begin by noting that from the argument above, we may assume that $K$ is normal in $\Gamma$. Given an $H$ in $\Gamma$, choose a retraction $r : A \longrightarrow H \cap K$, where $A$ has finite index in $K$. By the normality of $K$, $H$ acts by conjugacy on $A$ while stablising $H \cap K$, so that we may suppose intersect all these conjugates and restrict the original retraction, and we may suppose that $A$ is normalised by $H$.

Choose a faithful linear representation $\rho : H \cap K \to GL(V)$, for some finite dimensional vector space $V$. By composition with the retraction, we get a (nonfaithful) linear representation $A \to GL(V)$. Note that $A.H/A \cong H/(H \cap A)$ is finite, so that we may induce upwards to get a representation $\rho^* : A.H \to GL(V')$.

Since $A$ is normal in $A.H$, the restriction of the induced representation $\rho^*$ down to $A$ gives $[A.H : A]$ copies of $\rho$ which is therefore a faithful representation of $H \cap K$. It follows that when $\rho^*$ is restricted to $H$, it is faithful on a subgroup of finite index, namely $H \cap K$. This forces $\ker \rho^*$ to be finite.

We now remove this kernel as follows. Since $\Gamma$ is linear, it contains a normal torsionfree subgroup of finite index, and hence a homomorphism to a finite group which embeds all of the $\Gamma$ torsion.

By using the left regular representation of this finite group, we get a homomorphism $\pi : \Gamma \longrightarrow GL(V'')$ with finite image which embeds all of the $\Gamma$ torsion. Now the homomorphism

$$\xi = (\rho^* \times \pi) : A.H \longrightarrow GL(V' \times V'')$$

is faithful on the subgroup $H$ and restricting to the subgroup $\xi^{-1}(\xi(H))$ (which has finite index since it contains $\xi^{-1}(H \cap K)$ which is at least $A$) we obtain the desired retraction. $\square$

We note that linearity was really only used on the subgroup $H$ so we can argue similarly to obtain.

**Theorem 2.11** *Suppose that $\Gamma$ is a finitely generated group and $K$ a subgroup of $\Gamma$ of finite index. Then if $K$ has Property LR over its cyclic subgroups, then so does $\Gamma$.*

We can use this theorem as follows.

**Theorem 2.12** *Suppose that $A$ and $B$ are LR over cyclic groups. Then so is $A * B$.*

**Proof:** Given an element $\gamma$ in $A * B$, it can be uniquely expressed $a_1.b_1.a_2 \ldots a_k.b_k$ where all but possibly the first or last elements are nontrivial. If the element $\gamma$ lies in either $A$ or $B$, we project to the appropriate side and use the retraction coming from the hypothesis.

If not, using the fact that $A$ and $B$ are residually finite, we can map $A * B \to F_A * F_B$ where $F_A$ and $F_B$ are finite, so that none of the component elements of the image of $\gamma$ are trivial. In particular, the image of $\gamma$ has infinite order since it is in reduced form. Now free groups are LR over cyclic subgroups and the group $F_A * F_B$ is virtually free and so we can apply Theorem 2.11. $\square$

**Remark** It seems worthwhile pointing out that the situation actually improves upon taking free product: One needs only assume that $A$ and $B$ are residually finite and it follows from the proof that $A * B$ is $LR$ at least over the infinite cyclic groups which do not conjugate into a vertex group.

As remarked above it is well-known that $F_2 \times F_2$ is not LERF, however we have:

**Theorem 2.13** *Suppose that $A$ and $B$ are LR over cyclic subgroups. Then so is $A \times B$*

**Proof:** Let $(a, b) \in A \times B$ be an element of infinite order. Then both of $a$ and $b$ cannot have finite order, say that $b$ does not. Let $K_B$ be the retractor for $B$ over $< b >$ and let $A \times K_B \to < b >$ be the obvious map to $\mathbf{Z}$ exhibiting $A \times K_B$ as a retractor. $\square$

We have already mentioned the connection with the classical virtual Betti number problem. In fact we have more:

**Theorem 2.14** *Suppose that $M$ is a hyperbolic $n$-manifold for which $\pi_1(M)$ has LR over its cyclic subgroups. Then $M$ has infinite virtual Betti number.*

**Proof:** Suppose that the first Betti number of $M$ is $k$ and let $\gamma$ be an element lying in the kernel of the map $\pi_1(M) \to H_1(M)$. Let $q : \widetilde{M} \to M$ be a finite sheeted covering in which the lift of (some power of) $\gamma$ becomes an element of infinite order in $H_1(\widetilde{M})$.

By considering the transfer map, we see that with rational coefficients $H_1(\widetilde{M}) \cong H_1(M) \oplus ker(q_*)$ and the element $\gamma^r$ lies in $ker(q_*)$. It follows that $H_1(\widetilde{M})$ has rank at least $k + 1$. $\square$

**Remark.** Thus $LR$ proves infinite Betti number in a way which seems more natural than the traditional approach of finding a surjection to a nonabelian free group.

## 2.4 Retractions and geometry

Here we discuss the failure of Property LR for some classes of groups and subgroups. This seems interesting as the examples illustrate that Property LR "sees" geometric features that LERF/GFERF as yet cannot. Firstly, we remark that a group having Property LR is not implied by LERF. This follows from:

**Theorem 2.15** *Let* $\Gamma$ *be a hyperbolic* 3-*manifold group.*
   *Then there are no retractions onto virtual fibre groups.*

**Proof.** Let $S$ be a virtual fibre group and suppose that $V \to S$ is a retraction. Let $K$ be a subgroup of finite index in $\Gamma$ in which $S$ is a genuine fibre group. Then $K \cap V$ has finite index in $\Gamma$ and retracts to $S$ by restriction of the retraction from $V$. The group $S$ is Hopfian so the kernel of the retraction must miss $S$; however $S$ is normal in $K \cap V$ and two normal subgroups in a hyperbolic group cannot be disjoint. $\square$

Note that the fundamental group of the figure eight knot complement is LERF (see Theorem 3.4), and the methods of [3] shows that the fundamental group of the figure eight knot complement has a finite index subgroup that injects in an all right Coxeter group acting on $\mathbf{H}^6$. This, together with Theorem 2.6 and the fact that the property of LR over geometrically finite subgroups is preserved by super groups of finite index (cf. Theorem 2.10) shows that the fundamental group of the figure eight knot complement retracts to every geometrically finite subgroup.

## 2.5

We now turn our attention to lattices in rank 1 lie groups. Thus, let $X = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ or $\mathbf{O}$ be the real numbers, complex numbers, Hamiltonian quaternions or the Octonians respectively and $\mathbf{H}_X^n$ the hyperbolic space modelled on $X$. Then the full groups of isometries (holomorphic in the case of $\mathbf{C}$) are $PO_0(n, 1)$, $PU(n, 1)$, $PSp(n, 1)$ and $F_4^{-20}$ respectively. Denote these by $G(X)$. Apart from the cases of $\mathbf{H}_{\mathbf{R}}^n$, $n = 2, 3$, no example of a lattice in $G(X)$ is known to be LERF. Examples of lattices in $G(X)$ for $X = \mathbf{R}$ and $4 \leq n \leq 8$ are known to be GFERF (see [3] and [29]), but nothing is known beyond this. Given the fundamental difference between $PO_0(n, 1)$, $PU(n, 1)$ and $PSp(n, 1)$, $F_4^{-20}$, in that in the latter two, all lattices are superrigid and have Property T, and the former do not, one might expect lattices in $PSp(n, 1)$ and $F_4^{-20}$ to fail to be LERF or GFERF. The case of $PU(n, 1)$ seems unclear. However here we prove two theorems illustrating the failure of LR. The first is a straightforward consequence of the properties mentioned above.

**Theorem 2.16** *Let* $\Gamma$ *be a torsion-free lattice in* $PSp(n, 1)$, $n \geq 2$ *or* $F_4^{-20}$. *Then* $\Gamma$ *does not virtually retract to any non-trivial subgroup of infinite index.*

**Proof.** Let $\Delta$ be a finitely generated subgroup of $\Gamma$ of infinite index ($\neq 1$). If $\Gamma_\Delta$ is a retractor for $\Delta$ then we have an epimorphism $\theta : \Gamma_\Delta \to \Delta$. Now by assumption $\Delta$ is linear, and $\Gamma_\Delta$ is superrigid

by [14]. Since the homomorphism $\theta$ is not faithful, the image is therefore finite by superrigidity, hence a contradiction. □

Property T can also be used here to show that a group $\Gamma$ (as in Theorem 2.16) is not LR over a wide class of subgroups.

**Proposition 2.17** *Let $\Gamma$ be a finitely generated group with Property T. Then $\Gamma$ does not virtually retract to any infinite cyclic subgroup.*

**Proof:** This is clear, since a group with Property T cannot have any subgroup of finite index that admits a surjection onto $\mathbf{Z}$. □

Our second result distinguishes the behavior of retractions of complex hyperbolic from real hyperbolic examples.

**Theorem 2.18** *Let $\Gamma$ be a cocompact torsion-free lattice in $\mathrm{PU}(n,1)$. Let $\Delta < \Gamma$ be isomorphic to a lattice in $\mathrm{PO}_0(m,1)$ where $m \geq 3$. Then $\Gamma$ does not virtually retract to $\Delta$.*

**Proof:** Suppose to the contrary that $\Gamma$ virtually retracts to $\Delta$, that $\Gamma_\Delta$ is a retractor for $\Delta$ and $\theta : \Gamma_\Delta \to \Delta$ is the associated retraction homomorphism. Abusing notation we will identify $\Delta$ with its isomorphic image in $\mathrm{PO}_0(m,1)$.

Let $M = \mathbf{H}^n_{\mathbf{C}}/\Gamma_\Delta$ and $N = \mathbf{H}^m_{\mathbf{R}}/\Delta$. Note that since $M$ and $N$ are closed Riemannian manifolds of negative curvature, they are aspherical. Hence the homomorphism $\theta$ is induced by a continuous map, $f : M \to N$. By a fundamental result of Eels and Sampson [17], the homotopy class of the map $f$ contains a harmonic representative, which we continue to denote by $f$. Since $M$ is complex hyperbolic, it is Kahler, and so we can apply results of Carlson and Toledo [12] (see Corollary 3.7 and Theorems 7.1 and 7.2), the relevant parts we summarize as:

**Theorem 2.19** *In the notation above, then either $f(M)$ is a closed geodesic in $N$, or $f$ admits a factorization $f = h \circ g$ where $g : M \to S$ is a holomorphic map to a compact Riemann surface $S$, and $h : S \to N$ is a harmonic map.*

Applying this in our setting, notice the first possibility cannot occur since $\theta(\Gamma_\Delta) = \Delta \neq \mathbf{Z}$. To handle the second possibility we argue as follows. By definition the map $f$ induces $\theta$ and the second possibility affords a factorization $\theta = h_* \circ g_*$ (where $*$ denotes the induced homomorphism). Restricting $\theta$ to $\Delta$ gives the identity homomorphism by property LR, and so we deduce that on restricting to $\Delta$, $g_*$ is injective. However $g_*$ has image in a compact surface group, and $\Delta$ cannot inject in this as it is the fundamental group of a higher dimensional closed hyperbolic manifold. This contradiction completes the proof. □

**Remark:** In the case when $\Delta$ actually stablizes a totally geodesic copy of $\mathbf{H}^m_{\mathbf{R}}$, it is easy to arrange using standard arguments involving totally geodesic submanifolds (see [26]) that $\Gamma$ is $\Delta$-separable.

## 2.6

For convenience we include a collection of examples to which Theorem 2.18 applies. These are the *standard arithmetic lattices* acting on $\mathbf{H}^n_{\mathbf{C}}$ as described in [7].

Let $k \neq \mathbf{Q}$ be a totally real field with ring of integers $R_k$. Let $\alpha \in R_k$ be positive and all its galois conjugates negative. Let $K = k(i)$, $R_K$ the ring of integers of $K$ and let $V$ be a vector space over $K$ equipped with the Hermitian form $F_\alpha$ defined to be

$$z_0\overline{z_0} + z_1\overline{z_1} + \ldots + z_{n-1}\overline{z_{n-1}} - \alpha z_n\overline{z_n}, \qquad n \geq 2.$$

Note that this Hermitian form has signature $(n, 1)$ at the identity embedding of $k$ and signature $(n + 1, 0)$ at the other embeddings. As is shown in [7], the group $\Gamma_\alpha = \mathrm{PU}(F\alpha; R_K)$ defines a cocompact arithmetic lattice in $G(\mathbf{C})$.

Now the groups $\Gamma_\alpha$ contain as subgroups certain arithmetic groups arising from integral points of orthogonal groups. Namely, define the quadratic form $q_\alpha$ to be the form

$$x_0^2 + x_1^2 \ldots + x_{n-1}^2 - \alpha x_n^2, \qquad n \geq 2.$$

then it is easily seen that the group $\Delta_\alpha = \mathrm{PO}_0(q_\alpha; R_k)$ injects as subgroup of $\Gamma_\alpha$ acting cocompactly on a totally geodesic copy of $\mathbf{H}_\mathbf{R}^n$ inside $\mathbf{H}_\mathbf{C}^n$.

It is known that a Coxeter group cannot act as a lattice on $\mathbf{H}_\mathbf{C}^n$ for $n \geq 2$ (see [13] for a discussion). Theorem 2.18 provides the following corollary.

**Corollary 2.20** *Let $\Gamma < \mathrm{PU}(n, 1)$, $n \geq 3$ and suppose $\Gamma$ contains a subgroup $\Delta$ a lattice in $\mathrm{PO}_0(m, 1)$ stabilizing totally geodesic copy of $\mathbf{H}_\mathbf{R}^m$ with $m \geq 3$. Then $\Gamma$ does not inject as a quasi-convex subgroup of an all right Coxeter group.*

**Proof:** If $\Gamma$ admits such a injection in a Coxeter group $C$, then $\Delta$ will also be a quasi-convex subgroup of $C$. Hence Theorem 2.7 provides a retraction, and this is a contradiction to Theorem 2.18. □

## 2.7  Profinite groups.

As mentioned in §2, the statement that a group $G$ is $H$-separable is equivalent to $H$ being closed in the profinite topology on $G$. Thus in the context of Property $LR$, it is natural to ask whether there are obstructions to Property $LR$ which come from the topology of the profinite completion, since general considerations show that a surjective homomorphism $\theta : K \longrightarrow H$ gives rise to a continuous surjective homomorphism $\widehat{\theta} : \widehat{K} \longrightarrow \widehat{H}$ between the profinite completions. We offer two simple observations in this direction.

**Theorem 2.21** *Let $H$ be a closed subgroup of a profinite group $G$.*
  *Then the identity map $\mathrm{id} : H \to H$ extends to a continuous map $r : G \to H$.*

That is to say, the identity map on a closed subgroup always extends at least as a continuous map. Of course this has the immediate consequence:

**Corollary 2.22** *Let $H$ be a closed subgroup of a profinite group $G$.*
  *Then every continuous map $f : H \to X$ extends to a continuous map $F : G \to X$.*

The proof of 2.21 uses the following result:

**Theorem 2.23** *Let $G$ be a profinite group.*
  *Then there is a metric on $G$ which has the property that for every $g \in G$ and every non-negative real number $\alpha$, the set $\partial B_\alpha(g) = \{g \mid d(x, g) = \alpha\}$ is either empty or a single point*

**Proof:** Let $\{G/N_i\}_{i \geq 1}$ be a listing of the finite quotients which define the group $G$. Using the left regular representation, regard all these finite quotients as finite sets of $\{0, 1\}$ matrices. We will put a metric of the required type on the infinite product $\prod G/N_i$ and this will give a metric of the required type on $G$ by restriction.

For each $n \geq 1$ fix once and for all, a norm $\phi_n$ on the $n \times n$ matrices with the properties

- If $M$ and $N$ are matrices all of whose entries are 0 or $\pm 1$ and $\phi_n(M) = \phi_n(N)$, then $M = \pm N$

- $\phi_n(M)$ is approximately the number of $\pm 1$ entries in $M$ when this matrix has all entries $0$ or $\pm 1$.

- $\phi_n(M) \leq n^2$ for any matrix all of whose entries are $0$ or $\pm 1$.

For example, one can define such a norm by summing squares of the matrix entries, weighted by $n^2$ rationally independent transcendentals, each a little less than 1.

Let $\alpha_n$ be a sequence of positive reals which converges to 0 very rapidly, in the sense that it is monotonic decreasing and in addition that for every $t$, $\alpha_t >> \sum_{r>t} \alpha_r d(r)^2$ where $d(k)$ is the dimension in the $k$-th coordinate.

Define the distance between two sequences of matrices by

$$d((x_k), (y_k)) = \sum_k \alpha_k \phi_{d(k)}(x_k - y_k)$$

We claim that given $(x_k)$, there is at most one $(y_k)$ at any given distance from it, that is to say, given $(x_k)$ and a distance $d((x_k), (y_k))$, one can recover $(y_k)$.

The way one does this is the following. The magnitude of $d((x_k), (y_k))$ gives the first value $t$ for which $x_t \neq y_t$; in addition $d((x_k), (y_k))$ is then a mild perturbation of $\alpha_t \phi_{d(t)}(x_t - y_t)$, so that we can recover $y_t$. And so on.  $\square$

**Remark:** Notice that this metric gives the same topology as the Tychonoff topology on the product. One way to see this is that this metric and the more traditional metric have ball containments each way around for eg ball neighbourhoods of $g \in G$.

**Proof of 2.21.** As usual define $d(x, H)$ to be the infimum of the numbers $d(x, h)$ and by standard arguments this infimum is attained on $H$. By using the metric of the previous theorem, there is a *unique* point, denoted $r(x)$ (necessarily on $H$) for which $d(x, r(x)) = d(x, H)$.

We claim the map $G \to H$ defined by $x \to r(x)$ is continuous. The reason is this: Suppose that $x_n \to x$. By subconverging, we may suppose that the points $r(x_n)$ converge on $H$ to some point $a \in H$. The function $x \to d(x, H)$ is continuous, so $d(x_n, H) \to d(x, H)$ which is to say $d(x_n, r(x_n)) \to d(x, r(x))$. But we also have that $d(x_n, r(x_n)) \to d(x, a)$, so $d(x, a) = d(x, r(x))$. By uniqueness, $r(x) = a$ and $r$ is continuous, so that $r$ is the required extension.  $\square$

**Remark:** Given the remark following 2.23 above, we can use the same function $r$ and any metric on $G$ (in particular the usual Tychonoff metric or a group invariant metric) and see that $r$ is still continuous.

The above result shows that there is never any topological obstruction for retracting from the whole profinite group. The proof operated by constructing a special metric for which the map was given by the nearest point retraction. Our next result shows that this is the general situation: If genuine retractions exist, they are the nearest point retraction for special metrics on the profinite completion.

Before proving this, we make some preliminary remarks. Note that certain profinite groups have the property that *all* their finite index subgroups are open. This class includes finitely generated pro-p groups ([16] Theorem 1.17), and profinite completions of residually finite groups. In fact one can apparently conjecture all finitely generated profinite groups have this property - see [16] p. 32. For such groups we can show

**Theorem 2.24** *Suppose that $G$ is a profinite group which has the property that every subgroup of finite index in $G$ is open. Let $H$ be a closed subgroup of $G$ and let $r : V \to H$ be a retraction from some finite index subgroup $V$ of $G$.*

*Then the topology on $V$ is given by a left invariant metric $\rho$ with the property that $ker(r)$ is precisely the set of points $r^{-1}(I)$.*

**Proof:** The subgroup $\ker(r)$ is a closed subgroup of a profinite group, and therefore profinite; in particular it's compact. Choose a left invariant metric $\rho'_{\ker(r)}$ on it. Similarly choose a left invariant metric on $H$, denote this by $\rho_H$.

The group $H$ acts on $\ker(r)$ by conjugacy and we define a new metric by

$$\rho_{\ker(r)}(a,b) = \int_H \rho'_{\ker(r)}(hah^{-1}, hbh^{-1})d\mu$$

where $\mu$ is Haar measure on $\ker(r)$ and $a, b \in \ker(r)$. This new metric is visibly $H$-conjugacy invariant and it continues to be $\ker(r)$ left invariant, since for every $\lambda \in \ker(r)$

$$\rho_{\ker(r)}(\lambda a, \lambda b) = \int_H \rho'_{\ker(r)}(h\lambda a h^{-1}, h\lambda b h^{-1})d\mu = \int_H \rho'_{\ker(r)}(h\lambda h^{-1}hah^{-1}, h\lambda h^{-1}hbh^{-1})d\mu$$

$$= \int_H \rho'_{\ker(r)}(hah^{-1}, hbh^{-1})d\mu = \rho_{\ker(r)}(a,b)$$

Given two points of $G$, they can be uniquely written $\lambda_1 h_1$ and $\lambda_2 h_2$ and we define

$$\rho(\lambda_1 h_1, \lambda_2 h_2) = \rho_{\ker(r)}(\lambda_1, \lambda_2) + \rho_H(h_1, h_2)$$

One may check that this is a metric and in fact (using the fact that $\rho_{\ker(r)}$ is $H$ invariant) it is left invariant.

We claim that $\rho$ induces the correct topology on $V$. We argue as follows.

Let $X$ be any $\rho$-open neighbourhood of the identity in $V$. Pick a $g \in X$ and notice that by left invariance, $g^{-1}X$ is also $\rho$-open in $V$ so that by definition, there is an $\epsilon > 0$ for which $B_\epsilon(I) \subset g^{-1}X$.

Since the topology on $\ker(r)$ came from $\rho_{\ker(r)}$, the $\epsilon/2$ $\rho_{\ker(r)}$-ball neighbourhood of the identity inside $\ker(r)$ contains an open subgroup, K, of finite index in $\ker(r)$ which by shrinking if necessary, we may suppose is normalised by $H$. Similarly, the $\epsilon/2$ $\rho_H$-ball neighbourhood of the identity contains a subgroup, $L$ of finite index in $H$.

Now one may check that the subgroup $K \cdot L$ has finite index in $V$ and is therefore open in the profinite topology, since by the hypothesis on $G$, every subgroup of finite index in $V$ is open in the topology coming from $G$. Its points are all of the form $k \cdot \ell$ whose $\rho$-distance from the identity is $\rho(k \cdot \ell, I) = \rho_{\ker(r)}(1, k) + \rho_H(1, \ell) < \epsilon$. So, the open subgroup $K \cdot L$ is a subset of $B_\epsilon(I) \subset g^{-1}X$, so that $g \in g \cdot (K \cdot L) \subset X$, which is to say, every $\rho$-open set is a union of $(V, profinite)$ open sets and therefore is also open in the profinite topology on $V$.

Now we finish off as usual, the above considerations mean that the identity map induces a continuous map $(V, profinite) \to (V, \rho)$, the domain is compact the range is Hausdorff, so the map is a homeomorphism and $\rho$ is giving the correct topology.

Finally, notice that given any point $\lambda \in \ker(r)$, and any $h \in H$, $\rho(\lambda, h) = \rho_{\ker(r)}(\lambda, 1) + \rho_H(1, h)$ which is clearly minimised by taking $h = I$.

Moreover, if $\lambda_1 h_1$ is any point of $V$, then $\rho(\lambda_1 h_1, h) = \rho_{\ker(r)}(I, \lambda_1) + \rho_H(h_1, h)$ is minimised by taking $h = h_1$, so that $r^{-1}(I)$ is precisely $\ker(r)$. □

**Remark:** It is not in general possible to make the metric $\rho$ bi-invariant.

# 3  Application of extensions

In this section we record an application of extensions in the context of 3-manifold topology.

## 3.1

Recall that if $\Gamma$ is a finitely generated group, the **rank** of $\Gamma$ is the cardinality of a minimal generating set for $\Gamma$. By a **compression body group** we will mean a free product $A * B$ where $A$ is free of some rank $n \geq 0$ and $B$ is a free product of surface groups (i.e. the fundamental groups of closed surfaces). Note this is somewhat non-standard, in that we allow non-orientable closed surfaces. However this will be convenient in what follows.

**Proposition 3.1** *Let $M = \mathbf{H}^3/\Gamma$ be a non-compact orientable finite volume hyperbolic 3-manifold and assume that $\Gamma$ is LERF. Let $H$ and $K$ be finitely generated subgroups of $\Gamma$ with $H < K < \Gamma$ and where both $[\Gamma : K]$ and $[K : H]$ are infinite. Then there is a subgroup of finite index $K_0$ in $K$, and a finite group $G$ with a surjective homomorphism $\theta : K_0 \to G$ such that:*

- *$\theta$ admits an extension $\hat{\theta} : \Gamma_0 \to G$ with $\hat{\theta}(H) = 1$.*

- *If the rank of $H$ is $k$, then $\mathbf{H}^3/\ker\hat{\theta}$ has at least $k + 1$ cusps.*

**Proof.** The group $K$ has the structure of a (possibly trivial) free product $A * B$ where $A$ is a compression body group and $B = \pi_1(W_1) * \ldots * \pi_1(W_s)$, with each $W_i$ an orientable 3-manifold with $\pi_1(W_i)$ freely indecomposable. Since $M$ is orientable, $\Gamma$ contains no Klein bottle group. Hence the free product decomposition of $K$ cannot contain any Klein bottle group. We now break the proof up into a discussion of various cases depending on the above free product decomposition. We begin with the simplest cases.

$K$ **is $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$:** Recalling that $H$ is required to have infinite index, we see that the former case implies $H$ is trivial, the latter that $H$ is trivial or rank one.

The result holds easily for $H$ the trivial group (rank 0), since we can choose the trivial homomorphism of $K$ which extends over the whole group $\Gamma$. $M$ (and all its coverings) have at least one cusp and this completes the proof in this case.

If $K = \mathbf{Z} \oplus \mathbf{Z}$, and $H$ is infinite cyclic, then the group $K$ is conjugate into a maximal peripheral subgroup of $\Gamma$. By peripheral separability [28], there is a finite index subgroup $\Gamma_0$ of $\Gamma$ containing $K$, so that $\mathbf{H}^3/\Gamma_0$ has at least two cusps. In this case, we may again take the homomorphism of $K$ on to the trivial group and the trivial extension to $\Gamma_0$ to prove the proposition.

$K$ **is free of rank at least two:** Hence $H$ is a free subgroup of infinite index. Since free groups are LERF, we can find infinitely many finite index subgroups of $K$ containing $H$. Furthermore, we can arrange that $H$ is a free factor. Hence we can arrange that $H$ is a free factor of a subgroup $K_0 = H * Q$ of finite index in $K$, with $Q$ free of rank $N$ (chosen very much larger than 2). Since $Q$ surjects any group which is $N$ generator, applying Theorem 1.2, we have, for any finite group $G$ which is N-generator, a homomorphism $\theta : K_0 \to G$ with $\theta(H) = 1$ which may be extended to a finite index subgroup of $\Gamma$. This proves the first part of the proposition in this case. To complete the proof, we now specify a group $G$ so that the second part of Proposition 3.1 will also hold.

Fix $p > k$ a prime and choose $G = (\mathbf{Z}/p\mathbf{Z})^N$. As above, we have an extension of $\theta$ to a homomorphism $\hat{\theta}$ on a finite index subgroup $\Gamma_0$ of $\Gamma$ with $\hat{\theta}(H) = 1$. We claim that the cover corresponding to $\ker\hat{\theta}$ does have at least $k + 1$ cusps. To see this consider the preimage of a cusp $C$ in the cover corresponding to $\ker\hat{\theta}$. Let $P$ be the maximal peripheral subgroup of $\Gamma_0$ associated to $C$. Now $\hat{\theta}(P)$ is at worst $\mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$, which, by choice of $N$ (being very much larger than 2) has index at least $p$ in $G$. Now covering space theory shows that the number of preimages of the cusp in the cover corresponding to $\ker\hat{\theta}$ is $|G|/|\hat{\theta}(P)|$, and so this is at least $p$. By choice of $p$ this is at least $k + 1$ and so the result follows in this case.

$K$ **is a (non-free) compression body group:** Thus $K = A * B$ where $A$ is a free group and $B$

is a free product of non-trivial closed surface groups, all different from those of $\mathbf{RP}^2$ and the Klein bottle. The above considerations also show that we may suppose that $K$ does not consist of one copy of $\mathbf{Z} \oplus \mathbf{Z}$.

Now since $K$ is a subgroup of a LERF group, $K$ is also LERF (in fact since free products preserve LERF any compression body group is LERF). Since $H$ is assumed to have infinite index in $K$, we can find subgroups of arbitrarily large finite index in $K$ containing $H$. By the Kurosh subgroup theorem, these groups will also be compression body groups, and so every finite index subgroup of $K$ is of the form $F_n * Z$ where $F_n$ is a free group of some rank $n$, and $Z$ is a free product of a finite number of surface groups. To complete the proof in this case we use the following.

**Lemma 3.2** *Given an infinite tower of finite index subgroups of $F_n * Z$ as above, then the first Betti number goes to infinity in this tower.*

**Proof.** If the rank of the free part goes to infinity then this is clear. Thus assume the rank of the free part remains bounded and let $Z = \Sigma_{g_1} * \ldots * \Sigma_{g_t}$ where $\Sigma_{g_j}$ is a closed surface group of genus $g_j$ (orientable or non-orientable). By the Kurosh subgroup theorem, a subgroup of finite index in $F_n * Z$ splits as a free product of a free group and conjugates of subgroups of the surface subgroups. Since we are assuming the rank of the free part is bounded, we can therefore assume that the subgroups arising from conjugates of subgroups of $Z$ must be finite index subgroups of the surface groups, and so themselves are surface groups.

Now in the cases under consideration, $\chi(F_n * Z)$ is nonzero and since Euler characteristic is multiplicative in finite covers, a subgroup of index $N$ has Euler characteristic $N\chi(F_n * Z)$ and one sees easily that if the rank of the free part remains bounded while $N \to \infty$, then either the number of surface components or the genus (orientable or non-orientable) of one of these components must increase without bound. Either of these case proves the lemma. $\square$

With this lemma in hand, the proof of the proposition in this case is completed in a manner similar to the previous case. As before, let $p > k$ be a prime, choose $N$ very large, say with $N - k > 2$, and let $G = (\mathbf{Z}/p\mathbf{Z})^N$. Now from, Lemma 3.2 we can find a finite index subgroup $K_0$ of $K$ which contains $H$ and a homomorphism $\theta : K_0 \to G$. Note in this case $\theta(H)$ need not be trivial, but $\theta(H)$ is at worst $(\mathbf{Z}/p\mathbf{Z})^k$. Hence $G/\theta(H) \cong (\mathbf{Z}/p\mathbf{Z})^{N-k} \neq 1$ since $N - k > 2$. Indeed, by construction, $(\mathbf{Z}/p\mathbf{Z})^{N-k}$ contains a copy of $(\mathbf{Z}/p\mathbf{Z})^3$. Relabeling we can define a homomorphism $\theta : K_0 \to (\mathbf{Z}/p\mathbf{Z})^{N-k}$ in which $\theta(H) = 1$. By Theorem 1.2 we can find a finite index subgroup $\Gamma_0$, an extension of $\theta$ to $\Gamma_0$, and as in the previous case we consider the preimage of a cusp $C$ in the cover corresponding to $\ker \hat{\theta}$. The arguement is finished exactly as before.

$K = \pi_1(W)$ **is freely indecomposable:** Since $K < \Gamma$, then $K$ is LERF, and again, $H$ is contained in subgroups of $K$ of arbitrarily large finite index. The result will follow in this case on showing that the first betti number in such a tower also goes to infinity. Note that $\partial W \neq \emptyset$ and cannot consist only of tori. For if so, then the interior of $W$ will admit a complete hyperbolic structure of finite volume by Thurston's uniformization theorem for Haken manifolds. This implies $[\Gamma : K] < \infty$ which contradicts our assumption on $K$ (note also $\pi_1(W) \cong \mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$ has been dealt with by our previous considerations). Hence $\partial W$ has some component of genus at least 2. Now, a standard argument in 3-manifold topology shows the first Betti number of $W$ is at least $\frac{1}{2}\dim(H_1(\partial W))$. We claim that this must go to infinity in a tower of finite covers. This is clear if the number of boundary components goes to infinity (by the above remark). If the number of boundary components stays bounded, for some component of the boundary the genus must go to infinity (recall there is a component of genus at least 2). Hence this proves the claim. The argument is now completed exactly as in the previous cases.

The proof for the general case, that is, when $K = A * B$ where $A$ is a compression body group and

$B = \pi_1(W_1) * \ldots * \pi_1(W_s)$, with each $W_i$ an orientable 3-manifold with $\pi_1(W_i)$ freely indecomposable can be handled using a combination of the above methods. $\square$

## 3.2

Proposition 3.1 has the following conseqeunce.

**Theorem 3.3** *Let $M = \mathbf{H}^3/\Gamma$ be a non-compact orientable finite volume hyperbolic 3-manifold and $H$ a non-elementary subgroup of $\Gamma$ of infinite index generated by two parabolic elements. If $\Gamma$ is LERF, then there is a finite index subgroup $\Gamma_H < \Gamma$, and a homomorphism $\phi : \Gamma_H \to \mathbf{Z}$ for which $H$ is contained in $\ker \phi$.*

Before deducing Theorem 3.3 we note that given the hypothesis of Theorem 3.3, work of Canary (see [11], Proposition 8.4) shows that the cover corresponding to $H$ is tame. Tameness has recently been established by Agol [2] and independently Calegari and Gabai [10] without any hypothesis, but this shows some of the power of LERF and extensions. On the otherhand given the solution to tameness we have from [3] (see the discussion in §2.2).

**Theorem 3.4** *The Bianchi groups* $\mathrm{PSL}(2, O_d)$ *are LERF.* $\square$

Hence we can apply Theorem 3.3 to deduce:

**Corollary 3.5** *Let $M = \mathbf{H}^3/\Gamma$ be a non-compact orientable arithmetic hyperbolic 3-manifold and $H$ a non-elementary subgroup of $\Gamma$ of infinite index generated by two parabolic elements. Then there is a finite index subgroup $\Gamma_H < \Gamma$, a homomorphism $\phi : \Gamma_H \to \mathbf{Z}$ for which $H$ is contained in $\ker \phi$.* $\square$

**Proof of Theorem 3.3**

We shall assume from here on that $H = \langle x, y \rangle$ is generated by 2 parabolic elements $x$ and $y$. A theorem of Jaco and Shalen [25] says that any 2-generator subgroup of $\Gamma$ is either finite index, free or $\mathbf{Z} \oplus \mathbf{Z}$. Hence by assumption, $H$ is forced to be free of rank 2.

**Notation:** Let $\Gamma$ be a finite covolume Kleinian group and $t \in \Gamma$ a parabolic element. Denote by $P_t$ the maximal peripheral subgroup of $\Gamma$ containing $t$.

We may as well assume that $M$ has first Betti number at most 2 otherwise we will be done. For if $M$ is a compact orientable 3-manifold with incompressible boundary consisting of a disjoint union of $m$ tori, then the first Betti number is at least $m$. In which case, since $H$ has rank 2 there is a homomorphism $\Gamma \to \Gamma^{\mathrm{ab}} \to \mathbf{Z}$ that kills $H$.

Let $\mu \in \Gamma \setminus H$ be a parabolic element. Note that such elements are easily seen to exist, for since $H$ is free, for every parabolic element $t \in \Gamma$, there exists at least an infinite cyclic subgroup of $P_t$ which is not contained in $H$. In particular there are (primitive) parabolic elements of $\Gamma$ no power of which lie in $H$. Denote by $\Gamma_\mu$ the subgroup of $\Gamma$ generated by $\{x, y, \mu\}$.

Now observe that if there is a $\mu \in \Gamma \setminus H$ such that both $[\Gamma : \Gamma_\mu]$ and $[\Gamma_\mu : H]$ are infinite, then Proposition 3.1 immediately implies the result in this case—we get a cover with at least 3 cusps.

**Claim:** *If $\mu \in \Gamma \setminus H$ then $[\Gamma_\mu : H]$ is infinite.*

**Proof:** To see this, we argue as follows. If $\Gamma_\mu$ has finite index in $\Gamma$ this is clear since $H$ is free. Thus suppose that $\Gamma_\mu$ has infinite index in $\Gamma$ and $H$ has finite index in $\Gamma_\mu$. Now $H$ is free and so $\Gamma_\mu$ is a virtually free, torsion-free group, and hence it is free. Now $H$ is free of rank 2 and so Euler

16

characteristic forces the index $[\Gamma_\mu : H] = 1$. But $H = \Gamma_\mu$ contradicts $\mu \notin H$. $\square$

From the remarks above, we can assume that $[\Gamma : \Gamma_\mu]$ has finite index for all $\mu \notin H$. The remainder of the proof deals with this case.

## 3.3

Here we record some results that will be needed to complete the proof of Theorem 3.3. The first being from [18] and [32]. We include a proof for convenience.

**Lemma 3.6** *Let $M_1 = \mathbf{H}^3/\Gamma_1$ be 1-cusped finite volume hyperbolic 3-manifold, $M_2 = \mathbf{H}^3/\Gamma_1$ a 1-cusped finite volume hyperbolic 3-orbifold with a torus cusp, and $\Gamma_1 < \Gamma_2$. If $\Gamma_1$ is generated by parabolic elements, then the cover $M_1 \to M_2$ is regular. In the case when $M_2$ is a manifold, the cover is cyclic.*

**Proof.** Let $P$ be a maximal peripheral subgroup of $\Gamma_2$. Since $M_2$ has a torus cusp, then $P$ is abelian. Since both $M_1$ and $M_2$ have a single cusp, $\Gamma_2 = P.\Gamma_1$; that is every element in $\Gamma_2$ can be expressed as a product $p \cdot \gamma$ for some $p \in P$ and $\gamma \in \Gamma_1$. Let $g = p \cdot \gamma$ be such an element, and $x \in P \cap \Gamma_1$. Then since $P$ is abelian, notice that

$$g^{-1}xg = \gamma^{-1}p^{-1}xp\gamma = \gamma^{-1}x\gamma.$$

This latter element lies in $\Gamma_1$. Hence the normal closure of $P \cap \Gamma_1$ in $\Gamma_2$ is contained in $\Gamma_1$. Now $\Gamma_1$ is generated by parabolic elements, and so normally generated by $P \cap \Gamma_1$. It therefore follows that normal closure of $P \cap \Gamma_1$ in $\Gamma_2$ coincides with $\Gamma_1$. Hence $\Gamma_1$ is a normal subgroup of $\Gamma_2$ as required. In the case when $M_2$ is a manifold, it can be shown that $H_1(M_2; \mathbf{Z}) \cong \mathbf{Z}$, and that the covering group is isomorphic to $P/(P \cap \Gamma_1)$ which is therefore cyclic. $\square$

**Lemma 3.7** *Let $M = \mathbf{H}^3/\Gamma$ be a finite volume hyperbolic 3-manifold. Assume that $\Gamma$ is LERF and that $H$ is a finitely generated subgroup of infinite index in $\Gamma$ with the property that every subgroup $K$ of finite index in $\Gamma$ with $H < K$ is normal in $\Gamma$. Then $H$ is the fundamental group of a compact 2-manifold $S$ and $M$ is virtually fibered over the circle with fiber $S$.*

**Proof.** Since $\Gamma$ is LERF, we have from above that $H = \bigcap K$ where $K$ ranges over all subgroups of finite index containing $H$. Since each $K$ is normal in $\Gamma$ it follows that $H$ is normal in $\Gamma$, and by hypothesis $\Gamma/H$ is infinite. It follows from [24] Theorem 11.1 that $H$ is the fundamental group of a compact 2-manifold. The virtual fibration statement also follows from Theorem 11.1 of [24]. $\square$

A useful corollary of this in our setting is the following.

**Corollary 3.8** *Let $M$ and $H$ be as above, and assume further that $H$ is a free subgroup generated by two parabolic elements. Then there is a non-normal subgroup of finite index in $\Gamma$ containing $H$.*

**Proof.** If not, then by Lemma 3.7, $H$ is the fundamental group of a compact 2-manifold $S$ and $M$ is virtually fibered over the circle with fiber $S$. However, this is a contradiction. For since $H$ is generated by two parabolics, it follows that the compact 2-manifold $S$ is a twice-punctured disk. This has a unique hyperbolic structure, in particular it is aways totally geodesic. Therefore the group of such a manifold cannot arise as a virtual fiber group. $\square$

## 3.4

Returning to the proof of Theorem 3.3, we are assuming that $\Gamma_\mu$ has finite index in $\Gamma$.

Let $N_\mu = \mathbf{H}^3/\Gamma_\mu$ denote the cover of $M$ corresponding to $\Gamma_\mu$. Recall that we are supposing that $M$ has at most two cusps. We will first reduce the 1-cusped case to the 2-cusped case. To this end, suppose that $M$ has a single cusp.

**Claim 1:** *There is a 2-cusped cover $N$ of $M$ such that $H < \pi_1(N)$.*

**Proof of Claim 1:** Suppose that every cover $N$ of $M$ with $H < \pi_1(N)$ has 1 cusp. Since $\Gamma$ is LERF there are subgroups of arbitrarily large finite index containing $H$, and furthermore, that every subgroup containing $H$ contains a subgroup of the form $\Gamma_\mu$ for some $\mu$ in $\Gamma$. Also, notice that, since each $\Gamma_\mu$ is generated by parabolics, Lemma 3.6 shows that each of the covers $N_\mu \to M$ is a finite cyclic cover. Since every subgroup of finite index containing $H$ contains some $\Gamma_\mu$, it follows that all subgroups of finite index in $\Gamma$ containing $H$ are normal subgroups of $\Gamma$. However, this contradicts Corollary 3.8. Hence there exists a 2-cusped cover as required. $\square$

Thus we may as well now assume that $M$ is 2-cusped and establish the lemma. We can also assume that the groups $\Gamma_\mu$ are all 2-cusped, or again we are done. Since $\Gamma$ is LERF we can pass to a subgroup $\Gamma_0$ of arbitrarily large finite index (index $> 8$ will suffice for our purposes) which contains $H$. Denote by $M_0 = \mathbf{H}^3/\Gamma_0$, and let $\mu$ be a primitive parabolic element in $\Gamma_0 \setminus H$. Note that we can arrange that $\mu$ has the following properties:

(a) $\mu$ has length at least $2\pi$ in the Euclidean structure on the associated cusp torus, denoted by $T_\mu$. Call such a parabolic **long**.

(b) $\mu$-filling on $T_\mu$ produces an irreducible 3-manifold with incompressible torus boundary.

To see this, consider a maximal peripheral subgroup $P = < \alpha, \lambda > \subset \Gamma_0$. If $P \cap H = 1$, then it is clear we can find infinitely many long parabolics. Furthermore, by Theorem 1.3 of [20] (see also Theorem 3.1 of [21]), the number of slopes that produce manifolds that are reducible or in which the boundary compresses is at most 8. This allows us to choose the required $\mu$ in this case.

If $P \cap H$ is non-trivial, it is infinite cyclic, say $P \cap H = < t >$. Now $t = \alpha^p \lambda^q$ for integers $p$ and $q$ (not necessarily coprime). If either of $p$ or $q$ is zero, then it is clear how to find such an element. For example, if $q = 0$, then any element $\alpha^p \lambda^k$ for $(k, p) = 1$ will be primitive and eventually long. Such an element cannot lie in $H$. If neither $p$ or $q$ are zero, we can find infinitely many integers $k$ such that $(k + p, q) = 1$ and so $\alpha^k t$ is primitive and will be long for large $k$. As above, such an element cannot lie in $H$.

Note this provides infinitely many long $\mu$. Moreover, as above, an application of [20] allows us to choose a $k$ which produces an irreducible manifold and in which the boundary will remain incompressible. This completes the proof of $(b)$.

We can also assume that $P_\mu$, and $P_x$ (say) are not $\Gamma_0$-conjugate. For since there are two $\Gamma_0$-conjugacy classes of maximal peripheral subgroup, if $P_x$ and $P_y$ are $\Gamma_0$-conjugate we choose some $\mu$ from the other conjugacy class. If $P_x$ and $P_y$ are not $\Gamma_0$-conjugate we choose $\mu$ in the conjugacy class of $P_y$ and not in $P_x$.

**Claim 2:** *$P_x$ and $P_y$ cannot be $\Gamma_\mu$-conjugate.*

**Proof of Claim 2:** Assume to the contrary that $P_x$ and $P_y$ are $\Gamma_\mu$-conjugate. By choice of $\mu$, since there are two cusps, $P_\mu$ cannot be $\Gamma_\mu$-conjugate to $P_x$. Perform $\mu$-filling on the associated cusp

torus of $P_\mu$. By (b) above, this results in an irreducible 3-manifold with a single torus boundary component, which is incompressible. By (a), since $\mu$ is long, the resultant manifold will carry a metric of negative curvature ([22]). Let the manifold resulting from $\mu$-filling on $N_\mu$ be denoted by $N'_\mu$. It follows that $N'_\mu$ is a finite volume hyperbolic 3-manifold with a single cusp. Moreover, $x$ and $y$ will remain parabolic after filling, and so since $\mu$ has been killed, the fundamental group of $N'_\mu$ is generated by two parabolic elements. Appealing to a result of Adams [1] implies that $N'_\mu$ is homeomorphic to the complement of a hyperbolic 2-bridge knot in $S^3$. Now we can perform a $\mu$ orbifold filling on $M$, resulting in a 1-cusped orbifold $M'$ with a torus cusp. We obtain an orbifold covering $N'_\mu \to M'$. By Lemma 3.6, this covering is regular. In particular the covering group is a subgroup of the group of isometries of $N'_\mu$, a hyperbolic 2-bridge knot complement. However this has order at most 8 (see [19] for example). But the choice of covering degree was made very large and hence we have a contradiction.

Thus we can now assume that $P_x$ and $P_y$ are not $\Gamma_\mu$-conjugate. To complete the proof in this case we argue as follows.

By assumption, $P_\mu$ cannot be $\Gamma_\mu$-conjugate to $P_x$, it is therefore $\Gamma_\mu$-conjugate to $P_y$. Hence the cusps are represented by the peripheral subgroups $P_x$ and $P_y$. An application of LERF, together with another application of the observation above that every subgroup of finite index contains some $\Gamma_\nu$ allows us to pass to a finite cover and to arrange that $x$ is a primitive parabolic in some $\Gamma_{\mu_1} < \Gamma_\mu$.

Now perform $x$-filling on $N_{\mu_1}$. If the result is hyperbolic, we can argue as above to obtain a contradiction—for the covering degree of $N_{\mu_1}$ over the result of $x$-filling on $M$ is still large. Thus assume that the result of $x$-filling on $N_{\mu_1}$, denoted by $N'_{\mu_1}$, is non-hyperbolic. In this case, $y$ and $\mu_1$ remain peripheral. If the boundary torus is incompressible, then [5] Corollary 6 implies that $N'_{\mu_1}$ is a 2-bridge torus knot. We will deal with this case below and assume first that the boundary torus of $N'_{\mu_1}$ is compressible.

**Claim 3:** $\pi_1(N'_{\mu_1})$ is infinite cyclic.

**Proof of Claim 3:** $N'_{\mu_1}$ has a single torus boundary component which, by hypothesis, is compressible. Performing this compression results in a 2-sphere, if this 2-sphere bounds a homotopy ball we are done. If not, the 2-sphere splits the manifold into a connect sum $V \# W$ where $V$ is a solid torus and $W$ a compact 3-manifold. This yields a free product decomposition $\pi_1(N'_{\mu_1}) = \mathbf{Z} * B$ for some compact 3-manifold group $B$. Now $\pi_1(N'_{\mu_1})$ is generated by (the images of) $y$ and $\mu_1$, and so has rank at most 2. Hence we are done unless $B$ is a nontrivial cyclic group.

Observe that the homology of $N'_{\mu_1}$ is noncyclic when computed with appropriate finite field coefficients. In addition, since $y$ and $\mu_1$ are peripheral in $N_{\mu_1}$, they are peripheral in the surgered manifold, so that by a standard Poincaré duality argument, the subgroup they generate in the homology with finite field coefficients is at most a cyclic group. However, $y$ and $\mu_1$ generate the whole fundamental group of $N'_{\mu_1}$, a contradiction. Thus $N'_{\mu_1}$ is a homotopy solid torus as required by the claim.

We now complete the proof in the case that the boundary torus of $N'_{\mu_1}$ is compressible. By Corollary 3.8, $\Gamma_{\mu_1}$ has a non-normal subgroup $\Lambda$ of finite index containing $H$. Let $\nu$ be a parabolic element in $\Lambda$ which commutes with $\mu_1$. In particular $P_\nu$ is not $\Lambda$-equivalent to $P_x$. Now $\Lambda$ contains the group $\Gamma_\nu$ with finite index.

**Claim 4:** $N_\nu = \mathbf{H}^3/\Gamma_\nu$ *has at least 3 cusps, and so the conclusion of Theorem 3.3 holds.*

**Proof of Claim 4:** We can assume that $N_\nu$ has 2 cusps. Performing $x$-filling on $N_\nu$ determines a cover of $N'_{\mu_1}$. Since $\pi_1(N'_{\mu_1})$ is infinite cyclic, it follows that both have infinite cyclic fundamental group. We therefore deduce that $N_\nu \to N_{\mu_1}$ is a finite cyclic cover. Hence $N_\nu \to \mathbf{H}^3/\Lambda \to N_{\mu_1}$ is a cyclic cover, so that $\Lambda$ is a normal subgroup of finite index. This contradicts our choice of $\Lambda$. The claim follows. $\square$

**Claim 5:** *In the case when $N'_{\mu_1}$ is a 2-bridge torus knot, the conclusion of Theorem 3.3 holds.*

**Proof of Claim 5:** Note that $N'_{\mu_1}$, being a 2-bridge torus knot complement, it is a Seifert fibered space over the disk with 2 exceptional fibers of order 2 and $q$ with $q$ odd. Denoting the free product of the cyclic group of order 2 with a cyclic group of order $q$ by $\mathcal{F}_q$, we have epimorphisms:

$$\Gamma_{\mu_1} \to \pi_1(N'_{\mu_1}) \to \mathcal{F}_q.$$

Note that since $\mu_1$ and $y$ are peripheral, under the latter homomorphism the images remain peripheral; that is conjugate into the subgroup generated by the homotopy class of the boundary of the disk.

From [33], we can find infinitely primes $p$ and surjections $f_p : \mathcal{F}_q \to \mathrm{PSL}(2, \mathbf{F}_p)$ with $f_p(\mu_1)$ and $f_p(y)$ being elements of order $p$. Composing homomorphisms, we obtain for these $p$ a homomorphism $\Phi_p : \Gamma_{\mu_1} \to \mathrm{PSL}(2, \mathbf{F}_p)$ with $\Phi_p(x) = 1$, $\Phi_p(y)$ and $\Phi_p(\mu_1)$ of order $p$. Let $H_p = \Phi_p(H) = \Phi_p(y)$, and $M_p$ the cover of $N_{\mu_1}$ corresponding to $H_p$. By definition, $H < \pi_1(M_p)$ and we aim to show that $\pi_1(M_p)$ has first betti number at least 3. To do this we make use of following standard facts about the groups $\mathrm{PSL}(2, \mathbf{F}_p)$, where $p$ is prime and $\mathbf{F}_p$ denotes the field with $p$ elements.(see [35] Chapter 3, §6).

(i) For $p$ odd, $|\mathrm{PSL}(2, \mathbf{F}_p)| = \frac{p(p^2-1)}{2}$.
(ii) For all $p$ every abelian subgroup $A$ of $\mathrm{PSL}(2, \mathbf{F}_p)$ of odd order is cyclic.

Choose $p$ a very large odd prime, and consider the cover of $N'_{\mu_1}$ corresponding to $H_p$. By construction of the homomorphism $f_p$ and (ii) above, the image of the peripheral subgroup of $\pi_1(N'_{\mu_1})$ coincides with $H_p$, a cyclic group of order $p$. By (i) above, $H_p$ has index $\frac{(p^2-1)}{2}$, and so standard covering space theory implies that the cover of $N'_{\mu_1}$ corresponding to $H_p$ has $\frac{(p^2-1)}{2}$ cusps. Hence for large $p$ this is very much larger than 3. It now follows that $M_p$ has large first Betti number as required. $\square$

# 4  Final Comments and Questions

**1.** The hypothesis of Theorem 2.6 is a powerful technique for proving that a finitely generated group $\Gamma$ is GFERF. However, Theorem 2.6 shows that this technique actually shows much more. For example Theorem 2.6 shows that a group $\Gamma$ as in Theorem 2.6 has a finite index subgroup surjecting $\mathbf{Z}$. Of course we are simply using the consequence that $\Gamma$ is LR over its $\mathbf{Z}$ subgroups. For 3-manifolds, this implies the existence of a surface group. With these comments we pose.

**Question 4.1** *Suppose $\Gamma$ is an infinite word hyperbolic group which is LERF. Does $\Gamma$ contain any finite index subgroup surjecting onto $\mathbf{Z}$? Is it in fact LR over its cyclic groups?*

Considering only groups of hyperbolic isometries.

**Question 4.2** *Suppose $\Gamma < Isom(\mathbf{H}^n)$ is of finite covolume and LERF or GFERF. Does $\Gamma$ contain a finite index subgroup surjecting onto $\mathbf{Z}$? Or LR over its cyclic groups?*

**Question 4.3** *If the fundamental group of a closed hyperbolic 3-manifold is LERF or GFERF does the manifold contain an (immersed) essential surface?*

**2.** There is a notion of geometrically finite in the setting of discrete subgroups of $\mathrm{PU}(n,1)$, $\mathrm{PSp}(n,1)$ and $F_4^{-20}$ (see [4] and [15] for example). Hence we can define GFERF in these settings. As remarked lattices in $\mathrm{PSp}(n,1)$ or $F_4^{-20}$ have Property T, and so very different from lattices in $\mathrm{PO}_0(n,1)$ or $\mathrm{PU}(n,1)$. In particular there are no so no homomorphisms onto **Z** or splittings as a free product with amalgamation on finite index subgroups. This is in contrast to the first question above.

**Question 4.4** *Does there exist a lattice in* $\mathrm{PU}(n,1)$ *(resp.* $\mathrm{PSp}(n,1)$ *or* $F_4^{-20}$*) that is GFERF or LERF (or not GFERF or LERF)?*

**3.** More generally we can ask (and in contrast to Question 4.1).

**Question 4.5** *If* $\Gamma$ *is an infinite finitely generated group that is LERF, does* $\Gamma$ *fail to have Property T?*

Or put another way, does there exist an infinite group with Property T that is LERF.

# References

[1] C. C. Adams, *Hyperbolic 3-manifolds with two generators*, Comm. Anal. Geom. **4** (1996), pp. 181–206.

[2] I. Agol, *Tameness of hyperbolic 3-manifolds*, preprint.

[3] I. Agol, D. D. Long and A. W. Reid, *The Bianchi groups are separable on geometrically finite subgroups*, Ann. of Math. **153** (2001), pp. 599–621.

[4] B. Apanasov and X. Xie, *Geometrically finite complex hyperbolic manifolds*, International J. Math. **8** (1997), pp. 703–757.

[5] M. Boileau and R. Weidmann, *The structure of 3-manifolds with 2-generated fundamental group*, Preprint.

[6] F. Bonahon, *Bouts des variétés hyperboliques de dimension 3*, Annals of Math. **124** (1986), pp. 71–158.

[7] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), pp. 111–122.

[8] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature*, Grundlehren der Mathematischen Wissenschaften **319** Springer-Verlag, (1999).

[9] R. G. Burns, D. Karrass and D. Solitar, *A note on subgroups with separable finitely generated subgroups*, Bull. Australian Math. Soc. **36** (1987), pp. 153–160.

[10] D. Calegari and D. Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, preprint.

[11] R. D. Canary, *A covering theorem for hyperbolic 3- manifolds and its applications*, Topology **35** (1996), pp. 751–778.

[12] J. A. Carlson and D. Toledo, *Harmonic mappings of Kahler manifolds to locally symmetric spaces*, Publ. I. H. E. S. **69** (1989), pp173–201.

[13] D. Cooper, D.D. Long and A.W. Reid, *Infinite Coxeter groups are virtually indicable*, Proc. Edinburgh Math. Soc. vol. 41 (1998) pp. 303 - 313.

[14] K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. **135** (1992), pp. 165–182.

[15] K. Corlette, *Hausdorff dimension of limit sets, I*, Invent. Math. **102** (1990), pp. 521–542.

[16] J.D. Dixon *et alii*, *Analytic pro-p groups.* London Math. Soc. Lecture Notes 157 (1991).

[17] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), pp. 109–160.

[18] F. Gonzalez-Acuna and W. Whitten, *Imbeddings of three-manifold groups*, Memoirs of The A. M. S. **474**, (1992).

[19] C. McA. Gordon, R. Litherland and K. Murasugi, *Signatures of covering links*, Canad. J. Math. **33** (1981), pp. 381–394.

[20] C. McA. Gordon, *Boundary slopes of punctured tori in 3-manifolds*, Trans. A. M. S. **350** (1998), pp. 1713–1790.

[21] C. McA. Gordon, *Dehn filling: a survey*, Knot theory (Warsaw, 1995), pp. 129–144, Banach Center Publ., 42, Polish Acad. Sci., Warsaw, (1998).

[22] M. Gromov and W. P. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987), pp. 1–12.

[23] F. Haglund, *Finite index subgroups of graph products*, preprint.

[24] J. Hempel, *3-Manifolds*, Annals of Math Studies, **86**, Princeton Univ. Press (1976).

[25] W. Jaco and P. B. Shalen, *Seifert fibered spaces in 3-manifolds*, Memoirs of The A. M. S. **220**, (1979).

[26] D. D. Long, *Immersions and embeddings of totally geodesic surfaces*, Bull. London Math. Soc. **19** (1987), pp. 481–484.

[27] D. D. Long, *Engulfing and subgroup separability for hyperbolic groups*, Trans. Amer. Math. Soc. **308** (1988), pp. 849–859.

[28] D. D. Long and G. A. Niblo, *Subgroup separability and 3-manifold groups*, Math. Zeit. **207** (1991), pp. 209–215.

[29] D. D. Long and A. W. Reid, *The fundamental group of the double of the figure-eight knot exterior is GFERF*, Bull. London Math. Soc. **33** (2001), pp. 391–396.

[30] D. D. Long and A. W. Reid, *On subgroup separability in hyperbolic Coxeter groups*, Geom. Dedicata **87** (2001), pp. 245–260.

[31] G. A. Niblo and B. T. Williams, *Engulfing in word-hyperbolic groups*, Algebr. Geom. Topol. **2** (2002), pp. 743–755.

[32] A. W. Reid, *Arithmeticity of knot complements*, J. London Math. Soc. **43** (1991), pp. 171–184.

[33] R. Riley, *Parabolic representations of knot groups. I*, Proc. London Math. Soc. **24** (1972), pp. 217–242.

[34] G. P. Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. **17** (1978), pp. 555 - 565. See also *ibid Correction:* J. London Math. Soc. **32** (1985), pp. 217–220.

[35] M. Suzuki, *Group Theory I*, Grundlehren der Mathematischen Wissenschaften **247**, Springer-Verlag, (1982).

[36] W. P. Thurston, *The Geometry and Topology of 3-manifolds.* Princeton University mimeographed notes. (1979)

Department of Mathematics,
University of California
Santa Barbara, CA 93106

Department of Mathematics,
University of Texas
Austin, TX 78712