

CUSPS OF MINIMAL NON-COMPACT ARITHMETIC HYPERBOLIC 3-ORBIFOLDS

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ABSTRACT. In this paper we count the number of cusps of minimal non-compact finite volume arithmetic hyperbolic 3-orbifolds. We show that for each N , the orbifolds of this kind which have exactly N cusps lie in a finite set of commensurability classes, but either an empty or an infinite number of isometry classes.

1. INTRODUCTION.

In this paper we count the number of cusps of minimal non-compact finite volume arithmetic hyperbolic 3-orbifolds. An orbifold of this kind is isometric to \mathbf{H}^3/Γ , where \mathbf{H}^3 is hyperbolic upper half space and Γ is a maximal discrete arithmetic subgroup in $\mathrm{PGL}_2(k)$ for some imaginary quadratic field k .

It is well known (cf. §3 below) that the cusps of the orbifold \mathbf{H}^3/Γ correspond to Γ -equivalence classes of points of \mathbb{P}_k^1 under the action of $\mathrm{PGL}_2(k)$ on \mathbb{P}_k^1 . It was first noted by Bianchi [2] that $\mathbf{H}^3/\mathrm{PSL}_2(O_k)$ has h_k cusps where h_k is the class number of k . By work of Allan [1] and Schmidt [7], there is a unique maximal arithmetic subgroup $\Gamma_{\phi,\phi}$ of $\mathrm{PGL}_2(k)$ which contains $\mathrm{PSL}_2(O_k)$. Let $Cl(k)$ be the ideal class group of k , and let $h_{k,2}$ be the order of $Cl(k)/(2 \cdot Cl(k))$. It follows from the work of Vinberg in [11, §2] that $\mathbf{H}^3/\Gamma_{\phi,\phi}$ has $h_k/h_{k,2}$ cusps. (Some closely related results are proved by Elstrodt, Grunewald and Mennicke in [4, §7.2,7.4]). In particular, since there are only finitely many imaginary quadratic number fields of a fixed class number, for any given N there are only finitely orbifolds $\mathbf{H}^3/\Gamma_{\phi,\phi}$ as above which have N cusps.

The objective of this paper is to generalize the above result of Vinberg to an arbitrary maximal arithmetic subgroup Γ of $\mathrm{PGL}_2(k)$.

To state the main theorem, recall that in [3], Borel described for each pair (S, S') of finite disjoint sets of finite places of k a discrete finite covolume subgroup $\Gamma_{S,S'}$ of $\mathrm{PGL}_2(k)$. We recall the definition of $\Gamma_{S,S'}$ in §2. Borel showed that each maximal finite covolume discrete subgroup of $\mathrm{PGL}_2(k)$ is conjugate to $\Gamma_{S,S'}$ for some (S, S') .

The main result of this paper is:

Theorem 1.1. *Let $Cl(k)$ be the ideal class group of k . The number of cusps of $\mathbf{H}/\Gamma_{S,S'}$ is*

$$2^n \frac{h_k}{h_{k,2}}$$

where h_k is the class number of k , $h_{k,2}$ is the order of $Cl(k)/(2 \cdot Cl(k))$, $0 \leq n \leq \#S$ and 2^n is the order of the subgroup of $Cl(k)/(2 \cdot Cl(k))$ generated by the classes of prime ideals determined by the places in S .

This Theorem and work of Siegel in [8] leads to a proof of the following Corollary.

Corollary 1.2. *Let N be a positive integer, and let $C(N)$ be the set of isometry classes of minimal finite volume arithmetic hyperbolic three-orbifolds which have exactly N cusps.*

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- a. *Only finitely many commensurability classes are represented by the elements of $C(N)$.*
- b. *If $C(N)$ is not empty, there are infinitely many elements of $C(N)$ which are commensurable to each element of $C(N)$.*

The proof of part (a) of this Corollary is not effective, though it can be made effective up to at most one exceptional commensurability class using work of Tatuzawa in [9]. Finding an effective proof is equivalent to the problem of showing that there are only finitely many imaginary quadratic fields k such that $h/h_{k,2}$ is bounded above by a given constant. Such a proof appears to be beyond present methods.

This paper is organized in the following way. In §2 we recall Borel's definition of $\Gamma_{S,S'}$. In §3 we recall some well-known facts concerning cusps of non-compact arithmetic three-orbifolds. In §4 and §5 we analyze the cusps of certain orbifolds defined by congruence subgroups of $\Gamma_{S,S'}$. This leads to the proof of Theorem 1.1 in §6 - §8. The main techniques used in §4 - §8 are Borel's work, the Strong Approximation Theorem for SL_2 , and an argument of Swan [10] for constructing matrices satisfying various congruence conditions which send a prescribed point of \mathbb{P}_k^1 to another prescribed point. Corollary 1.2 is proved in §9.

2. BOREL'S SUBGROUPS.

Let k be an imaginary quadratic field, with ring of integers $O = O_k$. Let S and S' be finite disjoint subsets of the set of all finite places v of k . For each such v , let k_v be the completion of k at v . Let π_v be a uniformizer in the ring of integers O_v of k_v . Define $\mathcal{D}_v = \mathrm{Mat}_2(O_v)$, and let \mathcal{D}'_v be the maximal O_v -order of all matrices of the form

$$(2.1) \quad M = \begin{pmatrix} a & \pi_v b \\ \pi_v^{-1} c & d \end{pmatrix}$$

in which $a, b, c, d \in O_v$. Define $K_{1,v} = \mathrm{PGL}_2(O_v)$, so that $K_{1,v}$ is the image of \mathcal{D}_v^* in $\mathrm{PGL}_2(k_v)$. Let $K'_{1,v}$ to be the image of \mathcal{D}'_v^* in $\mathrm{PGL}_2(k_v)$. Finally, let $K_{2,v}$ be the group generated by $K_{1,v} \cap K'_{1,v}$ together with image in $\mathrm{PGL}_2(k_v)$ of the element

$$(2.2) \quad w_v = \begin{pmatrix} 0 & \pi_v \\ 1 & 0 \end{pmatrix}$$

Then $K_{1,v}$ and $K'_{1,v}$ are the stabilizers in $\mathrm{PGL}_2(k_v)$ of adjacent vertices of the Bruhat-Tits building of $\mathrm{SL}_2(k_v)$, and $K_{2,v}$ is the stabilizer the edge joining these vertices. In [3] Borel defines

$$(2.3) \quad \Gamma_{S,S'} = \{g \in \mathrm{PGL}_2(k) : g \in K_{2,v} \text{ (resp. } K'_{1,v}, \text{ resp. } K_{1,v}) \text{ if } v \in S \text{ (resp. } v \in S', v \notin S \cup S')\}$$

It is shown in [3, Prop. 4.4] the every maximal arithmetic discrete subgroup of $\mathrm{PGL}_2(k)$ is conjugate to $\Gamma_{S,S'}$ for some S and S' . Not all of the $\Gamma_{S,S'}$ need be maximal (cf. [3, §4.4]). By [3, Prop. 4.10, Thm. 4.6], the groups $\Gamma_{S,S'}$ for a fixed S lie in finitely many conjugacy classes inside $\mathrm{PGL}_2(k)$, while as S varies these groups lie in infinitely many distinct conjugacy classes.

3. CUSPS.

Suppose Γ is any discrete arithmetic subgroup of $\mathrm{PGL}_2(k)$ having finite covolume. An element $\sigma \in \Gamma$ is parabolic if it fixes a unique point of $\mathbb{P}_{\mathbb{C}}^1$, and such a fixed point is called a cusp of Γ (compare [6, p. 7-8]). The cusps of the orbifold \mathbf{H}^3/Γ are the Γ -equivalence classes of cusps of Γ .

Lemma 3.1. *The cusps of Γ are the points in $P_k^1 = k \cup \{\infty\}$, so that the cusps of \mathbf{H}^3/Γ are the orbits of Γ acting on P_k^1 .*

Proof: We first show that Γ has the same cusps as any group Γ' commensurable to Γ . For this, it will suffice to consider the case in which Γ' has finite index in Γ . Clearly the cusps of Γ' are cusps for Γ . Conversely, suppose z is a cusp of Γ , so z is the only point of $\mathbb{P}_{\mathbb{C}}^1$ fixed by a parabolic element $\sigma \in \Gamma$. Then σ^n is a parabolic element of Γ' fixing z when $n = [\Gamma : \Gamma']$, so z is also a cusp of Γ' . We can thus reduce to the case in which $\Gamma = \Gamma_{S,S'}$ for some S and S' .

Suppose z is a cusp of $\Gamma_{S,S'} \subset \mathrm{PGL}_2(k)$. Since z is the only fixed point of some $M \in \mathrm{GL}_2(k)$ acting on $\mathbb{P}_{\mathbb{C}}^1$, the quadratic formula implies that z must lie in \mathbb{P}_k^1 . Thus we now must show each $z \in \mathbb{P}_k^1$ is a cusp.

If b is a sufficiently divisible non-zero element of O , the matrix

$$(3.1) \quad M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

defines a parabolic element of $\Gamma_{S,S'}$ which fixes ∞ , so ∞ is a cusp of $\Gamma_{S,S'}$. Suppose now that $z \in k \subset \mathbb{P}_k^1$. There is then a matrix T in $\mathrm{GL}_2(k)$ such that $T \cdot \infty = z$. This implies z is a cusp of the discrete group $T\Gamma_{S,S'}T^{-1}$, since TMT^{-1} defines a parabolic element of this group fixing z . However, $T\Gamma_{S,S'}T^{-1}$ and $\Gamma_{S,S'}$ are commensurable, so they have the same cusps, which proves the Lemma.

In the following sections we analyze equivalence classes of cusps under the action of various subgroups Γ of $\Gamma_{S,S'}$.

4. THE PRINCIPAL CONGRUENCE SUBGROUP OF $\Gamma_{S,S'}$.

We consider in this section the following subgroup of $\Gamma_{S,S'}$.

Definition 4.1. Let I be the two-by-two identity matrix. Define $\Gamma(S, S')$ to be the subgroup of elements of $\Gamma_{S,S'} \subset \mathrm{PGL}_2(k)$ which are the images of matrices $M \in \mathrm{GL}_2(k)$ such that $M - I \in \pi_v \mathrm{Mat}_2(O_v)$ for $v \in S$, $M \in \mathcal{D}'_v$ for $v \in S'$ and $M \in \mathrm{GL}_2(O_v)$ for $v \notin S \cup S'$.

We will first describe the $\Gamma(S, S')$ -equivalent cusps of $\Gamma(S, S')$. By Lemma 3.1, this is the same as describing the cusps of $\mathbf{H}^3/\Gamma(S, S')$, and the orbits of $\Gamma(S, S')$ acting on \mathbb{P}_k^1 .

Define $\mathcal{I}(k)$ to be the multiplicative group of fractional ideals of k . For v a finite place of k , let $\mathcal{P}(v)$ be the prime ideal of O determined by v . If T is a finite set of finite places of k , define $\mathcal{P}(T) = \prod_{v \in T} \mathcal{P}(v)$. Define $L'(S)$ to be the set of triples (J, α_0, α_1) in which $J \in \mathcal{I}(k)$ and α_0 and α_1 are generators of $J/(\mathcal{P}(S) \cdot J)$ as a finite O -module. An element $\lambda \in k^*$ acts on $L'(S)$ by sending (J, α_0, α_1) to $(\lambda \cdot J, \lambda \cdot \alpha_0, \lambda \cdot \alpha_1)$. Define $L(S) = L'(S)/k^*$ to be the set of orbits in $L'(S)$ under this action of k^* .

Definition 4.2. Define a map $\Psi : \mathbb{P}_k^1 \rightarrow L(S)$ in the following way. Fix an element $t(S, S') \in \mathcal{P}(S')$ such that the ideal $t(S, S')O$ equals $\mathcal{P}(S') \cdot \mathcal{A}$ for some ideal \mathcal{A} prime to $\mathcal{P}(S \cup S')$. Suppose $(x_0 : x_1)$ are homogeneous coordinates for a point of \mathbb{P}_k^1 . Define J to be the fractional O -ideal $O \cdot x_0 + \mathcal{P}(S') \cdot x_1$ of k . Let $\beta_0 = x_0$ and $\beta_1 = t(S, S')x_1$, so that β_0 and β_1 are elements of J . Define α_i to be the image of β_i in $J/(\mathcal{P}(S) \cdot J)$ for $i = 0, 1$. Define

$$(4.1) \quad \Psi((x_0 : x_1)) = [(J, \alpha_0, \alpha_1)]$$

to be the class of (J, α_0, α_1) in $L(S)$. The other homogeneous coordinates for $(x_0 : x_1)$ have the form $(\lambda \cdot x_0 : \lambda \cdot x_1)$ for some $\lambda \in k^*$, so Ψ is well-defined.

Proposition 4.3. *The map Ψ is surjective, and its fibers are exactly the $\Gamma(S, S')$ -equivalent cusps of $\Gamma(S, S')$.*

Proof: Let us first check surjectivity. Suppose $(J, \alpha_0, \alpha_1) \in L'(S)$. We first claim that there is an $x_1 \in k^*$ such that $\mathcal{P}(S') \cdot x_1 \subset J$ and $t(S, S')x_1 \in J$ has class α_1 in $J/(\mathcal{P}(S) \cdot J)$. Such an x_1 exists because we can find an $x_1 \in \mathcal{P}(S')^{-1}J$ satisfying the appropriate congruence conditions at the places in S because S and S' are disjoint. Choose $x_0 \in J$ to have class α_0 in $J/(\mathcal{P}(S) \cdot J)$, and so that $O_v \cdot x_0 = O_v \cdot J$ for the finitely many finite places v of k which are not in S where $O_v \cdot \mathcal{P}(S')x_1$ is not equal to $O_v \cdot J$. We can find such an x_0 since these conditions amount to congruence conditions at a finite set of finite places of k . We show $\Psi((x_0 : x_1))$ is the class of (J, α_0, α_1) in $L(S)$. By construction, x_0 has class α_0 in $J/\mathcal{P}(S)J$, while $t(S, S')x_1$ has class α_1 in $J/(\mathcal{P}(S) \cdot J)$. Hence we only have to check that $J' = O \cdot x_0 + \mathcal{P}(S')x_1$ is equal to J . Clearly $J' \subset J$. Since $\alpha_0 \equiv x_0$ and $\alpha_1 \equiv t(S, S')x_1$ together generate $J/(\mathcal{P}(S) \cdot J)$ as an O -module, we have $O_v \cdot J' = O_v \cdot J$ if $v \in S$.

However, for $v \notin S$, we chose x_0 so that $O_v \cdot x_0 = O_v \cdot J$ if $O_v \cdot \mathcal{P}(S')x_1$ is not equal to $O_v \cdot J$. Thus $O_v \cdot J' = O_v \cdot J$ for all such v , and we conclude $J' = J$.

We now consider the fibers of Ψ . Suppose $(x_0 : x_1)$ and $(x'_0 : x'_1)$ are two points having the same image under Ψ . After multiplying x'_0 and x'_1 by a suitable $\lambda \in k^*$, we can assume the following is true:

$$(4.2) \quad J = O x_0 + \mathcal{P}(S')x_1 = O x'_0 + \mathcal{P}(S')x'_1$$

$$(4.3) \quad x_0 \equiv x'_0 \equiv \alpha_0 \pmod{\mathcal{P}(S)J}$$

and

$$(4.4) \quad t(S, S')x_1 \equiv t(S, S')x'_1 \equiv \alpha_1 \pmod{\mathcal{P}(S)J}.$$

We wish to show that there is a matrix $M \in \mathrm{GL}_2(k)$ such that

$$(4.5) \quad M \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix}$$

and

$$(4.6) \quad M - I \in \pi_v \mathrm{Mat}_2(O_v) \quad \text{for } v \in S,$$

$$(4.7) \quad M \in \mathcal{D}'_v{}^* \quad \text{for } v \in S',$$

$$(4.8) \quad M \in \mathrm{GL}_2(O_v) \quad \text{for } v \notin S \cup S'.$$

We adapt an argument of Swan in [10, Prop. 3.10] to construct M . There are two exact sequences of O -modules

$$(4.9) \quad 0 \longrightarrow \mathcal{B} \longrightarrow O \oplus \mathcal{P}(S') \xrightarrow{l'} J \longrightarrow 0$$

$$(4.10) \quad 0 \longrightarrow \mathcal{C} \longrightarrow O \oplus \mathcal{P}(S') \xrightarrow{l} J \longrightarrow 0$$

in which l and l' are defined for $(a, b) \in O \oplus \mathcal{P}(S')$ by

$$(4.11) \quad l(a, b) = ax_0 + bx_1 \quad \text{and} \quad l'(a, b) = ax'_0 + bx'_1.$$

Since J is a projective O -module, these sequences split, giving isomorphisms

$$(4.12) \quad O \oplus \mathcal{P}(S') = J \oplus \mathcal{B} \quad \text{and} \quad O \oplus \mathcal{P}(S') = J \oplus \mathcal{C}.$$

Again using the fact that O is a Dedekind ring, these isomorphisms imply that there is an isomorphism $\phi : \mathcal{B} \rightarrow \mathcal{C}$ of projective rank one O -modules.

Let s be a unit of O , and suppose $W \in \mathrm{Hom}_O(J, \mathcal{C})$. We define an O -linear map

$$(4.13) \quad \theta_{s,W} : O \oplus \mathcal{P}(S') = J \oplus \mathcal{B} \rightarrow J \oplus \mathcal{C} = O \oplus \mathcal{P}(S')$$

by

$$(4.14) \quad \theta_{s,W}(j \oplus a) = j \oplus (s\phi(a) + W(j))$$

for $j \in J$ and $a \in \mathcal{B}$. Then $\theta_{s,W}$ fits into a commutative diagram

$$(4.15) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{B} & \longrightarrow & O \oplus \mathcal{P}(S') & \xrightarrow{l'} & J & \longrightarrow & 0 \\ & & s\phi \downarrow & & \downarrow \theta_{s,W} & & \downarrow 1 & & \\ 0 & \longrightarrow & \mathcal{C} & \longrightarrow & O \oplus \mathcal{P}(S') & \xrightarrow{l} & J & \longrightarrow & 0 \end{array}$$

Since $s\phi$ is an isomorphism, and $1 : J \rightarrow J$ is the identity map, we conclude that $\theta_{s,W}$ is an automorphism. Furthermore, $\det_O(\theta_{s,W}) = s \cdot \det_O(\theta_{1,W}) = s \cdot \det_O(\theta_{1,0})$ is independent of the choice of W , so we can choose s (depending on ϕ) so that $\det(\theta_{s,W}) = 1$ for all W .

Define

$$(4.16) \quad M_{s,W} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

to be the matrix of $\theta_{s,W}$ when we view elements of $O \oplus \mathcal{P}(S') \subset k \oplus k$ as column vectors. Here

$$(4.17) \quad \begin{aligned} \alpha \in \text{Hom}_O(O, O) &= O, \\ \beta \in \text{Hom}_O(\mathcal{P}(S'), O) &= \mathcal{P}(S')^{-1}, \\ \gamma \in \text{Hom}_O(O, \mathcal{P}(S')) &= \mathcal{P}(S') \\ \delta \in \text{Hom}_O(\mathcal{P}(S'), \mathcal{P}(S')) &= O. \end{aligned}$$

Thus the transpose $M_{s,W}^{tr}$ of $M_{s,W}$ is an element of $\text{SL}_2(k)$ satisfying conditions (4.7) and (4.8), while $M_{s,W}^{tr} \in \text{SL}_2(O_v)$ for $v \in S$. The commutativity of (4.15) shows

$$(4.18) \quad x'_0 = l' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = l \left(M_{s,W} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = l \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \alpha x_0 + \gamma x_1$$

and

$$(4.19) \quad t(S, S')x'_1 = l \left(M_{s,W} \cdot \begin{pmatrix} 0 \\ t(S, S') \end{pmatrix} \right) = l \begin{pmatrix} \beta \cdot t(S, S') \\ \delta \cdot t(S, S') \end{pmatrix} = \beta \cdot t(S, S') \cdot x_0 + \delta \cdot t(S, S') \cdot x_1$$

This gives the matrix equation

$$(4.20) \quad M_{s,W}^{tr} \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix}$$

We now show that we can choose $W \in \text{Hom}_O(J, \mathcal{C})$ so that $M_{s,W}^{tr} = M$ will satisfy (4.6), i.e. so that $M - I \in \pi_v \text{Mat}_2(O_v)$ for $v \in S$. This will complete the proof that cusps having the same image under Ψ are $\Gamma(S, S')$ -equivalent.

For $v \in S$, let $k(v) = O/\mathcal{P}(v)$ and let J_v be the localization of J at v . Since $t(S, S') \in O_v^*$ and $t(S, S')x_1 \in J$, we have $x_1 \in J_v$ for $v \in S$. Define $\beta_{i,v}$ be the image of x_i in the one-dimensional $k(v)$ -vector space $J(v) = J_v/\mathcal{P}(v)J_v$. From (4.20), (4.3) and (4.4) we know that for $v \in S$, $M_{s,W}^{tr} \in \text{SL}_2(O_v)$ fixes the vector $\beta(v) = (\beta_{0,v}, \beta_{1,v})$ in $J(v) \oplus J(v)$. This $\beta(v)$ is not the zero vector, since α_0 and α_1 together generate $J/\mathcal{P}(S)J$ and $\alpha_0 = x_0 \bmod \mathcal{P}(S)J$ and $\alpha_1 = t(S, S')x_1 \bmod \mathcal{P}(S)J$. Thus the image $M_{s,W,v}^{tr}$ of $M_{s,W}^{tr}$ in $\text{SL}_2(k(v))$ lies in the stabilizer of $\beta(v)$, and this stabilizer has order $\#k(v)$ since $\beta(v)$ is non-zero. Letting v range over S , we see that the image of $M_{s,W}^{tr}$ in $T = \prod_{v \in S} \text{SL}_2(k(v))$ lies in a subgroup of matrices which has order $N = \prod_{v \in S} \#k(v)$. However, as W ranges over $\text{Hom}(J, \mathcal{C})$, the image of $M_{s,W}^{tr}$ in T also ranges over a set of N matrices, since each of J and \mathcal{C} are rank one projective O -modules. It follows that we can choose W so that $M_{s,W}^{tr}$ has image the identity element of T , as required.

The last statement we have to prove is that $\Gamma(S, S')$ -equivalent cusps have the same image under Ψ . Suppose $M = M_{s,W}^{tr}$ satisfies (4.20) and has the properties described in Definition (4.1). It will suffice to show (4.2), (4.3) and (4.4) hold. For (4.2), observe that the containments in (4.17) show

$$(4.21) \quad O x'_0 + \mathcal{P}(S')x'_1 = O(\alpha \cdot x_0 + \gamma \cdot x_1) + \mathcal{P}(S')(\beta \cdot x_0 + \delta \cdot x_1) \subset O x_0 + \mathcal{P}(S')x_1$$

Since M^{-1} also satisfies the conditions in (4.1) and takes the cusp $(x'_0 : x'_1)$ back to $(x_0 : x_1)$, we can interchange $(x'_0 : x'_1)$ and $(x_0 : x_1)$ to conclude that (4.2) holds. The proof of (4.3) and (4.4) is similar using the properties of M in Definition (4.1).

5. THE BOREL CONGRUENCE SUBGROUP OF $\Gamma_{S,S'}$.

We consider in this section the following subgroup of $\Gamma_{S,S'}$.

Definition 5.1. For v a finite place of k , let $B_v \subset \text{GL}_2(O_v)$ be the subgroup of invertible matrices of the form

$$(5.1) \quad \begin{pmatrix} a & \pi_v b \\ c & d \end{pmatrix}$$

in which $a, b, c, d \in O_v$. Define $\Gamma_0(S, S')$ to be the subgroup of elements of $\Gamma_{S,S'} \subset \text{PGL}_2(k)$ which are the images of matrices $M \in \text{GL}_2(k)$ such that that $M \in B_v$ for $v \in S$, $M \in \mathcal{D}'_v^*$ for $v \in S'$ and $M \in \text{GL}_2(O_v)$ for $v \notin S \cup S'$.

Note that the image of B_v in $\mathrm{PGL}_2(k_v)$ is the group $K_{1,v} \cap K'_{1,v}$ defined in §2. Thus $\Gamma_0(S, S') \subset \Gamma_{S, S'}$, while $\Gamma(S, S') \subset \Gamma_0(S, S')$.

Definition 5.2. Define $L_0(S)$ to be the set of pairs $([J], \{\beta_v\}_{v \in S})$ in which $[J]$ is an element of the ideal class group of k , and for each $v \in S$, β_v is either 0 or 1. Define $r : L(S) \rightarrow L_0(S)$ to be the map which sends a triple $(J, \alpha_0, \alpha_1) \in L'(S)$ representing an element of $L(S)$ to $([J], \{\beta_v\}_{v \in S})$, where $[J]$ is the ideal class of $J \in I(k)$, and $\beta_v = 0$ (resp. 1) if $\alpha_0 \equiv 0 \pmod{\pi_v J}$ (resp. if $\alpha_0 \not\equiv 0 \pmod{\pi_v J}$).

Proposition 5.3. *Let $\Psi : \mathbb{P}_k^1 \rightarrow L(S)$ be the map of Proposition 4.3. The composition $r \circ \Psi : \mathbb{P}_k^1 \rightarrow L_0(S)$ is surjective, and the fibers of this map are the $\Gamma_0(S, S')$ -equivalent cusps of $\Gamma_0(S, S')$.*

Proof: Recall that $L'(S)$ consists of the triples (J, α_0, α_1) in which $J \in I(k)$ and α_0 and α_1 are generators of $J/\mathcal{P}(S)J$. Since $\mathcal{P}(S) = \prod_{v \in S} \mathcal{P}(v)$, we see that we can choose α_0 and α_1 to have prescribed classes $\alpha_0(v), \alpha_1(v) \in J/\mathcal{P}(v)$ as v ranges over S provided that for no such v are both $\alpha_0(v)$ and $\alpha_1(v)$ trivial. This implies r is surjective, so $r \circ \Psi$ is surjective by Proposition 4.3.

Consider now the action of a matrix $M \in \mathrm{GL}_2(k)$ satisfying the hypotheses of Definition 5.1 on $\Psi(x_0 : x_1) = [(J, \alpha_0, \alpha_1)] \in L(S)$, where $(J, \alpha_0, \alpha_1) \in L'(S)$ is as in Definition 4.2 and $[(J, \alpha_0, \alpha_1)]$ is the class of (J, α_0, α_1) in $L(S)$. From $J = O \cdot x_0 + \mathcal{P}(S') \cdot x_1$ and the hypotheses on M we see that $J = O \cdot x'_0 + \mathcal{P}(S') \cdot x'_1$ when

$$(5.2) \quad \begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

Recall that α_0 (resp. α_1) is the image in $J/\mathcal{P}(S)$ of x_0 (resp. $t(S, S')x_1$). Suppose

$$(5.3) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Define

$$(5.4) \quad M' = \begin{pmatrix} a & b \cdot t(S, S')^{-1} \\ t(S, S')c & d \end{pmatrix}$$

We find from (5.2) that $\Psi(x'_0 : x'_1) = (J, \alpha'_0, \alpha'_1)$, where α'_0 and α'_1 are elements of $J/\mathcal{P}(S)$ given by the following residue classes $\alpha'_0(v), \alpha'_1(v) \in J_v/\mathcal{P}(v)J_v = J/\mathcal{P}(v)J$ for $v \in S$:

$$(5.5) \quad \begin{pmatrix} \alpha'_0(v) \\ \alpha'_1(v) \end{pmatrix} = M' \cdot \begin{pmatrix} \alpha_0(v) \\ \alpha_1(v) \end{pmatrix}.$$

The number $t(S, S') \in k$ is a unit at each $v \in S$, so $M' \in B_v$ for such v because $M \in B_v$. Thus

$$(5.6) \quad a, d \in O_v^*, \quad bt(S, S')^{-1} \in \pi_v O_v \quad \text{and} \quad t(S, S')c \in O_v$$

This implies $\alpha'_0(v) = 0$ if and only if $\alpha_0(v) = 0$. It follows that $r \circ \Psi(x_0 : x_1) = r \circ \Psi(x'_0 : x'_1)$, so $\Gamma_0(S, S')$ -equivalent cusps have the same image under $r \circ \Psi$.

To complete the proof of Proposition 5.3, we have to show that two points $(x_0 : x_1)$ and $(x'_0 : x'_1)$ with the same image under $r \circ \Psi$ are $\Gamma_0(S, S')$ -equivalent. After multiplying x'_0 and x'_1 by a suitable scalar, we can assume

$$(5.7) \quad J = O \cdot x_0 + \mathcal{P}(S') \cdot x_1 = O \cdot x'_0 + \mathcal{P}(S') \cdot x'_1$$

Furthermore, on defining $\alpha_0(v)$, $\alpha_1(v)$, $\alpha'_0(v)$ and $\alpha'_1(v)$ to be the images of x_0 , $t(S, S')x_1$, x'_0 and $t(S, S')x'_1$ in $J/\mathcal{P}(v)J$, we see that $\alpha_0(v) = 0$ if and only if $\alpha'_0(v) = 0$ for $v \in S$, since $r \circ \Psi(x_0 : x_1) = r \circ \Psi(x'_0 : x'_1)$. Furthermore, $\alpha_1(v) \neq 0$ if $\alpha_0(v) = 0$, and similarly $\alpha'_1(v) \neq 0$ if $\alpha'_0(v) = 0$. This implies there is a lower triangular matrix $m_v \in \mathrm{SL}_2(O_v/\pi_v O_v)$ such that

$$(5.8) \quad \begin{pmatrix} \alpha'_0(v) \\ \alpha'_1(v) \end{pmatrix} = m_v \cdot \begin{pmatrix} \alpha_0(v) \\ \alpha_1(v) \end{pmatrix}$$

We now use the Strong Approximation Theorem for SL_2 to conclude that there is $M \in \mathrm{SL}_2(k)$ which satisfies the hypotheses of Definition 5.1 such that when we write M in the form (5.3) and let

M' be as in (5.4), then $M' \in \mathrm{SL}_2(O_v)$ for $v \in S$ satisfies the congruence $M' \equiv m_v \pmod{\pi_v \mathrm{Mat}_2(O_v)}$. We conclude from this that

$$\Psi(M \cdot (x_0 : x_1)) = \Psi(x'_0 : x'_1)$$

so that $M \cdot (x_0 : x_1)$ and $(x'_0 : x'_1)$ are $\Gamma(S, S')$ -equivalent cusps by Proposition 4.3. Since $\Gamma(S, S') \subset \Gamma_0(S, S')$ and $M \cdot (x_0 : x_1)$ is $\Gamma_0(S, S')$ equivalent to $(x_0 : x_1)$ by our construction of M , this proves $(x_0 : x_1)$ and $(x'_0 : x'_1)$ are $\Gamma_0(S, S')$ -equivalent cusps.

Corollary 5.4. *The number of $\Gamma_0(S, S')$ -equivalence classes of cusps of $\Gamma_0(S, S')$ is $2^{\#S} h_k$, where h_k is the class number of k .*

6. $\Gamma_{S, S'}$ -INEQUIVALENT CUSPS.

In this section we will prove Theorem 1.1. The proof is based on the following two results, which will be proved in §7 and §8, respectively.

Proposition 6.1. *Let $C_0(S, S')$ be the set of $\Gamma_0(S, S')$ -equivalence classes of points of P_k^1 . Since $\Gamma_0(S, S') \subset \Gamma_{S, S'}$, the group $\Gamma_{S, S'}$ acts on $C_0(S, S')$. Each $\Gamma_{S, S'}$ -orbit in $C_0(S, S')$ has $[\Gamma_{S, S'} : \Gamma_0(S, S')]$ elements.*

Proposition 6.2. *Define $h_{k,2}$ to be the order of $Cl(k)/(2Cl(k))$ where $Cl(k)$ is the class group of k . Define 2^n to be the order of the subgroup of $Cl(k)/(2 \cdot Cl(k))$ generated by the classes of prime ideals determined by the places in S . Then $0 \leq n \leq \#S$ and*

$$(6.1) \quad [\Gamma_{S, S'} : \Gamma_0(S, S')] = 2^{\#S-n} h_{k,2}.$$

Theorem 1.1 is a consequence of these results in the following way. By Lemma 3.1 the set of $\Gamma_{S, S'}$ -orbits in $C_0(S, S')$ is the set of $\Gamma_{S, S'}$ -equivalence classes of cusps of $\Gamma_{S, S'}$. Corollary 5.4 together with Propositions 6.1 and 6.2 show this number is

$$(6.2) \quad \frac{2^{\#S} h_k}{2^{\#S-n} h_{k,2}} = 2^n \frac{h_k}{h_{k,2}}$$

as stated in Theorem 1.1.

7. PROOF OF PROPOSITION 6.1.

We will need several Lemmas.

Lemma 7.1. *To prove Proposition 6.1, it will suffice to show the following. Suppose*

$$(7.1) \quad \sigma \in \Gamma_{S, S'}, \quad (x_0 : x_1) \in P_k^1, \quad (x'_0 : x'_1) = \sigma \cdot (x_0 : x_1) \quad \text{and} \quad r \circ \Psi(x_0 : x_1) = r \circ \Psi(x'_0 : x'_1).$$

Then σ lies in $\Gamma_0(S, S')$.

Proof: This is clear from Proposition 5.3, which showed that the map $r \circ \Psi : P_k^1 \rightarrow L_0(S)$ has fibers equal to the elements of $C_0(S, S')$.

We will assume from now on that hypothesis (7.1) holds.

Definition 7.2. Let (J, α_0, α_1) be the triple associated in Definition 4.2 to the ordered pair (x_0, x_1) of elements of k which are not both 0. Thus $J = Ox_0 + \mathcal{P}(S')x_1$, and α_0 and α_1 are the classes of x_0 and $t(S, S')x_1$ in $J/\mathcal{P}(S)J$. The class $[(J, \alpha_0, \alpha_1)]$ of (J, α_0, α_1) in $L(S)$ is equal to $\Psi(x_0 : x_1)$. Write

$$\begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

for some matrix $M \in \mathrm{GL}_2(k)$ with image $\sigma \in \Gamma_{S, S'}$ in $\mathrm{PGL}_2(k)$. Let $(J', \alpha'_0, \alpha'_1)$ be the triple associated to (x'_0, x'_1) .

Lemma 7.3. *The element σ must be even at each $v \in S$, in the sense that $\det(M)$ has even valuation at each $v \in S$.*

Proof: Suppose to the contrary that σ is odd at some place $v \in S$. From the definition of $\Gamma_{S,S'}$ in §2, this implies that

$$(7.2) \quad M = \lambda_v \cdot w_v \cdot M_v$$

where $\lambda_v \in k_v^*$, w_v is the matrix

$$(7.3) \quad w_v = \begin{pmatrix} 0 & \pi_v \\ 1 & 0 \end{pmatrix}$$

and

$$(7.4) \quad M_v = \begin{pmatrix} a & \pi_v b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v)$$

for some $a, b, c, d \in O_v$. Consider the localization J_v of J at v . Since $\mathcal{P}(S')$ is prime to $\mathcal{P}(v)$, we have

$$(7.5) \quad J_v = O_v x_0 + O_v x_1 \subset k_v \quad \text{and} \quad J'_v = O_v x'_0 + O_v x'_1.$$

Since

$$(7.6) \quad \begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \lambda_v \cdot w_v \cdot M_v \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

we see from (7.3) and (7.4) that

$$(7.7) \quad x'_0 = \lambda_v \cdot \pi_v \cdot (cx_0 + dx_1) \quad \text{and} \quad x'_1 = \lambda_v \cdot (ax_0 + \pi_v bx_1).$$

Here $a, d \in O_v^*$, since M_v in (7.4) is in $\mathrm{GL}_2(O_v)$. We claim

$$(7.8) \quad J'_v = \lambda_v \cdot (\pi_v O_v x_1 + O_v x_0).$$

To show this, let $\mathrm{ord}_v : k_v \rightarrow \mathbf{Z} \cup \{\infty\}$ be the discrete valuation at v , normalized so that $\mathrm{ord}_v(\pi_v) = 1$. From (7.7) and (7.5) we have

$$J'_v = O_v x'_0 + O_v x'_1 \subset \lambda_v \cdot (\pi_v O_v x_1 + O_v x_0)$$

since $a, b, c, d \in O_v$. This containment must be an equality since (7.2) shows $\mathrm{ord}_v(\det(M)) = \mathrm{ord}_v(\lambda_v^2) + 1$, and this integer is the power of $\#O_v/\pi_v O_v$ appearing in the generalized index

$$[O_v x_0 + O_v x_1 : \lambda_v \cdot (\pi_v O_v x_1 + O_v x_0)].$$

The first case we now must consider is when $\mathrm{ord}_v(x_0) \leq \mathrm{ord}_v(x_1)$. In this case, (7.5) and (7.8) show

$$(7.9) \quad J_v = O_v x_0 \quad \text{and} \quad J'_v = \lambda_v \cdot O_v x_0$$

Thus $x_0 \not\equiv 0 \pmod{\pi_v J_v}$, while (7.7) shows $x'_0 \equiv 0$ in $J'_v/\pi_v J'_v$. This proves $\alpha_0(v) \neq 0$ but $\alpha'_0(v) = 0$. In view of the description of the map $r : L(S) \rightarrow L_0(S)$ in Definition 5.2, this forces $r([(J, \alpha_0, \alpha_1)]) = r \circ \Psi(x_0 : x_1)$ to be different from $r([(J', \alpha'_0, \alpha'_1)]) = r \circ \Psi(x'_0 : x'_1)$. This contradicts hypothesis (7.1), so we conclude that this hypothesis forces $\mathrm{ord}_v(x_0) > \mathrm{ord}_v(x_1)$. In this case (7.5) and (7.8) imply

$$(7.10) \quad J_v = O_v x_1 \quad \text{and} \quad J'_v = \lambda_v \cdot \pi_v \cdot O_v x_1.$$

Since $\mathrm{ord}_v(x_0) > \mathrm{ord}_v(x_1)$, we find that $x_0 \equiv 0 \pmod{\pi_v J_v}$, while (7.7) implies $x'_0 \not\equiv 0 \pmod{\pi_v J'_v}$. Thus we get $\alpha_0(v) = 0$ but $\alpha'_0(v) \neq 0$, again contradicting hypothesis (7.1). This contradiction proves Lemma 7.3.

Corollary 7.4. *The element $\sigma \in \Gamma_{S,S'}$ is represented by a matrix $M \in \mathrm{GL}_2(k)$ having the following properties. For each finite place v of k , there is an element $x_v \in k_v^*$ together with elements $a = a_v$, $b = b_v$, $c = c_v$ and $d = d_v$ of O_v such that*

$$(7.11) \quad M = x_v \cdot M_v \quad \text{and} \quad \det(M_v) \in O_v^*,$$

$$(7.12) \quad x_v \in O_v^* \quad \text{for all but finitely many places } v$$

$$(7.13) \quad M_v = \begin{pmatrix} a & \pi_v b \\ c & d \end{pmatrix} \quad \text{if } v \in S$$

$$(7.14) \quad M_v = \begin{pmatrix} a & \pi_v b \\ \pi_v^{-1} c & d \end{pmatrix} \quad \text{if } v \in S'$$

$$(7.15) \quad M_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{if } v \notin S \cup S'$$

Proof: By [3, Prop. 4.4(iii)], if v is a finite place of k such that $\Gamma_{S,S'}$ contains an element which is odd at v , then $v \in S$. We proved in Lemma 7.3 that σ must be even at each $v \in S$. Hence for each finite place v , there is an element $x_v \in k_v^*$ such that $2 \cdot \text{ord}_v(x_v) = \text{ord}_v(\det(M))$. On defining $M_v = x_v^{-1} \cdot M$, it now follows from the definition of $\Gamma_{S,S'}$ in (2.3) that M_v has properties (7.11) - (7.15).

Corollary 7.5. *With the notation of Corollary 7.4, let $\mathcal{B} = \prod_v \mathcal{P}(v)^{\text{ord}_v(x_v)}$. Then with the notation of (7.1), we have*

$$(7.16) \quad J' = Ox'_0 + \mathcal{P}(S')x'_1 = \mathcal{B} \cdot J = \mathcal{B} \cdot (Ox_0 + \mathcal{P}(S)x_1)$$

as fractional k -ideals.

Proof: Define

$$(7.17) \quad \begin{pmatrix} x_{v,0} \\ x_{v,1} \end{pmatrix} = M_v \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

for all v . Since $M = x_v \cdot M_v$, the localization J'_v at v satisfies

$$(7.18) \quad J'_v = (O_v x'_0 + \mathcal{P}(S')_v x'_1) = x_v \cdot (O_v x_{v,0} + O_v \mathcal{P}(S')_v x_{v,1})$$

However, the fact that $\det(M_v) \in O_v^*$ together with the form of M_v in (7.13) - (7.15) ensures that

$$(7.19) \quad O_v x_{v,0} + O_v \mathcal{P}(S')_v x_{v,1} = O_v x'_0 + \mathcal{P}(S')_v x'_1 = J_v$$

Combining (7.18) and (7.19) shows (7.16).

Completion of the proof of Proposition 6.1.

In hypothesis (7.1) we supposed $r \circ \Psi(x_0 : x_1) = r \circ \Psi(x'_0 : x'_1)$. This forces J and $J' = \mathcal{B} \cdot J$ to have the same ideal class as fractional k -ideals. Hence $\mathcal{B} = O \cdot \lambda$ is a principal ideal for some $\lambda \in k^*$. With the notation of Corollaries 7.4 and 7.5, we now see that if we choose $x_v = \lambda$ for all places v , then the matrix $M' = \lambda^{-1} \cdot M \in \text{GL}_2(k)$ has image M_v in $\text{GL}_2(k_v)$ for all v . This implies $M' \in \Gamma_0(S, S')$. Since M and M' have the same image σ in $\text{PGL}_2(k)$, we have $\sigma \in \Gamma_0(S, S')$, which completes the proof of Proposition 6.1 by Lemma 7.1.

8. PROOF OF PROPOSITION 6.2.

Let \mathcal{D}' be the maximal O -order $\cap_{v \notin S'} \mathcal{D}_v \cap_{v \in S'} \mathcal{D}'_v$ in $\text{Mat}_2(k)$, where \mathcal{D}_v and \mathcal{D}'_v are defined in §2. The set R_f of finite places of k which ramify in $\text{Mat}_2(k)$ is empty. Therefore the group Γ_{R_f} which Borel defines in [3, §8.4] is the image in $\text{PGL}_2(k)$ of the group $B_{R_f}^*$ of elements $\tau \in \text{GL}_2(k)$ such that $\det(\tau) \in O^*$. Define $\Gamma_{\mathcal{D}'^*}$ (resp. to $\Gamma_{\mathcal{D}'_1}$) to be the image in $\text{PGL}_2(k)$ of \mathcal{D}'^* (resp. the image of the group of $\tau \in \mathcal{D}'^*$ such that $\det(\tau) = 1$.) Borel shows in [3, Lemma 8.5] that $[\Gamma_{R_f} : \Gamma_{\mathcal{D}'_1}] = 2$, since in our case the unit group O^* is finite, cyclic and of even order and k has no real places. However, we also have $[\Gamma_{\mathcal{D}'^*} : \Gamma_{\mathcal{D}'_1}] = 2$, since \mathcal{D}'^* contains a diagonal matrix whose diagonal entries are 1 and a generator of O^* . Since

$$\Gamma_{\mathcal{D}'_1} \subset \Gamma_{\mathcal{D}'^*} \subset \Gamma_{R_f}$$

we conclude that $\Gamma_{\mathcal{D}'^*} = \Gamma_{R_f}$. Hence Borel's result in the Lemma of [3, §8.6] shows

$$(8.1) \quad [\Gamma_{\mathcal{D}'} : \Gamma_{\mathcal{D}'^*}] = h_{2,k}$$

where $\Gamma_{\mathcal{D}'}$ is the image in $\mathrm{PGL}_2(k)$ of the normalizer $\mathrm{Norm}(\mathcal{D}')$ of \mathcal{D}' in $\mathrm{GL}_2(k)$. For $v \in S$, let $k(v) = O_v/\pi_v O_v$, and let $b(v)$ be the subgroup of lower triangular matrices in $\mathrm{GL}_2(k(v))$. Definition 5.1 implies that $\Gamma_0(S, S')$ is the image in $\mathrm{PGL}_2(k)$ of the subgroup $\mathcal{D}'(S)^*$ of elements $M \in \mathcal{D}'^*$ such that the image of M in

$$\mathcal{D}'_v/\pi_v \mathcal{D}'_v = \mathcal{D}_v/\pi_v \mathcal{D}_v = \mathrm{Mat}_2(O_v/\pi_v O_v)$$

lies in $b(v)$ for each $v \in S$. Since each of the $1 + \#k(v)$ cosets of $b(v)$ in $\mathrm{GL}_2(k(v))$ is represented by an element of $\mathrm{SL}_2(k(v))$, the Strong Approximation Theorem for SL_2 implies

$$(8.2) \quad [\mathcal{D}'^* : \mathcal{D}'(S)^*] = \prod_{v \in S} (1 + \#k(v)).$$

Clearly $\mathcal{D}'^* \cap k^* = \mathcal{D}'(S)^* \cap k^*$ when we identify these groups with the diagonal matrices inside \mathcal{D}'^* and $\mathcal{D}'(S)^*$. Thus (8.2) gives

$$(8.3) \quad [\Gamma_{\mathcal{D}'^*} : \Gamma_0(S, S')] = \prod_{v \in S} (1 + \#k(v)).$$

The group $\Gamma_{\mathcal{D}'}$ is equal to $\Gamma_{\emptyset, S'}$ by [3, §4.9, eq. (4)]. Hence on letting

$$\Gamma_2 = \Gamma_{\mathcal{D}'} \cap \Gamma_{S, S'} = \Gamma_{\emptyset, S'} \cap \Gamma_{S, S'}$$

we have from [3, §5.3, eq. (7) and (8)] that

$$(8.4) \quad [\Gamma_{\mathcal{D}'} : \Gamma_2] = \prod_{v \in S} (1 + \#k(v))$$

(Note that there is a misprint in [3, §5.3, eq. (4)], since the product in that equation should be over places in S .) Putting together (8.1), (8.3) and (8.4) gives the generalized index relation

$$(8.5) \quad [\Gamma_2 : \Gamma_0(S, S')] = [\Gamma_{\mathcal{D}'} : \Gamma_{\mathcal{D}'^*}] \cdot [\Gamma_{\mathcal{D}'^*} : \Gamma_0(S, S')]/[\Gamma_{\mathcal{D}'} : \Gamma_2] = h_{2, k}.$$

We now define a homomorphism

$$(8.6) \quad F : \Gamma_{S, S'} \rightarrow \prod_{v \in S} (\mathbf{Z}/2)$$

by sending $\sigma \in \Gamma_{S, S'}$ to the vector having component 0 at $v \in S$ if σ is even at v and component 1 if v is odd at v . The kernel of F is

$$\Gamma_2 = \Gamma_{\emptyset, S'} \cap \Gamma_{S, S'}$$

so

$$(8.7) \quad [\Gamma_{S, S'} : \Gamma_2] = \#\mathrm{Image}(F)$$

Consider the homomorphism

$$(8.8) \quad T : \prod_{v \in S} (\mathbf{Z}/2) \rightarrow \mathrm{Cl}(k)/(2\mathrm{Cl}(k))$$

which sends the vector having component 1 at v and component 0 at the other places in S to the class of the prime ideal $\mathcal{P}(v)$. We will show that

$$(8.9) \quad \mathrm{Image}(F) = \mathrm{Kernel}(T).$$

Before proving (8.9) note that in the statement of Proposition 6.2, $\mathrm{Image}(T)$ has order 2^n . Thus (8.7) and (8.9) will show

$$(8.10) \quad [\Gamma_{S, S'} : \Gamma_2] = \#\mathrm{Image}(F) = \#\mathrm{Kernel}(T) = 2^{\#S}/\#\mathrm{Image}(T) = 2^{\#S-n}$$

Hence (8.5) and (8.10) prove (6.1), which will prove Proposition 6.2.

It remains to show (8.9). If $M \in \mathrm{GL}_2(k)$ represents $\sigma \in \Gamma_{S, S'}$, then $\det(M) \in k^*$ is even at all $v \notin S$, and $\mathrm{ord}_v(\det(M))$ is even (resp. odd) exactly if the component of $F(\sigma)$ at v is 0 (resp. 1). Since $\det(M)$ generates a principal ideal, it follows that the composition $T \circ F$ is trivial, so $\mathrm{Image}(F) \subset \mathrm{Kernel}(T)$.

To show equality in (8.9), it will now suffice to show the following. Suppose $\lambda \in k^*$ has $\text{ord}_v(\lambda) \equiv 0 \pmod{2\mathbf{Z}}$ for $v \notin S$. Then we need to show there is an element $\sigma \in \Gamma_{S,S'}$ which for $v \in S$ is odd at v if and only if $\text{ord}_v(\lambda)$ is odd. Without loss of generality, we can assume $\lambda \in O$. Fix a uniformizing element $\pi_v \in O_v$ for each place v . We can choose an element $c \in O$ satisfying the following finite system of congruences:

$$(8.11) \quad \text{If } \text{ord}_v(\lambda) = 2a_v + 1 \text{ is odd, then } c \equiv 0 \pmod{\pi_v^{a_v+1}O_v};$$

$$(8.12) \quad \text{If } \text{ord}_v(\lambda) = 2a_v \text{ is even, and } a_v > 0 \text{ or } v \in S \cup S', \text{ then } c \equiv \lambda \cdot \pi_v^{-a_v} - \pi_v^{a_v} \pmod{\pi_v^{a_v+1}O_v}.$$

We let σ' be the matrix

$$(8.13) \quad \sigma' = \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix}.$$

Clearly $\det(\sigma') = -\lambda$ is odd at exactly the places v of k where $\text{ord}_v(\lambda)$ is odd. We will show below that

$$(8.14) \quad \sigma' \text{ fixes an edge of } T_v \text{ if } v \in S \cup S' \text{ or } \text{ord}_v(\lambda) \neq 0.$$

Let us first prove that (8.14) implies equality in (8.9), which we have already proved will complete the proof of Proposition 6.2. If $v \in S$, then (8.14) states that σ' fixes an edge of T_v . If $v \in S'$ or $\text{ord}_v(\lambda) \neq 0$, then σ' must fix an edge of T_v pointwise, since (8.14) says σ' fixes an edge, and σ' is even at v . Finally, if $v \notin S \cup S'$ and $\text{ord}_v(\lambda) \neq 0$, then σ' lies in $\text{GL}_2(O_v)$, so σ' fixes the vertex of T_v that is fixed by every element of $\Gamma_{S,S'}$. The group $\text{SL}_2(k_v)$ acts transitively on the edges of T_v . Hence we can conclude from the Strong Approximation Theorem that a conjugate σ of σ' by an element of $\text{SL}_2(k)$ defines an element of $\Gamma_{S,S'}$. Since $\det(\sigma) = \det(\sigma')$ has odd valuation at exactly those v where $\text{ord}_v(\lambda)$ is odd, this implies equality holds in (8.9).

We now prove (8.14). Suppose first that v is a place for which $\text{ord}_v(\lambda) = 2a_v + 1$ is odd, so that v lies in S . Condition (8.11) implies that σ' acts in the following way on the lattices $\pi_v^{a_v}O_v \oplus O_v$ and $\pi_v^{a_v+1}O_v \oplus O_v$ in $k_v \oplus k_v$.

$$(8.15) \quad \sigma' \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix} \cdot \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} \lambda O_v \\ \pi_v^{a_v}O_v \end{pmatrix} = \pi_v^{a_v} \begin{pmatrix} \pi_v^{a_v+1}O_v \\ O_v \end{pmatrix}.$$

$$(8.16) \quad \sigma' \begin{pmatrix} \pi_v^{a_v+1}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix} \cdot \begin{pmatrix} \pi_v^{a_v+1}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} \lambda O_v \\ \pi_v^{a_v+1}O_v \end{pmatrix} = \pi_v^{a_v+1} \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix}.$$

These equalities show that σ' interchanges the homothety classes of $\pi_v^{a_v}O_v \oplus O_v$ and $\pi_v^{a_v+1}O_v \oplus O_v$. Hence σ' fixes the edge of T_v between these homothety classes (though it clearly does not fix this edge pointwise).

Now suppose that $\text{ord}_v(\lambda) = 2a_v$ is even and $a_v > 0$ or $v \in S \cup S'$. In all cases we have $a_v \geq 0$, since $\lambda \in O$. Condition 8.12 implies $c \equiv 0 \pmod{\pi_v^{a_v}O_v}$. Hence

$$(8.17) \quad \sigma' \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix} \cdot \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix} = \begin{pmatrix} \lambda O_v \\ \pi_v^{a_v}O_v \end{pmatrix} = \pi_v^{a_v} \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix}.$$

Thus σ' fixes the homothety class of $\pi_v^{a_v}O_v \oplus O_v$, so it will now suffice to show σ' fixes the homothety class of an O_v lattice L containing $\pi_v^{a_v}O_v \oplus O_v$ for which $L/(\pi_v^{a_v}O_v \oplus O_v)$ is O_v -isomorphic to $k(v) = O_v/\pi_v O_v$. We now compute

$$(8.18) \quad \begin{aligned} \sigma' \begin{pmatrix} \pi_v^{a_v-1} \\ \pi_v^{-1} \end{pmatrix} &= \begin{pmatrix} 0 & \lambda \\ 1 & c \end{pmatrix} \cdot \begin{pmatrix} \pi_v^{a_v-1} \\ \pi_v^{-1} \end{pmatrix} = \begin{pmatrix} \lambda \pi_v^{-1} \\ \pi_v^{a_v-1} + c \pi_v^{-1} \end{pmatrix} \\ &= \pi_v^{a_v} \cdot \begin{pmatrix} \lambda \pi_v^{-2a_v} \pi_v^{a_v-1} \\ (1 + c \pi_v^{-a_v}) \pi_v^{-1} \end{pmatrix} \\ &\equiv \pi_v^{a_v} \cdot \lambda \pi_v^{-2a_v} \cdot \begin{pmatrix} \pi_v^{a_v-1} \\ \pi_v^{-1} \end{pmatrix} \pmod{\pi_v^{a_v} \begin{pmatrix} \pi_v^{a_v}O_v \\ O_v \end{pmatrix}}. \end{aligned}$$

where the last congruence results from the condition on c in (8.12). Here $\lambda \cdot \pi_v^{-2a_v}$ is a unit of O_v , so we can take the lattice L to be the one generated as an O_v -module by $\pi_v^{a_v} O_v \oplus O_v$ and the vector $(\pi_v^{a_v-1}, \pi_v^{-1})$. This completes the proof of Proposition 6.2.

9. CUSPS AND CLASS NUMBERS.

We begin by giving an ineffective proof of part (1) of Corollary 1.2. Recall that $C(N)$ is the set of isometry classes of minimal finite covolume discrete arithmetic hyperbolic 3-orbifolds having exactly N cusps. The finite covolume discrete arithmetic subgroups of $\mathrm{PGL}_2(k)$ are commensurable. Hence to show the elements of $C(N)$ represent only finitely many distinct commensurability classes, it will suffice by Theorem 1.1 to show that there are only finitely many imaginary quadratic fields k such that $h_k/h_{k,2} \leq N$. Siegel proved in [8] that for each $\epsilon > 0$, there is an ineffective constant $c(\epsilon) > 0$ such that

$$(9.1) \quad h_k > c(\epsilon) |d_k|^{\frac{1}{2}-\epsilon}$$

where d_k is the discriminant of k . By a result of Tatzuzaawa, the constant $c(\epsilon)$ can be made effective except for at most one exceptional field k ; see [9] and [5]. Let n_k be the number of distinct prime factors of d_k . By genus theory, the two-rank of the ideal class group of k is equal to 2^{n_k-1} . Thus $h_{2,k} = 2^{n_k-1}$ and we get

$$(9.2) \quad \frac{h_k}{h_{k,2}} > c(\epsilon) \frac{|d_k|^{\frac{1}{2}-\epsilon}}{2^{n_k-1}} > c(\epsilon) \prod_{p|d_k} \frac{p^{\frac{1}{2}-\epsilon}}{2}.$$

The fact that there are only finitely many k for which $h_k/h_{k,2} < N$ is clear from (9.2), since $-d_k$ is either a square-free positive integer or 4 times such an integer, and if $\epsilon < 1/2$ then there are only finitely many primes p for which $\frac{p^{\frac{1}{2}-\epsilon}}{2} < 2$.

Suppose now that X is an element of $C(N)$. To show part (2) of Corollary 1.2, we must show that there are infinitely many elements of the commensurability class of X which also lie in $C(N)$. By Borel's work, $X = \mathbf{H}^3/\Gamma_{S,S'}$ for some imaginary quadratic field k and some maximal discrete subgroup $\Gamma_{S,S'} \subset \mathrm{PGL}_2(k)$. Theorem 1.1 shows

$$(9.3) \quad 2^n \frac{h_k}{h_{k,2}} = N$$

where 2^n is the order of the subgroup of $Cl(k)/2Cl(k)$ generated by the places in S . We now let S_0 be a set of n places whose images in $Cl(k)/2Cl(k)$ generate a subgroup of order 2^n ; such an S_0 exists by the Chebotarev density theorem. Let W be the set of finite places v of k such that

$$(9.4) \quad \mathcal{P}(v) \cdot \mathcal{P}(S_0) = \mathcal{P}(S_0 \cup \{v\})$$

is principal. The Chebotarev density theorem also implies W is infinite. We claim that for $v \in W$, the group $\Gamma_{S_0 \cup \{v\}, \emptyset}$ contains an element σ_v which is odd at $S_0 \cup \{v\}$. To construct σ_v , let λ_v be a generator for the ideal in (9.4). We can then take σ_v to be

$$(9.5) \quad \sigma_v = \begin{pmatrix} 0 & \lambda_v \\ 1 & 0 \end{pmatrix}.$$

We now see from [3, Prop. 4.4(iii)] that if $\Gamma_{S_0 \cup \{v\}, \emptyset}$ is not maximal, then it is conjugate to subgroup of a maximal discrete subgroup of the form $\Gamma_{S_0 \cup \{v\}, S'(v)}$ for some finite set of places $S'(v)$ which is disjoint from $S_0 \cup \{v\}$. We conclude that for each $v \in W$, there is a maximal discrete group $\Gamma_{S_0 \cup \{v\}, S'(v)}$ which contains an element which is odd at v . Furthermore, the fact that (9.4) is principal implies that $\{\mathcal{P}(v') : v' \in S_0\}$ and $\{\mathcal{P}(v') : v' \in S_0\} \cup \{\mathcal{P}(v)\}$ generate the same subgroup of $Cl(k)/2Cl(k)$, which by hypothesis has order 2^n . Thus Theorem 1.1 shows $\mathbf{H}^3/\Gamma_{S_0 \cup \{v\}, S'(v)}$ has exactly N cusps, were N is as in (9.3). The orbifolds $\mathbf{H}^3/\Gamma_{S_0 \cup \{v\}, S'(v)}$ and $\mathbf{H}^3/\Gamma_{S_0 \cup \{v'\}, S'(v')}$ are not isometric for distinct v and v' in $W - S_0$, since the group $\Gamma_{S_0 \cup \{v\}, S'(v)}$ contains no elements

which are odd at v' and similarly with the roles of v and v' reversed (cf. [3, Prop. 4.4(ii)]). This completes the proof that $C(N)$ contains infinitely many elements which are commensurable to X .

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