

# Computing varieties of representations of hyperbolic 3-manifolds into $\mathrm{SL}(4, \mathbb{R})$

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## 1 Introduction

Following the seminal work of M. Culler and P. Shalen [Culler & Shalen, 1983], and that of A. Casson [Akbulut & McCarthy, 1990], the theory of representation and character varieties of 3-manifolds has come to be recognized as a powerful tool, and has duly assumed an important place in low-dimensional topology. Among the many papers that have appeared in this context, we mention [Culler et al., 1987, Cooper et al., 1994, Boyer & Zhang, 1998]. Most of the work carried out to date is concerned with representations into Lie groups of  $2 \times 2$  matrices, owing mainly to connections with actions on trees and the isometry groups of hyperbolic space in dimensions 2 and 3, but also owing to the extreme difficulty of computations beyond the realm of such matrices.

This paper was originally motivated by the following question: under what circumstances can one take the hyperbolic structure on a closed hyperbolic 3-manifold and deform it to a real projective structure? In the language of representations, this amounts to beginning with an  $\mathrm{SO}^+(3, 1)$ -representation  $\phi_0$  of the fundamental group of the manifold, given by the hyperbolic structure, and then endeavouring to compute the component of the  $\mathrm{SL}(4, \mathbb{R})$ -representation variety containing  $\phi_0$ . Using a computer, it is relatively easy to see that for many closed hyperbolic 3-manifolds there are linear obstructions to deforming, but when these obstructions vanish it is of considerable interest to see whether genuine deformations exist. The

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purpose of this article, then, is to describe a method for the exact computation of the representation varieties of closed hyperbolic 3-manifolds into  $\mathrm{SL}(4, \mathbb{R})$ . For simplicity we restrict our attention to orientable manifolds with 2-generator fundamental groups.

The technique is sufficiently practical that we have used it to compute 21 varieties of this type exactly, and have investigated numerically the first 4500 closed orientable manifolds with 2-generator groups in the Hodgson-Weeks census [Hodgson & Weeks, 2000]. Numerical evidence strongly suggests that only 52 of these 4500 manifolds admit non-trivial deformations of  $\phi_0$  into  $\mathrm{SL}(4, \mathbb{R})$ . When these deformations do occur, they lead to a number of interesting constructions, including families of real projective structures on the manifold and families of discrete faithful representations into  $\mathrm{PU}(3, 1)$ , the group of orientation-preserving isometries of 3-dimensional complex hyperbolic space. These and other theoretical aspects are considered in more depth in [Cooper et al., 2005]; in this paper we concentrate on the computational aspects of the investigation.

The presence of an embedded totally geodesic surface in the manifold guarantees the existence of a well-established type of deformation known as *bending*, but owing to their small volume the census manifolds cannot contain such surfaces [Kojima & Miyamoto, 1991]. The underlying reason for the sporadic occurrence of these deformations is still a mystery.

In order to make the paper reasonably self-contained, there now follows a section summarizing the background material.

## 2 Background

### 2.1 The Minkowski model for hyperbolic space

A hyperbolic structure on a 3-manifold  $M$  corresponds to a discrete faithful representation of the fundamental group of  $M$  into the group of isometries of 3-dimensional hyperbolic space  $\mathbb{H}^3$ . In the case where  $M$  is closed, by Mostow rigidity the structure is unique up to isometry (if  $M$  is non-compact and of finite volume, we also have uniqueness if we add the requirement that the geometric structure be complete).

In this article we shall focus our attention on closed orientable hyperbolic 3-manifolds  $M$ . Thus the geometric structure on  $M$  corresponds to a discrete faithful representation  $\phi_0 : \pi_1(M) \rightarrow \mathrm{Isom}^+ \mathbb{H}^3$ , unique up to conjugacy; here  $\mathrm{Isom}^+ \mathbb{H}^3$  denotes the group of orientation-preserving isometries of  $\mathbb{H}^3$ . In the upper-half space model for  $\mathbb{H}^3$ , the group  $\mathrm{Isom}^+ \mathbb{H}^3$  is naturally identified with  $\mathrm{PSL}(2, \mathbb{C})$ , the group of Möbius transforma-

tions of the boundary, but for us it will be more propitious to work in the *Minkowski model* for  $\mathbb{H}^3$ , as our intention is to consider  $\text{Isom}^+ \mathbb{H}^3$  as a subgroup of  $\text{SL}(4, \mathbb{R})$  and search for deformations of the composite homomorphism  $\pi_1(M) \xrightarrow{\phi_0} \text{Isom}^+ \mathbb{H}^3 \hookrightarrow \text{SL}(4, \mathbb{R})$ .

*Minkowski space of dimension  $n + 1$* , denoted  $\mathbb{M}^{n+1}$ , is the real vector space  $\mathbb{R}^{n+1}$  endowed with a quadratic form  $Q$  of signature  $(n, 1)$ , which we take without loss of generality to be

$$Q(x_0, x_1, \dots, x_n) = -x_0^2 + x_1^2 + \dots + x_n^2 .$$

The group of isometries of  $\mathbb{M}^{n+1}$  is the *Lorentz group*  $\text{O}(n, 1)$ ; it consists of those  $(n+1) \times (n+1)$  matrices  $A$  satisfying  $A^{-1} = FA^tF$ , where the superscript  $t$  denotes transpose, and where  $F$  is the diagonal  $(n+1) \times (n+1)$  matrix with entries  $-1, 1, \dots, 1$ . An excellent reference is [Epstein & Penner, 1988].

Let  $\mathbf{x}$  denote  $(x_0, x_1, \dots, x_n) \in \mathbb{M}^{n+1}$ . In the Minkowski model, *real hyperbolic  $n$ -space* is the “upper sheet” of the hyperboloid  $\mathbb{H}^n = \{\mathbf{x} : Q(\mathbf{x}) = -1\}$ , namely the set of points of this hyperboloid for which  $x_0 > 0$ . We note that  $\mathbb{H}^n$  is asymptotic to the *light cone*  $C = \{\mathbf{x} : Q(\mathbf{x}) = 0\}$ , and that  $Q(\mathbf{x}) > 0$  if and only if the vector  $\mathbf{x}$  points in a direction outside  $C$ . It is not hard to see that the tangent vectors at any point of  $\mathbb{H}^n$  satisfy this condition, whence the  $(3, 1)$ -form of  $\mathbb{M}^{n+1}$  is positive definite on the tangent bundle of  $\mathbb{H}^n$ ; indeed it induces the expected Riemannian metric of constant negative curvature. The *boundary* of  $\mathbb{H}^n$  in the Minkowski model is the space whose points are rays of the light cone  $C$ . The group  $\text{Isom}^+ \mathbb{H}^n$  is the subgroup  $\text{SO}^+(n, 1)$  of  $\text{O}(n, 1)$  consisting of linear transformations that (i) have determinant 1, and (ii) preserve the sheets of the hyperboloid  $\{\mathbf{x} : Q(\mathbf{x}) = -1\}$ .  $\text{SO}^+(n, 1)$  has index 4 in  $\text{O}(n, 1)$ , and is the component subgroup of the identity.

Let us specialize to the case  $n = 3$ . The assignment

$$(t, x, y, z) \mapsto \begin{bmatrix} t + z & x + iy \\ x - iy & t - z \end{bmatrix}$$

defines an  $\mathbb{R}$ -vector space isomorphism from  $\mathbb{R}^4$  to the space  $\mathcal{H}^2$  of  $2 \times 2$  Hermitian matrices over the complex numbers; the quadratic form in  $\mathcal{H}^2$  corresponding to  $Q$  under this isomorphism is simply the negative of the determinant. We therefore have an isomorphism  $\mathbb{M}^4 = (\mathbb{R}^4, Q) \approx (\mathcal{H}^2, -\det)$ . It is then easy to construct an isomorphism from  $\text{PSL}(2, \mathbb{C})$  to the group of isometries of  $\mathbb{M}^4$ : we simply assign to each matrix  $A \in \text{SL}(2, \mathbb{C})$  the map  $H \mapsto A^*HA$  ( $H \in \mathcal{H}^2$ ), where  $*$  denotes Hermitian transpose. We may

construct an explicit isomorphism from  $\mathrm{PSL}(2, \mathbb{C})$  to  $\mathrm{SO}^+(3, 1)$  by choosing a specific basis for  $\mathcal{H}^2$ , for example

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \right\},$$

and then computing the matrix of the linear transformation  $H \mapsto A^*HA$  with respect to this basis.

## 2.2 Eigensystems of isometries of hyperbolic space

The isomorphism given above may be used to determine the eigenvectors and corresponding eigenvalues of the various kinds of isometry of  $\mathbb{H}^3$  in the Minkowski model. The situation may be summarized as follows.

**(i) The eigensystem of a loxodromic.** Let  $g \in \mathrm{Isom}^+ \mathbb{H}^3$  be a loxodromic; then  $g$  acts freely on  $\mathbb{H}^3$ , and fixes two points on the boundary of  $\mathbb{H}^3$ . In the upper-half space model, the boundary is identified with the extended complex plane, and with suitable choice of coordinates we may assume that the fixed points of  $g$  are  $0, \infty$ . Then  $g$  corresponds to a dilation  $z \mapsto az$  on the boundary, where the complex number  $a$  is the *dilation factor* of  $g$  (see [Maskit, 1980]). We may think of  $g$  as the composition of a “pure translation” with axis the geodesic joining  $0$  and  $\infty$ , together with a rotation (elliptic) about that axis through an angle  $\arg(a)$ . The isometry  $g$  corresponds to the element of  $\mathrm{PSL}(2, \mathbb{C})$  represented by the matrix  $\begin{bmatrix} a^{1/2} & 0 \\ 0 & 1/a^{1/2} \end{bmatrix}$ , where  $a^{1/2}$  is either of the square roots of  $a$ .

Application of the above isomorphism yields the following matrix in  $\mathrm{SO}^+(3, 1)$ :

$$\begin{bmatrix} p & q & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & r & s \\ 0 & 0 & -s & r \end{bmatrix}, \quad \text{where} \quad \begin{cases} p = \frac{1}{2} \left( |a| + \frac{1}{|a|} \right), \\ q = \frac{1}{2} \left( |a| - \frac{1}{|a|} \right), \\ r = \Re \left( \frac{a}{|a|} \right), \\ s = \Im \left( \frac{a}{|a|} \right). \end{cases}$$

The eigenvalues of this matrix are  $|a|, \frac{1}{|a|}, e^{i\theta}, e^{-i\theta}$ , where  $\theta = \arg(a)$ . The eigenspaces of the real eigenvalues  $|a|, \frac{1}{|a|}$  are the two rays of the light cone identified with the fixed points of  $g$ . Indeed the subspace spanned by these two rays meets  $\mathbb{H}^3$  precisely in the axis of  $g$ . The pairing of each eigenvalue with its inverse corresponds to the fact that the axis of  $g$  admits

two orientations. It can happen that  $\theta = 0$ , in which case the isometry is a pure translation.

**(ii) The eigensystem of an elliptic.** Let  $g$  be an elliptic isometry. Since we may regard  $g$  as a degenerate loxodromic, with dilation factor on the unit circle, the corresponding matrix in  $\mathrm{SO}^+(3, 1)$  will have eigenvalues  $1, e^{i\theta}, e^{-i\theta}$ , where  $\theta$  is the angle of rotation, as before. The eigenspace of the eigenvalue 1 is the 2-dimensional subspace of  $\mathbb{M}^4$  containing the two rays on the light cone corresponding to the ends of the axis of  $g$ .

**(iii) The eigensystem of a parabolic.** Let  $g$  be parabolic; then  $g$  has one fixed point on the boundary, and  $g$  effects a Euclidean translation along each horosphere centred at this fixed point. If we choose coordinates in the upper-half space model so that the fixed point is  $\infty$ ,  $g$  corresponds to the element of  $\mathrm{PSL}(2, \mathbb{C})$  represented by  $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ , where the complex number  $a$  describes the Euclidean translation. The corresponding matrix in  $\mathrm{SO}^+(3, 1)$  is:

$$\begin{bmatrix} 1+p & p & r & s \\ -p & 1-p & -r & -s \\ r & r & 1 & 0 \\ s & s & 0 & 1 \end{bmatrix}, \quad \text{where} \quad \begin{cases} p = \frac{1}{2}|a|^2, \\ r = \Re a, \\ s = \Im a. \end{cases}$$

This matrix has a single eigenvalue 1 with algebraic multiplicity 4 and geometric multiplicity 2. The eigenspace is spanned by a vector (in this case  $(-1, 1, 0, 0)$ ) pointing along the ray on the light cone corresponding to the single fixed point of  $g$ , and an orthogonal vector (in this case  $(0, 0, -\Im a, \Re a)$ ), whose direction encodes the direction of the Euclidean translation.

### 3 Trace calculus

The ultimate aim of this paper is to compute representation varieties of  $\pi_1(M)$  into  $\mathrm{SL}(4, \mathbb{R})$ . Since any representation can be composed with an inner automorphism of the target group to produce another, we are content to find just one representation in each such equivalence class. In essence we are computing a variety  $\mathcal{V}$  that is an embedded copy of the character variety in the full representation variety.

If the variety  $\mathcal{V}$  has dimension  $n$ , then the image of each generator of  $\pi_1(M)$  is a matrix whose entries are algebraic functions of  $n$  independent parameters  $u_1, \dots, u_n$ . The matrix entries at a generic point of the variety can therefore be considered to lie in a field  $\mathbb{F}$  of transcendence degree  $n$

over  $\mathbb{R}$ , and  $\mathcal{V}$  is then specified by a single “tautological” representation  $\Psi$  into  $\mathrm{SL}(4, \mathbb{F})$ . Individual representations are obtained from  $\Psi$  by evaluating at specific points  $(u_1, \dots, u_n)$  in parameter space. Clearly  $\Psi$  depends on a choice of parametrization of  $\mathcal{V}$ .

Two important fields in this context are (i) the field  $K$  generated by the entries of image matrices, and (ii) the subfield  $T$  of  $K$  generated by the traces of image matrices. By conjugating judiciously, we can guarantee that the field  $K$  (hence also  $T$ ) is algebraic of finite degree over the purely transcendental extension  $\mathbb{Q}(u_1, \dots, u_n)$  of the rationals. The trace field  $T$  is independent of the choice of conjugation, and in practice it is often easy to guess  $T$ . However, in order to compute the variety  $\mathcal{V}$  we shall need to specify generators for  $K$  over the base field  $\mathbb{Q}(u_1, \dots, u_n)$ ; the elementary proposition given below will be helpful in that regard.

In order to state Proposition 1 it will be convenient to introduce some notation. Let  $a = (a_{ij})$  be any  $n \times n$  matrix over a commutative ring  $R$ , and let  $\sigma = (n_1, \dots, n_k)$  be any cyclically ordered sequence of distinct numbers from  $\{1, 2, \dots, n\}$  (thus we regard the sequences  $(n_1, n_2, n_3, \dots, n_k)$ ,  $(n_2, n_3, \dots, n_k, n_1)$  as being identical). Then we define  $a_\sigma$  to be the following element of  $R$ :

$$a_\sigma = a_{n_1 n_2} a_{n_2 n_3} \cdots a_{n_k n_1} .$$

The proof of Proposition 1 is greatly facilitated by having a computer algebra system to hand, for example Mathematica or Maple.

**Proposition 3.1.** *Let  $G$  be a subgroup of  $\mathrm{SL}(4, \mathbb{C})$  generated by matrices  $a, b$ , where  $b$  is a diagonal matrix*

$$b = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} .$$

*Suppose further that*

- (i) *the  $\lambda_i$  are all distinct;*
- (ii)  *$\mathrm{tr}(b) \neq 0$ ;*
- (iii)  *$(\mathrm{tr}(b))^3 \neq \mathrm{tr}(b^3)$ .*

*Let  $T$  be the trace field of  $G$ , and let  $K$  be the field obtained by adjoining  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  to  $T$ . Then for each cycle  $\sigma$  of length 1, 2 or 3 in  $\{1, 2, 3, 4\}$ ,  $a_\sigma \in K$ .*

*Proof.* First we deal with the case where  $\sigma$  has length 1, *i.e.* the diagonal entries  $a_{ii}$  ( $1 \leq i \leq 4$ ). It is readily checked that the four traces  $\text{tr}(a)$ ,  $\text{tr}(ab)$ ,  $\text{tr}(ab^2)$ ,  $\text{tr}(ab^3)$  are all linear expressions in the entries  $a_{ii}$  over the field  $K$ . We therefore have a system of linear equations for the  $a_{ii}$  over  $K$ , and it is easily verified (using *e.g.* Mathematica) that the determinant of the matrix of coefficients is  $\prod_{i < j} (\lambda_i - \lambda_j)$ . Therefore, from the hypothesis that the  $\lambda_i$  are distinct, the system has a (unique) solution and the  $a_{ii}$  have been shown to lie in the field  $K$ .

Next, consider the six products corresponding to cycles of length 2, namely

$$a_{12}a_{21}, a_{13}a_{31}, a_{14}a_{41}, a_{23}a_{32}, a_{24}a_{42}, a_{34}a_{43} .$$

Let  $B$  denote the inverse of  $b$ . The trace of any word in the generators  $a, b$  involving precisely two occurrences of  $a$  (and no occurrences of  $a^{-1}$ ) is a linear expression in these six products with coefficients in the field  $T(\lambda_1, \lambda_2, \lambda_3, \lambda_4, a_{11}, a_{22}, a_{33}, a_{44})$ , which we now know to be equal to  $K$ . The six traces

$$\text{tr}(aa), \text{tr}(aab), \text{tr}(aaB), \text{tr}(aabb), \text{tr}(abab), \text{tr}(aBab)$$

therefore give rise to a linear system in the six products over this field, and one can verify that the determinant of the matrix of coefficients is  $4 \prod_{i < j} (\lambda_i - \lambda_j)^2$ . Again, this determinant is non-zero, and it follows that the six products corresponding to cycles of length 2 are in the field  $K$ .

The eight products corresponding to cycles of length 3 are dealt with similarly, using the traces of the words

$$aaa, aaab, aaabb, aaBab, aabaB, aaBabb, abaBabb, abaBBabb .$$

This time the determinant of the  $8 \times 8$  matrix of coefficients is

$$\left( \prod_{i < j} (\lambda_i - \lambda_j)^4 \right) \text{tr}(b) ((\text{tr}(b))^3 - \text{tr}(b^3)) ,$$

and the result follows.  $\square$ .

**Remark 3.1.** The choice of words at each stage of the proof of Proposition 1 is certainly not unique, but the eight words used for the last stage need to be chosen quite carefully in order that the matrix of coefficients should have full rank.

**Remark 3.2.** If  $M$  is a closed, orientable, hyperbolic 3-manifold, then each non-trivial element  $g \in \pi_1(M)$  is loxodromic. Let  $\phi_0 : \pi_1(M) \rightarrow \mathrm{SO}^+(3, 1)$  be the representation given by the geometric structure; then the eigenvalues of  $\phi_0(g)$  are distinct if and only if  $g$  is not a pure translation. Therefore, if  $b$  is the diagonalization of  $\phi_0(g)$  and  $g$  has a non-trivial rotational component, then at least condition (i) of Proposition 1 is met. Condition (ii) is automatically met as the trace of  $\phi_0(g)$  cannot equal zero; this follows directly from the fact, explained in §2.2, that  $\phi_0(g)$  has eigenvalues  $|a|, \frac{1}{|a|}, e^{i\theta}, e^{-i\theta}$  with  $|a| \neq 1$ . Condition (iii) has also been met for all manifolds we have investigated. Furthermore, these three properties are “open conditions”, so if they are satisfied for  $\phi_0(g)$ , they are also satisfied for  $\phi_1(g)$ , given that  $\phi_1$  is sufficiently close to  $\phi_0$ .

**Remark 3.3.** It often happens in practice that the characteristic polynomial of the image of  $g \in \pi_1(M)$  under the tautological representation  $\Psi$  is *reciprocal*, meaning that its roots are in reciprocal pairs; equivalently, the coefficients of the polynomial form a palindromic sequence. If  $\phi_0(g)$  is not a pure translation, and if  $\phi$  is an evaluation of  $\Psi$  close to the  $\mathrm{SO}^+(3, 1)$  representation  $\phi_0$ , then the two “rotational” eigenvalues  $\lambda_3, \lambda_4$  of  $\phi(g)$  are non-real. If  $\lambda_2 = 1/\lambda_1$  and  $\lambda_4 = 1/\lambda_3$ , then we have  $\lambda_3 + 1/\lambda_3 \in T(\lambda_1)$ , whence  $\lambda_3, 1/\lambda_3$  are roots of a quadratic over the field  $T(\lambda_1)$ . Therefore  $K$  has degree 2 over  $T(\lambda_1)$ , and  $T(\lambda_1) = K \cap \mathbb{R}$ .

The hypotheses of the following Corollary are obviously not best possible, but the Corollary suits our purpose in obtaining representations for which the matrix entry field is algebraic over  $\mathbb{Q}(u_1, \dots, u_n)$ .

**Corollary 3.1.** *Let  $G, K$  be as in Proposition 1, and let us impose the extra hypothesis that all off-diagonal entries of  $a$  are non-zero. Let  $c$  be the matrix*

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{12}a_{23} & 0 \\ 0 & 0 & 0 & a_{12}a_{23}a_{34} \end{bmatrix}.$$

*Then the field generated by the matrix entries of  $c G c^{-1}$  is precisely  $K$ .*

*Proof.* The group  $c G c^{-1}$  is generated by  $cac^{-1} = a'$  say, and  $cbc^{-1} = b$ . Let  $K'$  be the field generated by the matrix entries of  $c G c^{-1}$ . Since conjugation by  $c$  has not affected traces or the diagonal matrix  $b$ ,  $K$  is contained in  $K'$ . On the other hand, the (1, 2), (2, 3), (3, 4) entries of the matrix  $a'$  are all equal to 1, whence the matrix entries of  $a'$  are all expressible as products of the  $a'_\sigma$  and their inverses. We now apply Proposition 1 to the group  $\langle a', b \rangle$ ,

deducing that each entry of  $a'$  is in  $K$ ; it then follows at once that  $K' \subset K$ .  
 $\square$

## 4 Canonical form for representations

In this section we explain how to conjugate a representation  $\phi : \pi_1(M) \rightarrow \mathrm{SL}(4, \mathbb{R})$  into a convenient “standard” form. As always, we are assuming that  $M$  is a closed, orientable, hyperbolic 3-manifold and that its fundamental group is generated by a pair of elements  $\alpha, \beta$ . Let  $a, b$  denote the images of  $\alpha, \beta$  respectively under  $\phi$ , and let  $a_0, b_0$  denote the corresponding images under the “base” representation  $\phi_0 : \pi_1(M) \rightarrow \mathrm{SO}^+(3, 1)$ . Since all non-trivial elements of  $\pi_1(M)$  are loxodromic, the matrix  $b_0$  is diagonalizable; also, as explained in §2.2,  $b_0$  has two distinct real eigenvalues and two other mutually conjugate eigenvalues. If  $\phi$  is sufficiently close to  $\phi_0$ , continuity together with the fact that the characteristic polynomial of  $b$  has real coefficients ensures that the matrix  $b$  enjoys the same properties. We make the further assumption that the diagonalization of  $b$  satisfies conditions (i), (ii), (iii) of Proposition 1. As explained in Remark 3.2 above, condition (ii) is automatically met for representations  $\phi$  close to  $\phi_0$ , and in all observed cases the other two conditions are also met; should this happen not to be so, one could resort to changing the generating set for  $\pi_1(M)$ .

Let the two real eigenvalues of  $b$  be  $\lambda_1, \lambda_2$ , and let the other two eigenvalues be  $\lambda_3, \overline{\lambda_3}$ . Since  $b$  is an automorphism of  $\mathbb{R}^4$ , we may choose real eigenvectors  $\mathbf{v}_1, \mathbf{v}_2$  for  $\lambda_1, \lambda_2$  respectively, and eigenvectors  $\mathbf{v}_3, \overline{\mathbf{v}_3}$  for  $\lambda_3, \overline{\lambda_3}$ . Since these four eigenvectors form a linearly independent set over  $\mathbb{C}$ , the set of real vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + \overline{\mathbf{v}_3}, i(\mathbf{v}_3 - \overline{\mathbf{v}_3})\}$  is also linearly independent over  $\mathbb{C}$ , hence also over  $\mathbb{R}$ . Therefore each vector in  $\mathbb{R}^4$  is uniquely expressible as a real linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + \overline{\mathbf{v}_3}, i(\mathbf{v}_3 - \overline{\mathbf{v}_3})$ , or alternatively as a linear combination  $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \overline{k_3}\overline{\mathbf{v}_3}$  where  $k_1, k_2$  are real.

Let us now consider the matrices  $a_1, b_1 \in \mathrm{SL}(4, \mathbb{C})$  representing the same linear transformations as  $a, b$  respectively, but with respect to the basis of eigenvectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \overline{\mathbf{v}_3}\}$ . Then  $b_1$  is diagonal, and from the discussion in the previous paragraph, the entries of the upper left  $2 \times 2$  submatrix of  $a_1$  are real. Assuming that the off-diagonal entries of  $a_1$  are non-zero, we see that we may adjust  $\mathbf{v}_2$  by a real scalar so that the  $(1, 2)$  entry of  $a_1$  is 1. We may independently adjust the last two eigenvectors by scalars so that the  $(2, 3)$  and  $(3, 4)$  entries are also 1. Then, from Corollary 1.1, all entries of  $a_1$  lie in the field  $K$  obtained by adjoining the eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  to the trace field  $T$ .

We can improve matters slightly by conjugating  $a_1, b_1$  to new matrices  $a_2, b_2$  by means of a further change of basis, namely to  $\{\mathbf{v}_1, \mathbf{v}_2, a(\mathbf{v}_1), a(\mathbf{v}_2)\}^1$ . Note that each of these basis vectors is real, so the resulting matrices have real entries. Indeed they have the convenient form

$$a_2 = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \quad b_2 = \begin{bmatrix} \lambda_1 & 0 & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

Since the transition matrix corresponding to this latest change of basis has columns

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad a_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

its entries are in  $K$ , and we infer that the entries of  $a_2, b_2$  are in  $K \cap \mathbb{R}$ . In particular, if the characteristic polynomial of  $b$  is reciprocal, by Remark 3.3  $a_2, b_2$  are matrices over the field  $T(\lambda_1)$ .

**Remark 4.1.** Matrices with reciprocal characteristic polynomials are particularly desirable, as their eigenvalues admit relatively simple expressions in terms of the coefficients. Indeed, the roots of  $1 + px + qx^2 + px^3 + x^4$  are

$$\frac{1}{4} \left( -p - \sqrt{p^2 - 4q + 8} \pm \sqrt{2 \left( p^2 - 2q - 4 + p\sqrt{p^2 - 4q + 8} \right)} \right),$$

$$\frac{1}{4} \left( -p + \sqrt{p^2 - 4q + 8} \pm \sqrt{2 \left( p^2 - 2q - 4 - p\sqrt{p^2 - 4q + 8} \right)} \right),$$

as is easily verified by substituting  $y = x + 1/x$ .

## 5 Infinitesimal and actual deformations

Since a representation of  $\pi_1(M)$  into a linear group  $G$  is determined by its effect on the generators of  $\pi_1(M)$ , we may consider the representation variety  $\text{Hom}(\pi_1(M), G)$  as being a subspace of the  $k$ -fold product of  $G$ , where  $k$  is

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<sup>1</sup>In practice, this set of vectors has always been found to be linearly independent within a neighbourhood of  $\phi_0$ .

the number of generators. In particular, if  $k = 2$  and  $G = \mathrm{SL}(4, \mathbb{R})$ , we consider the variety to be a subspace of  $\mathrm{SL}(4, \mathbb{R}) \times \mathrm{SL}(4, \mathbb{R})$ , which in turn embeds naturally into  $\mathbb{R}^{32}$  if we use matrix entries as coordinates. Since  $M$  is closed,  $\pi_1(M)$  has deficiency zero (recall that the *deficiency* of a finitely presented group is the minimum over all finite presentations of ( $\#$  generators  $- \#$  relators) ).

Given a closed orientable hyperbolic 3-manifold  $M$ , the first step in deciding whether  $\phi_0 : \pi_1(M) \rightarrow \mathrm{SO}^+(3, 1)$  deforms into  $\mathrm{SL}(4, \mathbb{R})$  is to linearize the problem and see whether there exist perturbations  $\phi : \pi_1(M) \rightarrow \mathrm{GL}(4, \mathbb{R})$  of  $\phi_0$  that preserve the group relations to first order. Such perturbations are called *infinitesimal deformations* or *deformations to first order*, and form the so-called *Zariski tangent space* at  $\phi_0$ .

There now follows a brief review of the Zariski tangent space. Our discussion is specific to the current context, but is easy to generalize. Suppose that we have a group  $G = \langle x, y \mid R_1, R_2 \rangle$  and that we wish to investigate the possibility that a representation  $\rho : G \rightarrow \mathrm{SL}(4, \mathbb{R})$  can be smoothly deformed. If such a deformation were possible, there would exist a smooth path of representations  $(x_t, y_t)$  for which  $(x_0, y_0)$  is the given initial representation  $(\rho(x), \rho(y))$ . Clearly, we must have  $R_1(x_t, y_t) = I$ ,  $R_2(x_t, y_t) = I$  all along the path, but this condition is usually hard to work with; indeed, it is the main concern of this article. It is much easier to focus on the corresponding linearized condition at  $t = 0$ , thereby obtaining a necessary (but not sufficient) condition for the existence of deformations. Writing  $x_t = v_t x_0$ ,  $y_t = w_t y_0$ , we see that a smooth path  $(x_t, y_t)$  corresponds to pair of paths  $v_t, w_t$  in the Lie group  $\mathrm{SL}(4, \mathbb{R})$  through the identity. Let  $\mathfrak{v}, \mathfrak{w}$  be the tangent vectors at  $I$  to the paths  $v_t, w_t$ , respectively. The vectors  $\mathfrak{v}, \mathfrak{w}$  are elements of the Lie algebra  $\mathfrak{sl}(4, \mathbb{R})$ , and satisfy a certain linear condition easily obtainable from the group relators  $R_1, R_2$  by taking Fox derivatives. The *Zariski tangent space* for the deformation problem is the subspace of  $\mathfrak{sl}(4, \mathbb{R}) \oplus \mathfrak{sl}(4, \mathbb{R})$  consisting of all ordered pairs  $(\mathfrak{v}, \mathfrak{w})$  satisfying this linear condition. Intuitively, these ordered pairs can be considered as “infinitesimal deformations”. There is of course no guarantee that a given vector in the Zariski tangent space is integrable, *i.e.* that it corresponds to an actual deformed representation; indeed, there is in general an infinite sequence of obstructions that have to be surmounted (see [Kapovich, 2000], p.71 for a discussion of this issue).

The Zariski tangent space is relatively easy to compute in Mathematica or Maple as the kernel of a Jacobian matrix  $J$ ; since usually we are only interested in its dimension, we merely compute the nullity of  $J$ . The method is illustrated in §7 for the fundamental group of the manifold Vol3.

At this point it is necessary to discuss a fundamental issue regarding the practicalities of computation. A computer has two distinct modes of computation, namely integer and floating-point, each with its advantages and drawbacks. Assuming freedom from programming error, integer computations are exact, and their output may be used directly in a mathematical proof. However, in many situations, for example if one wishes to solve a non-linear equation by means of an iterative method based on analytical principles, integer computations are not appropriate. Floating-point computations, on the other hand, are inherently inexact, however many decimal places of accuracy are used, since the result of any arithmetic operation is rounded off before being stored for further processing. With floating-point calculations it is possible to assert that a result is accurate within certain specified bounds if one keeps track of the propagation of round-off errors. Therefore, for example, with sufficient working accuracy and with due cause, one can assert that the result of a computation is non-zero; however, one can never justifiably assert that the result of a computation is exactly zero.

The computation of the dimension of the Zariski tangent space takes place in floating-point mode. Therefore, strictly speaking, by computing the nullity of the Jacobian  $J$  we are only computing an upper bound for the dimension. However, for theoretical reasons we do have a lower bound. There are two kinds of “inessential” infinitesimal deformations (abbreviated in this paragraph as “iid’s”) that we must exclude from our count. First, composition with inner automorphisms of  $\mathrm{GL}(4, \mathbb{R})$  gives rise to  $16 - 1 = 15$  dimensions of iid’s (we subtract 1 from 16 as the centre of  $\mathrm{GL}(4, \mathbb{R})$  has dimension 1, and absolute irreducibility of the representation guarantees that only scalar matrices can commute with the generators). Secondly, suppose that  $H_1(M)$  has a free summand of rank  $r$ . Then there exists an epimorphism  $\eta$  of  $\pi_1(M)$  to the direct sum of  $r$  copies of  $\mathbb{Z}$ , and we have  $r$  independent 1-parameter families of iid’s  $\phi_{i,\lambda} : g \mapsto \lambda^{w_i(g)} \cdot \phi_0(g)$ , where the parameter  $\lambda$  is a non-zero real number, and the “weight function”  $w_i$  is the composition of  $\eta$  with projection to the  $i$ th summand of  $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$  ( $1 \leq i \leq r$ )<sup>2</sup>. Manifolds for which  $\nu(J)$  is found to equal  $15 + r$  are *rigid*, in the sense that  $\chi(\phi_0)$  is an isolated point in the character variety. This assertion is justified by the fact that  $15 + r$  is both an upper bound (as a result of the computation) and a lower bound (from the theory). The excess of  $\nu(J)$  over  $15 + r$  is the potential dimension of the character vari-

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<sup>2</sup>Of course these are not infinitesimal deformations into  $\mathrm{SL}(4, \mathbb{R})$ , but for computational expediency we first consider all infinitesimal deformations into  $\mathrm{GL}(4, \mathbb{R})$  and then take an appropriate subspace.

ety of deformations of  $\phi_0$  into  $\mathrm{SL}(4, \mathbb{R})$ ; as already mentioned, quite apart from the uncertainty due to roundoff error, there is no guarantee that an infinitesimal deformation is integrable to an actual deformation.

It is possible in principle to calculate  $\nu(J)$  exactly, as  $J$  depends only on the base representation  $\phi_0 : \pi_1(M) \rightarrow \mathrm{SO}^+(3, 1)$ . However, in order to be able to paint a broad picture relatively quickly, we have chosen to compute  $\nu(J)$  in floating-point mode, using an approximation of  $J$  to 1000 decimal places. To obtain this accuracy, we start with the representation  $\rho : \pi_1(M) \rightarrow \mathrm{SL}(2, \mathbb{C})$  given by SnapPea to machine accuracy, and then increase the accuracy of  $\rho$  to the required level by means of a few iterations of Newton’s method, using the group relations, before converting to a very accurate approximation of  $\phi_0 : \pi_1(M) \rightarrow \mathrm{SO}^+(3, 1)$ . In cases where the computed dimension of the Zariski tangent space exceeds the known dimension of the representation variety, the stated dimension of the Zariski tangent space is therefore not rigorous, albeit almost certainly correct.

This computation was carried out for the first 4500 2-generator closed orientable manifolds in the Hodgson-Weeks census [Hodgson & Weeks, 2000], and it was found that only 61 of these manifolds admit non-trivial infinitesimal deformations. Of these, 21 have been shown rigorously to admit actual deformations, and there is compelling numerical evidence that a further 31 do. Of the remaining 9 manifolds, 3 have been proved to be rigid using a certain third order obstruction explained in [Cooper et al., 2005], and numerical evidence strongly suggests that the remaining 6 are rigid. These results are set out in Table 1. A check mark in the last column indicates that the variety has been computed exactly, and has been shown rigorously to have the stated dimension; absence of a check mark should be interpreted as “compelling numerical evidence only”. Details of the computation for the manifold  $m007(3, 1)$  are given in section 7 of this paper.

The outcome for  $v2678(2, 1)$  is of interest. This manifold apparently has a 5-dimensional space of essential infinitesimal deformations, whereas the  $\mathrm{SL}(4, \mathbb{R})$  character variety apparently has two 3-dimensional branches meeting in a 1-dimensional subvariety containing  $\chi(\phi_0)$ . It would follow that  $\chi(\phi_0)$  is not a smooth point of the variety. It is also interesting that from the first 2000 manifolds in the census,  $v2678(2, 1)$  appears from numerical evidence to be the only example admitting deformations into  $\mathrm{SO}(4, 1)$ . K. Scannell [Scannell, 2000] has proved that  $m036(-3, 2)$ , the double cover of  $\mathrm{Vol}3$ , admits infinitesimal deformations into  $\mathrm{SO}(4, 1)$ ; however, numerical evidence suggests that these are not integrable.

manifold	volume	inf.	actual	manifold	volume	inf.	actual
m007(3, 1)	1.014941	1	1 ✓	s912(0, 1)	4.059766	2	2
m036(-3, 2)	2.029883	1	1 ✓	m401(-2, 3)	4.059766	2	2
m034(-4, 1)	2.195964	1	1 ✓	v825(4, 1)	4.059766	1	1
m160(-3, 2)	2.595387	1	1 ✓	m358(1, 3)	4.059766	1	1
m082(1, 3)	2.786804	1	1 ✓	m368(-4, 1)	4.059766	1	0
m078(5, 1)	2.816179	1	1 ✓	s778(-3, 2)	4.059766	2	2 ✓
m100(2, 3)	2.882494	1	1 ✓	s779(1, 2)	4.059766	2	2
m149(-4, 1)	3.044824	1	0 ✓	m395(-2, 3)	4.059766	2	2
m188(2, 3)	3.044824	1	1 ✓	s440(-1, 3)	4.059766	1	1
m247(-1, 3)	3.044824	1	1 ✓	v2678(2, 1)	4.116968	5	3
m159(2, 3)	3.044824	1	0 ✓	s500(4, 1)	4.116968	1	1
m115(5, 2)	3.060334	1	1 ✓	v2334(-1, 2)	4.116968	2	0
m121(-4, 3)	3.195780	1	1 ✓	s490(-4, 1)	4.116968	1	1
m336(-1, 3)	3.663862	2	2 ✓	s668(4, 1)	4.221804	1	1
m303(-1, 3)	3.663862	1	1 ✓	s518(-1, 4)	4.400901	1	1 ✓
s572(1, 2)	3.663862	1	1 ✓	v2817(-3, 1)	4.407345	1	1
m293(4, 1)	3.663862	1	0 ✓	s533(1, 4)	4.422687	1	1
s645(-1, 2)	3.663862	1	1 ✓	m402(2, 3)	4.436783	1	1
m312(-1, 3)	3.663862	2	2 ✓	v1461(1, 3)	4.598034	1	1
s778(-3, 1)	3.663862	1	1 ✓	s636(-1, 4)	4.598853	1	1 ✓
m304(5, 1)	3.663862	1	1	s618(1, 4)	4.598853	1	1 ✓
s682(-3, 1)	3.663862	3	2	v1222(-5, 1)	4.626243	1	1
s350(-4, 1)	3.663862	1	0	v1251(4, 3)	4.686034	1	1
m294(4, 1)	3.663862	1	0	s666(-4, 3)	4.686034	1	1
s495(1, 2)	3.663862	1	0	v2413(-3, 2)	4.686034	2	0
s235(-3, 4)	3.794090	1	1	v1695(-5, 1)	4.834441	1	1
m290(-3, 4)	3.818259	1	1	v1860(2, 3)	4.974542	1	1
m350(-1, 3)	3.861814	1	1	v1847(-4, 3)	5.016110	1	1
m360(-2, 3)	3.861814	1	1	v1845(-5, 2)	5.017640	1	1
s287(3, 4)	3.896345	1	1	v3283(-3, 1)	5.171469	1	1
m346(2, 3)	3.933297	1	1				

Table 1: Infinitesimal and actual deformations of closed manifolds

## 6 Summary of the computational procedure

We are now ready to give an outline of the entire computational procedure. A feature of the method is that the trace field, matrix entry field and exact matrix entries are derived by informed guesswork based on numerical data; it is only at the very last step that the existence of the representation variety is proved, by checking formally that the proposed matrices  $\Psi(\alpha), \Psi(\beta)$  satisfy the two relations of  $\pi_1(M)$ . Initial data needed to get started, *i.e.* generators and relations for  $\pi_1(M)$  and  $\phi_0$  accurate to a few decimal places, can easily be obtained using SnapPea [Weeks, 1990] or Snap [Coulson et al., 2000].

**Step 1.** Compute the dimension of the Zariski tangent space at the  $\mathrm{SO}^+(3, 1)$  representation  $\phi_0$ , using a highly accurate approximation to the Jacobian matrix; the computational details are best understood by reading the account of this step in §7. If the manifold is found to be rigid, then there are no deformations and we quit.

**Step 2.** Apply a small random perturbation to  $\phi_0$  by slightly modifying the matrices  $\phi_0(\alpha), \phi_0(\beta)$ , and then perform Newton's method to try to converge to a representation  $\phi_1 : \pi_1(M) \rightarrow \mathrm{SL}(4, \mathbb{R})$  not conjugate to  $\phi_0$  (the test is to compare characteristic polynomials of matrices  $\phi_0(g), \phi_1(g)$  for various elements  $g \in \pi_1(M)$ ). The 32 unknowns at each stage of the Newton process are the adjustments to the 32 matrix entries needed to cancel out the residuals to first order. The two defining relations of the group provide 32 constraints for these unknowns, but in the case where  $H_1(M)$  has a free direct summand it will be necessary to add constraints  $\det(\phi_1(\alpha)) = 1, \det(\phi_1(\beta)) = 1$ . Since there are now more equations than unknowns, we use the QR-decomposition of the matrix of coefficients (*i.e.* the Jacobian matrix) to find a least squares solution to the linear system.

It can happen that the manifold is rigid despite the existence of a non-trivial Zariski tangent; this phenomenon manifests itself here by the Newton process refusing to converge<sup>3</sup>. To prove rigidity one then has to compute a higher order obstruction, for example the third order obstruction described in [Cooper et al., 2005]. Referring to Table 1, in this way it was proved that m149(-4, 1), m159(2, 3), m293(4, 1) are all rigid. The six outstanding cases listed in Table 1 have not been checked rigorously, but there is strong evidence that they are rigid.

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<sup>3</sup>This is probably because the vanishing of the first order obstruction to the existence of a variety implies the vanishing of the second order obstruction also; see for example [Cooper et al., 2005].

If the Newton process converges satisfactorily to a representation  $\phi_1$  distinguished from  $\phi_0$  by examination of characteristic polynomials, we move on to Step 3. We note that because we are using floating-point arithmetic, we cannot yet assert definitely that we have found a genuine representation  $\phi_1$ .

**Step 3.** Find a suitable parametrization of the character variety. The character variety can always be parametrized by means of coefficients of characteristic polynomials, but the aim is to find parameters  $u_1, \dots, u_n$  for which the trace field has small degree over the field  $\mathbb{Q}(u_1, \dots, u_n)$ . This usually involves a modest amount of experimentation. From Step 1 we have an upper bound on the number of parameters  $n$ .

From Step 2 we already know of at least one trace that varies as one moves away from  $\phi_0$ , say  $\text{tr}(\phi(g))$  where  $g \in \pi_1(M)$ . Add a constraint to the Newton process of Step 2, declaring that this trace is some rational number reasonably close to the value of this trace at  $\phi_0$ . This might make the convergence of the Newton process less robust, in which case it will be necessary to control the step-length, by multiplying the adjustments at each stage by some dynamically controlled scale factor. Once one has achieved convergence, see by means of LLL [Lenstra et al., 1982] (or an alternative, *e.g.* PSLQ [Bailey & Ferguson, 1991]) whether all other traces now appear to be algebraic numbers, and if so, note the degree and complexity of their minimal polynomials. If some trace appears not to be algebraic, select it as an additional parameter, add an extra constraint declaring it to equal an appropriate rational number, and repeat the process. We note that the proof of Proposition 1 provides a list of traces that is sufficient for this purpose.

Eventually, we should achieve a numerical approximation to a representation where LLL declares that all traces are algebraic. In practice, one would not wish to compute a variety for which  $n > 2$ , although we have used this method to work out the deformation variety for one 3-dimensional example, the cusped manifold m007. In the last section of this paper, we take the reader in some detail through a 1-dimensional example, namely the closed manifold m007(3, 1) known as Vol3.

**Step 4.** Compute the trace field  $T$ , as an extension of finite degree over the field  $\mathbb{Q}(u_1, \dots, u_n)$ . This is best explained by means of a worked example, but broadly speaking the method is to compute generators for the trace field evaluated at each point of a cubic lattice in parameter space, using LLL, and then obtain generators for  $T$  over  $\mathbb{Q}(u_1, \dots, u_n)$  using poly-

nomial interpolation<sup>4</sup>. The points of the lattice should be chosen to have rational coordinates, where the denominators are not too large, and should be reasonably close together. If  $n > 1$  it will be necessary to interpolate in each of the  $n$  coordinate directions, so as to obtain a polynomial in the  $n$  variables  $u_1, \dots, u_n$ . In practice a row of data points  $(x_1, y_1), \dots, (x_k, y_k)$  for the interpolation may not lie on the graph of the desired polynomial, but  $(x_1, \lambda_1 y_1), \dots, (x_k, \lambda_k y_k)$  will lie on the graph for integers  $\lambda_i$  that are small relative to the  $y_i$ . Determining these “multipliers”  $\lambda_i$  is perhaps the trickiest part of the entire process, but skill comes with practice! It is usually self-evident when the correct  $\lambda_i$  have been found, as the degree of the interpolating polynomial is then much smaller than the number of data points.

Once the trace field  $T$  has been computed, choose once and for all a basis  $\tau_1, \dots, \tau_m$  for  $T$  as a vector space over  $\mathbb{Q}(u_1, \dots, u_n)$ .

**Step 5.** Choose an element of  $\pi_1(M)$  whose image is to fulfill the role of the matrix  $b$  in §3. If at all possible, to avoid field extensions of uncomfortably large degree,  $b$  should have a reciprocal characteristic polynomial. Fortunately such elements have proved to be available for all varieties that we have computed. Once this matrix has been chosen, decide on a basis for the field  $K$  generated by the matrix entries over  $\mathbb{Q}(u_1, \dots, u_n)$ . This basis will of course be a function of the parameters.

**Step 6.** Express each trace used in the proof of Proposition 1 as a linear combination  $\sum_{i=1}^n f_i \tau_i$ , where the  $f_i$  are rational functions of the parameters  $u_1, \dots, u_n$ . Again one uses polynomial interpolation, but for determining the coefficients  $f_i$  at each lattice point the required tool is a facility for detecting integer relations. Pari’s *linddep* is such a function [Batut et al., 1990]. Neither Mathematica nor Maple has a built-in integer relation facility, but notebooks/worksheets are available that will perform this task.

**Step 7.** Write a program that uses the method of §4 to compute generating matrices  $a, b$  in standard form, to a large number of decimal places, for each point of a lattice in parameter space. The “large number” just referred to depends on the complexity of the situation, but typically 2000 decimal places are appropriate, so that the integer relation detector *linddep* can produce results that are not spurious. The number of points in the lattice depends also on the complexity, but usually we have found that an  $n$ -cube of edge-length

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<sup>4</sup>Here we use polynomial interpolation of the most basic kind, namely fitting a polynomial of least degree to finitely many data points. Functions for this purpose are provided in all standard computer algebra systems.

50 is sufficient. Fortunately this method of computing numerical approximations to representations is remarkably fast, and even the computation of the 2500 representations for a  $50 \times 50$  lattice can usually be accomplished in an hour or two.

**Step 8.** Determine an exact expression for each matrix entry in terms of the parameters, using *linddep* and polynomial interpolation, as in Step 4.

**Step 9.** Use the formal algebra capabilities of Maple or Mathematica to verify that the exact matrices obtained in the previous step satisfy the group relations.

## 7 The manifold Vol3

The first manifold in the census to admit  $SL(4, \mathbb{R})$ -deformations is the third manifold listed in the census, known as “Vol3”. Its denotation in the census is m007(3, 1), meaning that it is obtained by (3, 1)-surgery on the cusped manifold m007. m007 in turn is the seventh manifold in the census of cusped manifolds obtained by gluing together up to five ideal tetrahedra (for historical reasons, the prefixes “s”, “v” are used for 6, 7 ideal tetrahedra respectively). Surgery coefficients are given relative to the basis {[shortest curve], [second shortest curve]} for the first homology group of the cusp cross-section.

SnapPea gives the following presentation for the fundamental group of Vol3:

$$\pi_1(\text{Vol3}) = \langle a, b \mid aabbABAbb, aBaBabaab \rangle ,$$

and the following numerical approximation to a lift to  $SL(2, \mathbb{C})$  of the discrete faithful representation into  $PSL(2, \mathbb{C})$ :

$$a \mapsto \begin{bmatrix} 0.853230696696 - 1.252448658070i & -0.500000000000 + 0.866025403784i \\ 0.159374980683 - 2.137255282203i & 0.371514174695 + 1.959555439256i \end{bmatrix}$$

$$b \mapsto \begin{bmatrix} 0.500000000000 + 0.866025403784i & -0.286522831781 + 0.594411651593i \\ 0.658037006476 + 1.365143787662i & 0.000000000000 + 0.000000000000i \end{bmatrix}$$

Here, for notational convenience, we are using the upper-case letters  $A, B$  to denote  $a^{-1}, b^{-1}$  respectively. Since we shall only be considering representations of  $\pi_1(\text{Vol3})$  into linear groups within matrix algebras, we shall take the liberty of considering  $a, b$  as matrices and write the group relations as

$$r_1(a, b) := aabb - BBaba = 0 \quad , \quad r_2(a, b) := aBaBa - BAAAB = 0 \quad .$$

We begin by using Newton's method to improve the accuracy of the  $\mathrm{SL}(2, \mathbb{C})$  representation. The procedure is to compute the residuals  $y_1, y_2$  of  $r_1, r_2$ , namely their actual starting values (which are already close to the zero  $2 \times 2$  matrix), and then solve a linear system to find changes  $da, db$  in the matrices  $a, b$  that cancel out these residuals to first order. For this we compute formal expressions for the changes  $dr_1, dr_2$  in  $r_1, r_2$  effected by changing  $a, b$  to  $a+da, b+db$ ; we then solve the system  $dr_1 = -y_1, dr_2 = -y_2$ .

"Differentiating"  $mM = I$ , we obtain  $(dm)M + m(dM) = 0$ , whence  $dM = -M(dm)M$ . Therefore

$$\begin{aligned} dr_1 &= ((da)abb + a(da)bb + aa(db)b + aab(db)) \\ &\quad - ((-B(db)B)Baba + B(-B(db)B)aba + BB(da)ba \\ &\quad + BBa(db)a + BBab(da)) , \\ dr_2 &= ((da)BaBa + a(-B(db)B)aBa + aB(da)Ba \\ &\quad + aBa(-B(db)B)a + aBaB(da)) \\ &\quad - ((-B(db)B)AAAB + B(-A(da)A)AAB + BA(-A(da)A)AB \\ &\quad + BAA(-A(da)A)B + BAAA(-B(db)B)) . \end{aligned}$$

The 8 unknowns of the linear system are the entries of the matrices  $da, db$ , and entry-by-entry comparison of  $dr_i$  with  $-y_i$  ( $i = 1, 2$ ) provides 8 equations in these unknowns. All this is easy to program in Mathematica or Maple, and after a small number of iterations we arrive at an  $\mathrm{SL}(2, \mathbb{C})$ -representation accurate to 1000 decimal places. We note in the case where  $H_1(M)$  has a free summand, it is necessary to add constraints  $\det(a) = 1, \det(b) = 1$  and then use a QR-decomposition on the resulting rectangular matrix of coefficients.

The accurate  $\mathrm{SL}(2, \mathbb{C})$ -representation that we have just obtained is now converted to an  $\mathrm{SO}^+(3, 1)$ -representation, using the method explained in §2.1. Let  $a_0, b_0$  denote the images in  $\mathrm{SO}^+(3, 1)$  of the group generators. The Jacobian matrix  $J$ , whose nullity we need to compute for Step 1, is obtained using the above expressions for  $dr_1, dr_2$ , with  $a_0, b_0$  in place of  $a, b$ . Since  $a_0, b_0$  have in total 32 entries,  $J$  has size  $32 \times 32$ . We then find that the rank of  $J$  is 16, whence the nullity of  $J$  is also 16 and from the discussion of §5 there is one dimension's worth of essential infinitesimal deformations<sup>5</sup>.

We now proceed to Step 2. We perturb  $a_0, b_0$  very slightly to matrices  $a_1, b_1$ , and then try to converge to a representation using Newton's method.

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<sup>5</sup>Strictly speaking, because we are working in floating-point mode, we may only assert at present that the space of essential infinitesimal deformations has dimension at most 1.



$i$	$p_i(x)$
1	$5181915799 - 1490060879x + 104060401x^2$
2	$5389356866 - 1549614193x + 108243216x^2$
3	$5602964647 - 1610935039x + 112550881x^2$
4	$5822860162 - 1674058049x + 116985856x^2$
5	$6049165607 - 1739018191x + 121550625x^2$
6	$6282004354 - 1805850769x + 126247696x^2$
7	$6521500951 - 1874591423x + 131079601x^2$
8	$6767781122 - 1945276129x + 136048896x^2$
9	$7020971767 - 2017941199x + 141158161x^2$
10	$83041 - 10000x$

Table 2: Minimal polynomials of  $\text{tr}(abAB)$  at selected points of  $\mathcal{V}$

and we suspect strongly that the middle term of the characteristic polynomial of  $b_1$  is  $-(v^2 - 1)$  (A keen observer might have noticed this earlier).

We now try to identify the trace field. A quick search reveals that the trace of the commutator  $a_1b_1A_1B_1$  appears to be irrational, and that, according to LLL, for rational  $v$  it appears to be a root of a quadratic over  $\mathbb{Q}$  (see the Appendix for a brief description of LLL). We run the Newton program again for  $v = 2 + \frac{1}{100+i}$  ( $1 \leq i \leq 10$ ), and print out the minimal polynomials of this trace, as given by LLL. The results are displayed in Table 2.

We proceed to interpolate the first nine polynomials, assuming that the anomalous degree of the polynomial for  $i = 10$  is caused by accidental rationality of  $\text{tr}(a_1b_1A_1B_1)$  at  $v = 2 + \frac{1}{110}$ . The polynomials are only defined up to integer multiples; however, the coefficients appear to lie on a “nice” curve, so it probably will not be necessary to find “multipliers”  $\lambda_i$ .

Indeed, interpolation reveals that the minimal polynomial for this trace is

$$1 - (v^4 - 2)x + (2v^4 + 4v^2 + 1)x^2 ,$$

with discriminant  $v^2(v^2 + 2)^2(v^2 - 4)$ . We are led to surmise that the trace field is

$$T = \mathbb{Q}(v)(\alpha) , \text{ where } \alpha = \sqrt{v^2 - 4} ;$$

some support for this conjecture is provided by computation of several other traces. Incidentally, we note that for  $v = 2 + \frac{1}{110}$ ,  $\sqrt{v^2 - 4}$  is the rational number  $\frac{21}{110}$ , explaining the anomalous polynomial of degree 1 for  $i = 10$ .

We now proceed to Step 5, where we choose the matrix to be diagonalized. Since the characteristic polynomial of  $b_1$  is reciprocal, satisfies the conditions of Proposition 1 and has roots that are relatively simple expressions in  $v$ , we select  $b_1$  for this purpose (in fact  $a_1$  would have done equally well). Two of the eigenvalues of  $b_1$  are real for  $v > 2$ ; they are

$$\lambda_1 = \frac{1}{4} \left( 1 + \sqrt{5 + 4v^2} + \sqrt{2 \left( -5 + 2v^2 + \sqrt{5 + 4v^2} \right)} \right) ,$$

$$\lambda_2 = \frac{1}{4} \left( 1 + \sqrt{5 + 4v^2} - \sqrt{2 \left( -5 + 2v^2 + \sqrt{5 + 4v^2} \right)} \right) .$$

If it should transpire that our guess for the trace field is correct, we can already predict from the discussion of §4 that the field generated by the matrix entries will be  $K = \mathbb{Q}(v)(\alpha, \lambda_1)$ , and that a vector space basis for  $K$  over  $\mathbb{Q}(v)$  will be  $\{1, \gamma, \beta, \beta\gamma, \alpha, \alpha\gamma, \alpha\beta, \alpha\beta\gamma\}$ , where  $\alpha = \sqrt{v^2 - 4}$ ,  $\beta = \sqrt{4v^2 + 5}$ ,  $\gamma = \sqrt{2(-5 + 2v^2 + \beta)}$ . We note that  $K$  is not a Galois extension of  $\mathbb{Q}(v)$ , as for  $v > 2$   $K$  does not contain the two non-real roots of the minimal polynomial of  $\lambda_1$ . It has an automorphism  $\sigma_1$  negating  $\alpha$  and fixing  $\gamma$ , and an automorphism  $\sigma_2$  fixing  $\alpha$  and negating  $\gamma$ ; these automorphisms commute and generate the automorphism group of  $K : \mathbb{Q}(v)$ , which is therefore a Klein group of order 4.

Moving on to Step 6, we would now like to produce representations of high numerical accuracy (2000 decimal places to be safe) for a sequence of values of the parameter  $v$ , say  $v = 2 + \frac{1}{110+i}$  ( $1 \leq i \leq 20$ ) (recall that we wish to avoid  $v = 2 + \frac{1}{110}$ ). In the case of Vol3 it is feasible to obtain these directly from the Newton program; however, typically this is too slow, and a much better approach is to use the method of Proposition 1.

Thus our immediate task is to identify each of the 18 traces used in that Proposition as elements of the trace field  $T = \mathbb{Q}(v)(\alpha)$ . For this we run the Newton program as we did earlier for the trace of  $a_1 b_1 A_1 B_1$ , but this time, for each trace  $t$ , we run the integer relation detector *lindep* on the vector  $(1, \alpha, t)$ , and interpolate the resulting coefficients over the data points. In this way we obtain a generic relation  $p_0(v) + p_1(v)\alpha + p_2(v)t = 0$ , where each  $p_i$  is a polynomial in  $v$  with integer coefficients, and we record that  $t = -\frac{p_0(v)}{p_2(v)} - \frac{p_1(v)}{p_2(v)}\alpha$ . The results for the 18 traces are set out in Table 3.

These are incorporated into a short program that produces (numerical approximations to) the canonical forms  $a_2, b_2$  defined in §4 for the generating matrices, by solving the linear systems given in the proof of Proposition 1 and then conjugating as described in §4.

$\text{tr}(a) = v$	$\text{tr}(aaa) = 3v + v^3$
$\text{tr}(ab) = v$	$\text{tr}(aaab) = v$
$\text{tr}(abb) = v$	$\text{tr}(aaabb) = v$
$\text{tr}(abbb) = v^3$	$\text{tr}(aaBab) = (1/2)(-2v + v^3 + (2 + v^2)\alpha)$
$\text{tr}(aa) = v^2$	$\text{tr}(aabaB) = (1/2)(-2v + v^3 - (2 + v^2)\alpha)$
$\text{tr}(aab) = 1$	$\text{tr}(aaBabb) = (1/2)(2v - v^3 + v^5 + (2 - v^2 - v^4)\alpha)$
$\text{tr}(aaB) = -1 + 2v^2$	$\text{tr}(abaBabb) = (1/2)(2v - v^3 + v^5 + (2 - v^2 - v^4)\alpha)$
$\text{tr}(aabb) = 1$	$\text{tr}(abaBBabb) = -3v + 2v^3 + 2v^5$
$\text{tr}(abab) = v^2$	
$\text{tr}(aBab) = 1$	

Table 3: Traces of selected words in the fundamental group of Vol3

The penultimate stage of the process is to apply *linddep* and polynomial interpolation to our data, so as to obtain exact expressions for the matrix entries. The  $(2, 3)$  entry of the matrix  $a_2$  gives a good idea as to what is involved here. Let us denote this entry  $x$ . For each  $v = 2 + \frac{1}{110+i}$  ( $1 \leq i \leq 20$ ), we apply *linddep* to the 9-component vector  $(1, \gamma, \beta, \beta\gamma, \alpha, \alpha\gamma, \alpha\beta, \alpha\beta\gamma, x)$ . The output from Pari is a sequence of 20 9-component vectors  $\mathbf{w}_i$  ( $1 \leq i \leq 20$ ), where each component is an integer of approximately 17 digits. The first component of each vector is the integer  $y_i$  in Table 4, from which the prospect of fitting a reasonable polynomial admittedly looks bleak.

$i$	$y_i$	$\lambda_i$	$i$	$y_i$	$\lambda_i$
1	9671015960804800	1	11	16193135112008400	1
2	-510168310445250	-20	12	-94497839870700	-180
3	-1195546153052400	-9	13	-17860109073398800	-1
4	2835383895633100	4	14	4686438400061250	4
5	11949122303635440	1	15	2185299690910560	9
6	-349544299928850	-36	16	-5156780125192300	-4
7	-2649179302985600	-5	17	-4325046517516800	-5
8	-3484248880566300	-4	18	-629535323374650	-36
9	-1628653880584800	-9	19	23742505309322800	1
10	3852395824372810	4	20	6216057796580460	4

Table 4: Interpolation data for the  $(2, 3)$  entry of the generator  $a$

But we have to bear in mind that the vectors output by *linddep* are

homogeneous, and that Pari will reduce each vector so that the GCD of its components is 1. Note that  $y_1, y_5, y_{11}, y_{13}, y_{19}$  look (up to sign) as if they stand a fair chance of lying on the correct polynomial curve, so we begin by fitting a polynomial  $p(x)$  to these five data points. Our desired polynomial will probably have higher degree than  $p(x)$ , but we hope that  $p(x)$  will be a close approximation. This is given support by the fact that  $p(i)/y_i$  is very close to being an integer for all data points, so we take these integers as our multipliers  $\lambda_i$ . The list of the 20 multipliers  $\lambda_i$  is given above, alongside the  $y_i$ .

We now interpolate the components of the vectors  $\lambda_i \mathbf{w}_i$  at all 20 data points, obtaining the following 9-component vector of polynomials in  $v$ :

$$\begin{aligned}
p_1 &= 10v(v^2 - 1)(v^2 - 4)(4v^2 + 5) \\
p_2 &= -v(v^2 + 2)(v^2 - 4)(4v^2 + 5) \\
p_3 &= -2v(v^2 - 1)(v^2 - 4)(2v^2 + 15) \\
p_4 &= -v(v^2 + 2)(v^2 - 10) \\
p_5 &= 2v^2(v^2 - 1)(4v^2 + 5) \\
p_6 &= -v^2(v^2 + 2)(4v^2 + 5) \\
p_7 &= -2v^2(v^2 - 1)(2v^2 - 5) \\
p_8 &= (v^2 + 2)(7v^2 - 10) \\
p_9 &= -32(v^2 - 1)(v^2 - 5)(4v^2 + 5)
\end{aligned}$$

Since the degree of each of these polynomials is much smaller than the degree of 19 that one would obtain generically, we are confident that we have the right answer, although this will not be proved until the final formal check of the two group relations.

Therefore we record that  $x = -\frac{1}{p_9} \sum_{i=1}^8 p_i \nu_i$ , where  $\{\nu_1, \dots, \nu_8\}$  is the basis we have chosen for the matrix entry field over  $\mathbb{Q}(v)$ , and we are done. The other entries of  $a_2, b_2$  are computed similarly.

Once exact expressions for all entries of the generators  $a_2, b_2$  have been computed, it remains for Maple or Mathematica to check that they are correct by verifying formally that the group relations are satisfied, *i.e.* that  $r_1(a_2, b_2) = r_2(a_2, b_2) = 0$ . This can take a moderate amount of time; the key is to simplify aggressively at each stage of the computation.

**Remark 7.1.** The real hyperbolic representation occurs at  $v = 2$ . There are in fact two non-conjugate  $\mathrm{SL}(4, \mathbb{C})$  representations for each  $v \in \mathbb{C}, v \neq 2$ , corresponding to the two square roots of  $\alpha^2 = v^2 - 4$ . There is a symmetry of the variety interchanging these two representations, induced by the *contragredient* automorphism of  $\mathrm{GL}(4, \mathbb{C})$  that assigns to each matrix the transpose of its inverse.

**Remark 7.2.** Further experimentation can sometimes result in a conjugate family of representations with simpler expressions for the matrix entries. In the case of Vol3, a significant improvement may be obtained by exploiting symmetries of the manifold. The symmetry group of Vol3 is semihedral of order 16 [Hodgson & Weeks, 1994], and in particular there is a rotational symmetry  $\rho$  of order 4, fixing the axis of  $aBaba$  and inducing an automorphism  $\rho_*$  of  $\pi_1(\text{Vol3})$  sending  $a$  to  $BA$  and  $b$  to  $aba$ . Thus we may form the orbifold fundamental group  $\pi_1(\text{Vol3}/\langle\rho\rangle)$  as a supergroup of  $\pi_1(\text{Vol3})$  by adding a new generator  $u$  together with relations  $u^4 = 1$ ,  $Uau = BA$ ,  $Ubu = aba$ . It follows easily that  $u^2a$  has order 2, and that  $\pi_1(\text{Vol3}/\langle\rho\rangle)$  is generated by  $u, u^2a$ . Starting from the above family of representations and considering  $\pi_1(\text{Vol3}/\langle\rho\rangle)$  as a carefully chosen group of isometries of 3-dimensional complex hyperbolic space, for a suitable range of values of the parameter  $v$  (see [Cooper et al., 2005]), the following much simpler curve of representations was found:

$$\Psi(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{v^2-4}{v^2+8}} & \sqrt{\frac{-2v^2-4}{v^2+8}} \\ 0 & 0 & \sqrt{\frac{-2v^2-4}{v^2+8}} & -\sqrt{\frac{v^2-4}{v^2+8}} \end{bmatrix},$$

$$\Psi(u^2a) = \begin{bmatrix} p & 0 & q & 0 \\ 0 & p^* & 0 & q^* \\ q & 0 & -p & 0 \\ 0 & q^* & 0 & -p^* \end{bmatrix},$$

where

$$p = \frac{1}{4} \left( v + \sqrt{v^2 + 8} \right), \quad q = -\frac{1}{2\sqrt{2}} \sqrt{4 - v^2 - v\sqrt{v^2 + 8}},$$

$$p^* = \frac{1}{4} \left( v - \sqrt{v^2 + 8} \right), \quad q^* = -\frac{1}{2\sqrt{2}} \sqrt{4 - v^2 + v\sqrt{v^2 + 8}}.$$

**Remark 7.3.** Earlier, we showed that the trace field contained  $\mathbb{Q}(v)(\alpha)$ , but did not prove equality. However, we can show equality as follows. Let  $K$  be the matrix entry field used in the main part of this section, and let  $K'$  be the matrix entry field of Remark 8.2. Since the trace field is invariant under conjugation, it must lie in  $K \cap K' = \mathbb{Q}(v)(\alpha)$ , QED.

## 8 Appendix: using LLL

For readers not familiar with this wonderful tool, here is a very brief description. LLL is available in Pari as the function **algdep()**, in Mathematica as **Recognize[]**, and in Maple as **MinimalPolynomial()**. Here we describe the Pari version.

For **algdep()**, one enters as input a complex number  $t$  in floating point format, a maximum degree  $d \geq 1$  and a working precision  $p$ . In return, one receives a polynomial in  $\mathbb{Z}[x]$  of degree at most  $d$ , of which  $t$  is a root to within the working precision. For example, the command

```
algdep(1.41421356 , 2 , 6)
```

yields the answer  $x^2 - 2$ .

Of course, the input  $t$  is always rational if interpreted literally, and one should regard as spurious any polynomial whose coefficients are large relative to the declared working precision  $p$ . A good, conservative rule of thumb is that an answer is trustworthy if the total number of digits over all coefficients is at most  $\frac{2p}{3}$ , and that the answer is suspect otherwise. For example, one should reject

```
algdep(3.1415926535 , 10 , 10)  $\longrightarrow$   $3x^{10} - 9x^9 - 5x^7 + 9x^5 - 5x^4 + 3x^3 + 6x^2 + 6x + 1$  .
```

The time taken for **algdep()** to complete its task appears to depend exponentially on the degree  $d$ , and also on the precision  $p$ ; however, it works well for, say,  $d = 20$  and  $p = 1000$ .

The related **linddep()** function of Pari accepts as input a vector  $[t_1, \dots, t_n]$  of floating-point numbers and a working precision  $p$ , and returns a vector  $[\lambda_1, \dots, \lambda_n]$  of integers for which  $\sum \lambda_i t_i = 0$  to within the precision  $p$ . For example, if we set  $t = 2^{1/2} + 3^{1/3}$ , the command

```
linddep([1 , t , t^2 , t^3 , t^4 , t^5 , 2^{1/2}] , 100)
```

instantly yields the answer

```
[-1092 , 879 , -468 , -320 , 27 , 48 , -755] ,
```

from which one deduces that

$$\sqrt{2} = \frac{1}{755} (-1092 + 879t - 468t^2 - 320t^3 + 27t^4 + 48t^5) .$$

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