Constructing thin subgroups commensurable with the figure-eight knot group

S. Ballas & D. D. Long [*]

October 29, 2014

Abstract

In this paper we find infinitely many lattices in $SL(4, \mathbf{R})$ each of which contains thin subgroups commensurable with the figure-eight knot group.

1 Introduction.

Following Sarnak (see [23]), one says that a subgroup Δ of Γ is *thin* if Δ has infinite index in Γ, and is Zariski dense. Since it is rather easy to exhibit Zariski dense subgroups of lattices that are free products, the case of most interest is that the thin group Δ is finitely generated and does not decompose as a free product.

In this note, we shall exhibit subgroups of the fundamental group of the figure eight knot as subgroups of infinitely many incommensurable lattices. A precise statement will be given shortly.

We begin by briefly describing where the lattices in question arise. They are constructed from Theorem 6.55 of [24] by a rather general construction which involves L, a real quadratic extension of $\mathbf{Q}$ and D, a central simple division algebra of degree d over L. However in our situation we may assume that $D = L$, and we state only this special case.

In this paper, the field L is a real quadratic extension of $\mathbf{Q}$ and if $A \in SL(4, L)$, we will denote by A^* the matrix obtained by taking the transpose of the matrix obtained from A by applying τ (the non-trivial Galois automorphism) to all its entries. Then one has:

Theorem 1.1. *Suppose that L is a real quadratic extension of $\mathbf{Q}$, with Galois automorphism τ. Suppose that $b_1, ..., b_4$ are nonzero elements of $\mathbf{Z}$. Setting $J = \mathrm{diag}(b_1, ..., b_4)$, then the group*

$$SU(J, \mathcal{O}_L, \tau) = \{A \in SL(4, \mathcal{O}_L) \mid A^* J A = J\}$$

is a lattice in $SL(4, \mathbf{R})$.

We also note from [24] Proposition 6.55 that, in the case being considered here (when $D = L$), the corresponding forms will represent zero non-trivially, and so the lattices produced are non-uniform.

We shall show:

Theorem 1.2. *Let Γ denote the fundamental group of the figure eight knot and $L_d = \mathbf{Q}(\sqrt{d})$ for d a positive, square free integer.*

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Then for infinitely many values of d, there is a subgroup of finite index in Γ, H_d, and a faithful, Zariski dense representation into a lattice:

$$r_d : H_d \longrightarrow SU(J_d, \mathcal{O}_{L_d}, \tau_d)$$

The power of this statement lies in the fact that r_d is faithful and Zariksi dense; much of the work of the paper is devoted to this aspect. Once one knows this, it follows immediately that the image group $r_d(H_d)$ has infinite index in the lattice $SU(J_d, \mathcal{O}_{L_d}, \tau_d)$, since deep results of Margulis (see Chapter 13 of [24]) imply that such lattices do not have subgroups which admit homomorphisms onto **Z**. Moroever, subgroups of finite index in Γ are freely indecomposable, so that it follows from 1.2 that the image is thin.

This paper is organized as follows. The starting point is a pair of representations of Γ constructed independently by Ballas [3] (which we denote ϕ_t) and Thistlethwaite (private communication) (which we denote ρ_v). These representations are conjugate when $v = 2t$ and while one could work with just one of them, each presents sufficiently interesting features that it seems worthwhile to include both. Explicit generators for both representations are included in Appendix A. We begin in §2 with the algebraic considerations needed to construct the representations r_d of Theorem 1.2 from the Ballas-Thistlethwaite representation. The main work here is proving that the representation has an integral character in an appropriate sense; the remaining ingredients in constructing r_d are fairly standard once this has been proved.

The results of §3 and §4 are geometric and lie deeper. Specifically, §3 is devoted to the proof that the representations r_d are faithful infinitely often using results from real projective geometry.

In §4 we show that, with the exception of the representation corresponding to the complete hyperbolic structure, ϕ_t has Zariski dense image in $SL(4, \mathbf{R})$.

2 Constructing r_d.

This section is devoted to outlining the computations necessary to exhibit the representations r_d of Theorem 1.2. With a view to a point that arises in the sequel (see Theorem 3.2), we begin by proving the following:

Theorem 2.1. *The representations ρ_v are absolutely irreducible for real $v > 0$.*

Proof. If there is an invariant subspace of dimension one, then any commutator must have eigenvalue $= 1$. One can check that the commutator $[x^2, y]$ has such an eigenvalue only when v is a root of $v^2 + v + 1$. The same considerations applied to the transpose representation show there can be no invariant subspace of ρ_v of dimension three.

The case of a two dimensional invariant subspace is somewhat more subtle. One computes that the characteristic polynomial of the longitude is $(-Q + v)^3(-1 + Qv^3)/v^3$, so that if there were such an invariant subspace, either $\rho_v(\lambda)$ or $\rho_v^T(\lambda)$ would have the $1/v^3$ eigenvalue appearing in that subspace. It follows that the orbit of that eigenvector would be two dimensional. However, one can compute that this happens for neither of these representations. $\square$

Our next claim concerns the traces of the ρ_v.

Lemma 2.2. *The representation of the figure eight knot given by ρ_v has traces lying in $\mathbf{Z}[v, 1/v]$.*

Proof. This involves some computation, as it must. We indicate the mathematics behind the idea, with a sample of its implementation at [25].

Regarding v as a real transcendental, using Burnside's theorem [18, p. 648 Cor. 3.4], or by inspection, one can find elements $g_1, \ldots, g_{16} \in \Gamma$ which are a basis for the vector space of 4×4 matrices $M(4, \mathbf{R})$; we always choose g_1 to be the identity matrix, which we denote by I.

Let $g_1^*, \ldots, g_{16}^*$ be the dual basis with respect to trace, i.e. $tr(g_i.g_j^*) = \delta_{ij}$. One can use the action of the figure eight knot group on this dual basis by left multiplication to obtain a 16-dimensional representation of the group, i.e. if $\gamma \in \Gamma$, then its action is defined by

$$\gamma.g_i^* = \sum_j \alpha_{ij}(\gamma)g_j^*$$

Taking traces in this equation, we get

$$tr(\gamma.g_i^*) = \sum_j \alpha_{ij}(\gamma)tr(g_j^*)$$

Notice that since we have chosen $g_1 = I$, we have that $tr(g_j^*) = tr(g_1.g_j^*) = \delta_{1j}$, in particular these are all rational integers. Writing $I = \sum \tau_j g_j^*$, notice that $\tau_i = tr(g_i)$ by duality; one verifies that these traces only have denominators which are powers of V.

Moreover, multiplying by γ and taking traces, we have

$$tr(\gamma) = \sum_j tr(g_j)tr(\gamma.g_j^*) = \sum_{j,k} tr(g_j)tr(g_k^*)\alpha_{jk}(\gamma) = \sum_j tr(g_j)\alpha_{j1}(\gamma)$$

The upshot of these two computations is the following: If with some choice of basis, one computes that the denominators of the entries for the associated 16×16 regular representation, then this collection of denominators contains the denominators for the traces of the original collection of matrices of Γ.

Therefore, if one could find a basis for which this construction gave $\mathbf{Z}[v, 1/v]$-matrices $(\alpha_{ij}(\gamma))$, then this would prove the result claimed by the lemma. However, this appears to be hard. We bypass this difficulty by constructing two representations via two different choices of basis $\{g_i\}$. For the first choice one sees all the matrix entries have denominators v, $(-1+v)$ and $(1+v)$, for the second, one sees denominators v, $(1+3v+v^2)$ and $(1+3v+4v^2)$. Since traces are not dependent on choice of basis, the denominators of the trace of the original representation of Γ must lie in the intersection of these two sets, i.e. powers of v. $\square$

Corollary 2.3. *If one specializes v to be a unit in any number field, then the resulting representation has integral trace.*

Since when one sets $v = 2t$ the representation ρ_v is conjugate to the representation ϕ_t and this representation has entries lying in $\mathbf{Q}(t) = \mathbf{Q}(v/2)$, we are now in a position to apply the following lemma with $k = \mathbf{Q}(v/2)$.

Theorem 2.4. *Suppose that $G < SL(4, k)$ is a finitely generated group with the property that $tr(\gamma) \in \mathcal{O}_k$ for every $\gamma \in G$.*
    Then G has a subgroup of finite index contained in $SL(4, \mathcal{O}_k)$.

Proof. Consider

$$\mathcal{O}G = \{\Sigma\, a_i\gamma_i \mid a_i \in \mathcal{O}_k, \gamma_i \in G\ \}$$

where the sums are finite. It is shown in [5] (see Proposition 2.2 and Corollary 2.3), that $\mathcal{O}G$ is an order of a central simple subalgebra $B \subset M(4, k)$ defined over k. Now while $\mathcal{O}G$ need not be an order in $M(4, k)$, it is known that it is contained in some maximal order $\mathcal{D}$ of $M(4, k)$ (cf. [22] p. 131, Exercise 5 and the proof of Lemma 2.3 of [19]).

Now it is a standard fact that the groups of elements of norm 1 in orders contained in $M(4, k)$ are commensurable (since the intersection of two orders is an order and the unit groups of orders

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will be irreducible lattices in $SL(4, \mathbf{R}) \times SL(4, \mathbf{R})$, see [24] Chapter 15I). In particular, $SL(4, \mathcal{O}_k)$ and $\mathcal{D}^1$ are commensurable. Let $\Delta = SL(4, \mathcal{O}_k) \cap \mathcal{D}^1$, which has finite index in both groups. Then $G \leq \mathcal{D}^1$, so that $G \cap \Delta$ has finite index in G and lies inside $SL(4, \mathcal{O}_k)$ as required. $\square$

What have achieved to this point is that when one specializes v to be any unit of a number field k, there is a representation, namely $\phi_{v/2}$, of a subgroup of finite index in Γ whose image has entries lying in $\mathcal{O}_k$.

We now turn our attention to the unitary aspects required by Theorem 1.1. There is an obvious involution on the matrices in the image of ρ_v given by transposing and mapping $v \to 1/v$. We denote this operation by $A \to A^*$. A routine computation on the generators reveals:

Lemma 2.5. *There is a matrix Q_v with $\det(Q_v) \neq 0$ for which*

$$A^* . Q_v . A = Q_v$$

for all $A \in \phi_{v/2}(\Gamma)$.

Since it's required by the result of Theorem 1.1, for the rest of this paper we assume $k = \mathbf{Q}(\sqrt{d})$ is a real quadratic number field equipped with involution τ_d. Specialising v to be a unit of this field with the property that $\tau_d(v) = 1/v$ (e.g. one can take the square of a random unit of $\mathbf{Q}(\sqrt{d})$; we call these the *positive units*), we see that Lemma 2.5 and Theorem 2.4 taken together prove:

Theorem 2.6. *For each positive square free integer d, and a positive unit u in the ring of integers of $\mathbf{Q}(\sqrt{d})$, there is a representation of a subgroup of finite index $H_{d,u}$ in Γ*

$$r_{d,u} : H_{d,u} \longrightarrow SU(Q_v, \mathcal{O}_k, \tau_d)$$

There is one final consideration that must be addressed, namely the form described by Witte is diagonal and Q_v is not. However this concern is addressed as follows. It can easily be shown by using the Gram-Schmidt process for example, that there is a change of basis matrix $M_d \in GL(\mathbf{Q}(\sqrt{d}))$ for which $M_d^* . Q_v . M_d$ is a diagonal form Δ_v. Then it is a standard argument (see for example Lemma 2.2 of [2]) that the groups $SU(Q_v, \mathcal{O}_k, \tau_d)$ and $SU(\Delta_v, \mathcal{O}_k, \tau_d)$ are commensurable, so at possibly the expense of passing to a further subgroup of finite index, we obtain a representation

$$r_{d,u} : H_{d,u} \longrightarrow SU(\Delta_v, \mathcal{O}_k, \tau_d)$$

as required by Theorem 1.1.

3 Projective considerations: r_d is faithful.

This section contains the proof that one can find many representations r_d which are faithful. It is basically geometric in nature and relies upon the fact that these representations are associated to convex real projective structures on the figure eight knot.

Throughout we denote the complement of the figure-eight knot in $\mathbf{S}^3$ by M and denote its fundamental group by Γ. We begin with some considerations of a fairly general nature. We recall that a group is said to be *non-radical* if there is no infinite normal nilpotent subgroup. The figure eight knot group (and indeed any finite volume hyperbolic manifold group) is non-radical. The following is a classical theorem of Zassenhaus.

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Theorem 3.1. *(See Kapovich [16, Thm 8.4]) Suppose that G is a finitely generated non-radical group and $\mathcal{L}$ any linear Lie group and that $\{\sigma_n\}$ is a convergent sequence of discrete, faithful representations of G into $\mathcal{L}$, say $\sigma_n \to \sigma_\infty$.*

Then σ_∞ is discrete and faithful.

We will also need the following:

Theorem 3.2. *(Goldman-Choi [14]) Let G be a finitely generated non-radical group.*

Suppose that Ω_n is a sequence of properly convex open domains in $\mathbf{RP}^d$ and $\sigma_n : G \to SL(d+1, \mathbf{R})$ a convergent sequence of discrete faithful representations for which $\sigma_n(G) < Aut(\Omega_n)$. Denote the limit representation by σ_∞.

Then if σ_∞ is irreducible, it preserves some properly convex open subset of $\mathbf{RP}^d$.

We define a subset of the representation variety $\Omega(G; \mathbf{RP}^d) < Hom(G, SL(d+1, \mathbf{R}))$ to be the set of representations $\sigma : G \to SL(d+1, \mathbf{R})$ satisfying

- σ is discrete and faithful

- There is a properly convex, open domain $\Omega_\sigma \subset \mathbf{RP}^d$ for which $\sigma(G) < Aut(\Omega_\sigma)$

In the case that G is the fundamental group of e.g. a hyperbolic d-manifold, the set $\Omega(G; \mathbf{RP}^d)$ is nonempty, since it contains the representation corresponding to the complete structure $\rho_1 = \phi_{\frac{1}{2}}$.

Theorem 3.3. *The path component of the set $\mathcal{G} = \{ \, v \mid \rho_v \in \Omega(\Gamma; \mathbf{RP}^3) \}$ which contains 1 is open and closed.*

In particular, once Theorem 3.3 is proven, we have that $\mathcal{G}$ is some interval, and therefore $\mathcal{G} = (0, \infty)$ since the only place the representations ρ_v fail to be defined is at $v = 0$. This implies that any specialization of v in $(0, \infty)$ is discrete and faithful, so that taken in conjunction with Theorem 2.6, we will have proved all of Theorem 1.1, barring the fact the image is Zariski dense.

Proof of Theorem 3.3. The fact that the set is closed follows from a concatenation of results proved above: Theorem 3 implies that the endpoint of a path of discrete faithful representations is discrete and faithful, then Theorem 3.2 implies the resulting representation is the holonomy of a properly convex structure, since we proved in 2.1 that the limit representation is irreducible.

Openness is a good deal more subtle. By applying a theorem of Cooper and Long ([10, Thm 0.4]) we find that a small deformation of the holonomy of the complete hyperbolic structure of M whose restriction to some (hence any) peripheral subgroup has a common fixed point in $\mathbf{RP}^3$ will itself be the holonomy of a projective structure on M. Furthermore, the cusp of this projective structure is foliated by projective rays with a common endpoint.

A priori, this projective structure need not be properly convex. However the main result of [12] shows that if the cusp of the resulting projective structure satisfies slightly stronger hypotheses then the deformed projective structure will be properly convex. Roughly speaking, the additional hypothesis is that the cusp of the deformed projective structure must admit a second foliation by "strictly convex" hypersurfaces that is transverse to the previously mentioned foliation by projective rays.

In [4] it is shown that for all $t \in (0, \infty)$ the representation ρ_t satisfy the hypotheses of the previously mentioned theorems. Since the two families of representations are conjugate this implies that for all $u \in (0, \infty)$, ρ_u also satisfy the same hypotheses. As a result we find that for u sufficiently close to 1 that ρ_u is the holonomy of a properly convex projective structure on M. The holonomy of such a structure is necessarily discrete and faithful and so for u sufficiently close to 1 we find that $\rho_u \in \Omega(\Gamma; \mathbf{RP}^3)$. We thus conclude that $\mathcal{G}$ is open. $\square$

4 Zariski Denseness

In this section we analyze the Zariski closure of the groups $\phi_t(\Gamma)$, which we denote G_t. We begin by with some background and results which can be found in Benoist [6]. A comprehensive summary of the necessary background in algebraic groups, notions of proximality, and Zariski closures can be found in [20, 21].

We begin by defining *proximality* in the context of groups, group actions, and representations. In all cases proximality is related to the existence of unique attracting fixed points.

Let G be a subgroup of $SL(n, \mathbf{R})$ and let g be an element of G. We say that g is *proximal* if it has a unique eigenvalue of largest modulus. In this case it is easy to see that this eigenvalue is real. If in addition this eigenvalue of largest modulus is positive then we say that g is *positive proximal*. A group is called *proximal* if it contains a proximal element and *positive proximal* if **every** proximal element is positive proximal.

A group action of G on $\mathbf{RP}^{n-1}$ is *proximal* if for any pair of points $x, y \in \mathbf{RP}^{n-1}$ there is a sequence $\{g_m\}$ of elements in G such that

$$\lim_{m \to \infty} g_m \cdot x = \lim_{m \to \infty} g_m \cdot y.$$

If G is a connected semi-simple Lie group then a representation $\rho : G \to SL(n, \mathbf{R})$ is *proximal* if the weight space corresponding to highest restricted weight is 1-dimensional. More specifically, if we let $G = KAN$ be an Iwasawa decomposition of G, where K is a maximal compact subgroup, A is a maximal abelian subgroup, and N is a maximal nilpotent subgroup, then the set

$$\{x \in \mathbf{R}^n \mid n(x) = x \ \forall n \in N\}$$

is a line.

We now discuss some relations between these notions. We say that a group $\Gamma \subset SL(n, \mathbf{R})$ is *strongly irreducible* if every finite index subgroup of Γ is irreducible. The following two theorems relate the various notions of proximality.

Theorem 4.1 ([15, Thm 2.9])**.** *Let G be a subgroup of $SL(n, \mathbf{R})$ then the following are equivalent*

1. *G is strongly irreducible and proximal.*

2. *G is irreducible and its action on $\mathbf{RP}^{n-1}$ is proximal.*

Theorem 4.2 ([1, Thm 6.3])**.** *Let G be a semi-simple Lie group with finite center and let $\rho : G \to SL(n, \mathbf{R})$ be an irreducible representation. Then the following are equivalent.*

1. *ρ is proximal.*

2. *$\rho(G)$ is a proximal.*

Given a strongly irreducible and proximal subgroup $G \subset SL(n, \mathbf{R})$ we can define the *limit set* of G, which we denote by Λ_G as the closure of the set of attractive fixed points in $\mathbf{RP}^{n-1}$ of the proximal elements of G. In [15, Thm 2.3] it is shown that the action of G on Λ_G is minimal (i.e. any non-empty closed G-invariant subset of $\mathbf{RP}^{n-1}$ contains Λ_G).

When G is a Zariski closed semi-simple subgroup of $SL(n, \mathbf{R})$ we can describe Λ_G more explicitly. Since G is proximal, Λ_G is non-empty and so let $x \in \Lambda_G$. Consider the orbit $G \cdot x$ of x. First, observe that $G \cdot x \subset \Lambda_G$ by G-invariance. Furthermore, since $G \cdot x$ is the orbit of an algebraic group acting algebraically on a variety we have that $G \cdot x$ is a smooth subvariety of $\mathbf{RP}^{n-1}$. If we let H be the Zariski closure of $G \cdot x$ then we see that $G \cdot x$ is open in H and so $H \backslash (G \cdot x)$ is a Zariski closed subset of $\mathbf{RP}^{n-1}$. Since Zariski closed sets are closed in the standard topology, minimality implies that

$H\backslash(G \cdot x)$ is either empty or Λ_G. However, $x \notin H\backslash(G \cdot x)$ and so $H\backslash(G \cdot x)$ is empty. As a result we see that $G \cdot x$ is a non-empty closed G-invariant subset and thus $G \cdot x = \Lambda_G$. By minimality of the action on the limit set we also find that Λ_G is the *unique* closed orbit for the action of G on $\mathbf{RP}^{n-1}$.

We can say even more. Let $G = KAN$ be an Iwasawa decomposition and let x_N be the (unique) point in $\mathbf{RP}^{n-1}$ corresponding to a highest weight vector. By our choice of x_N we see that the groups A and N both fix x_N and so $G \cdot x_N = K \cdot x_N$. Thus the orbit of x_N is closed and so $G \cdot x_N = \Lambda_G$ (i.e. the limit set of G is the orbit of a compact group.)

We can now identify the Zariski closure of $\phi_t(\Gamma)$, which we denote G_t. The representation $\phi_{1/2}$ corresponds to the holonomy of the complete finite volume hyperbolic structure on the figure-eight knot and so by the Borel density theorem we have $G_{1/2} = SO(3, 1)$. The main goal of this section is to show that when $t \neq 1/2$ that $G_t = SL(4, \mathbf{R})$. The proof is based on the following heuristic.

A Lie subgroup of $SL(4, \mathbf{R})$ with a large orbit in $\mathbf{RP}^3$ must be large.

Theorem 4.3. *If $t \neq 1/2$ then $G_t = SL(4, \mathbf{R})$. In particular, for $t \neq 1/2$, $\phi_t(\Gamma)$ is Zariski dense.*

Proof. Let $t \neq 1/2$. We have previously seen that ρ_t is an absolutely irreducible representation whose image preserves a properly convex open subset of $\mathbf{RP}^3$. We first show that $\rho_t(\Gamma)$ is strongly irreducible. Suppose for contradiction that $\rho_t(\Gamma)$ is not strongly irreducible. By work of Benoist [6, Lem 2.9] we can find a finite index subgroup of Γ that splits as a non-trivial direct sum. However, any finite index subgroup of Γ is the fundamental group of a finite volume hyperbolic 3-manifold and such groups never admit non-trivial direct sum decompositions.

Let G_t^0 be the connected component of G_t containing the identity. We claim that G_t^0 is semi-simple. To see this observe that by [15, Lem 2.6] we find that G_t^0 acts irreducibly on $\mathbf{R}^4$. As a result the action of G_t^0 turns $\mathbf{R}^4$ into a simple $\mathbf{R}[G_t^0]$ module. Let R_t be the unipotent radical of G_t^0. Since R_t is unipotent and solvable the Lie-Kolchin theorem implies that there is a non-trivial $\mathbf{C}[R_t]$-submodule, $E_{\mathbf{C}}$, of similtaneous 1-eigenvectors of R_t in $\mathbf{C}^4$. A simple computation shows that $E_{\mathbf{C}}$ is conjugation invariant and so there is a non-trivial $\mathbf{R}[R_t]$-submodule, $E_{\mathbf{R}} \subset \mathbf{R}^4$, of similtaneous 1-eigenvectors of R_t whose complexification is $E_{\mathbf{C}}$. However, since R_t is normal in G_t^0 we see that $E_{\mathbf{R}}$ is a non-trivial $\mathbf{R}[G_t^0]$-submodule. However, simplicity implies that this submodule must be all of $\mathbf{R}^4$. Therefore R_t acts trivially on $\mathbf{R}^4$ and is thus trivial. We conclude that G_t^0 is reductive. Furthermore, since G_t^0 is proximal and irreducible we see that it has trivial center and is thus semi-simple.

The group $\phi_t(\Gamma)$ is proximal and so Theorem 4.2 implies that the representation induced by the inclusion of G_t^0 into $SL(4, \mathbf{R})$ is also proximal.

Next we show that $\Lambda_{G_t^0}$ contains a codimension-1 submanifold. Let Γ_p be a peripheral subgroup of Γ. By work of [4, §6] it is possible to conjugate so that $\phi_t(\Gamma_p)$ to a lattice in the 2-dimensional abelian Lie group H of matrices of the form

$$\begin{pmatrix} 1 & 0 & s & s^2/2 - t \\ 0 & e^t & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus we see that the Zariski closure of $\phi_t(\Gamma_p)$ (and hence G_t) contains H. A generic orbit of H can be written in homogenous coordinates as

$$\{[-\log(|x|) + y^2/2 + c : \epsilon x : y : 1] \mid x > 0\},$$

where $\epsilon \in \{\pm 1\}$ and $c \in \mathbf{R}$ (see Figure 1). Since G_t^0 is irreducible we see that $\Lambda_{G_t^0}$ contains a point, z of one of these orbits and so $H \cdot z \subset G \cdot z \subset \Lambda_{G_t^0}$. Furthermore, since $\Lambda_{G_t^0}$ is closed we see that it contains the closure of this orbit, which is the boundary of a properly convex domain $\Omega \subset \mathbf{RP}^3$.
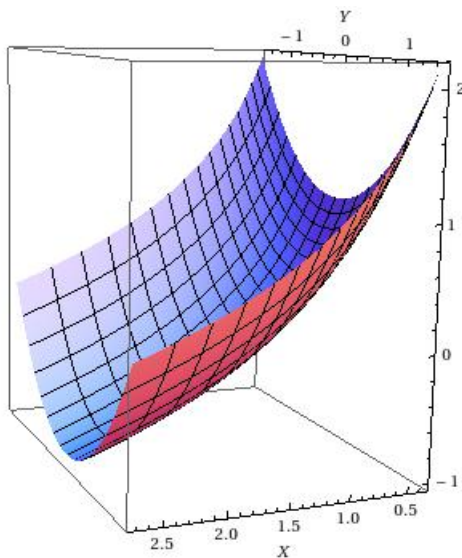
Figure 1: A generic orbit of H

There are now two cases to consider: either $\Lambda_{G_t^0}$ has codimension 0 and is thus equal to $\mathbf{RP}^3$ or $\Lambda_{G_t^0}$ is a codimension 1 submanifold of $\mathbf{RP}^3$ and is thus equal to $\partial\Omega$. In the first case work of Benoist [6, Lem 3.9 & Cor 3.5] shows that $G_t^0 = SL(4, \mathbf{R})$, in which case we are done.

We now rule out the second case. In this case we see that there is a point $w \in \mathbf{RP}^3$ such that $\Lambda_{G_t^0} = K \cdot w = \partial\Omega$, where K is a maximal compact subgroup of G_t^0. As a result we see that (up to a subgroup of index 2) K preserves Ω. Since K is compact we see that there is a point $x_0 \in \Omega$ such that both x_0 and its dual point $x_0 \in \Omega^{*1}$ are both fixed by K. The hyperplane dual to x_0^* provides us an affine patch containing Ω and for which x_0 is the center of mass of Ω. In these affine coordinates K acts by affine transformations fixing x_0.

If we let $\mathcal{E}$ be a John ellipsoid[2] for Ω centered at x_0 in this affine patch, then we see that K preserves $\mathcal{E}$. Since $\mathcal{E}$ is has maximal volume $\partial\mathcal{E}$ has nonempty intersection with $\partial\Omega$. Since K acts transitively on $\partial\Omega$ we see that $\partial\Omega \subset \partial\mathcal{E}$.

For dimensional reasons we see that $\partial\Omega$ has non-empty interior in $\partial\mathcal{E}$. Since $\partial\Omega$ is a K-orbit we see that it is an open subset of $\partial\mathcal{E}$. We conclude that $\partial\Omega = \partial\mathcal{E}$ and thus Ω is an ellipsoid. Since $\phi_t(\Gamma)$ preserves this ellipsoid we see that it is conjugate to a subgroup of $SO(3, 1)$. This is a contradiction since, for example the image of the longitude has an eigenvalue whose inverse is not also an eigenvalue.

$\square$

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[1] see [20, §9] for a definition of dual point of properly convex domains

[2] A John ellipsoid for a convex subset Ω of affine space is an ellipsoid of maximal Euclidean volume contained in Ω with the same center of mass

A The matrices

The discrete faithful representation corresponding to the hyperbolic structure occurs at $v = 1$.

$$\rho_v(x) = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{2}(1+1/v) & \frac{1}{\sqrt{12}}(1-1/v) \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}(1+1/v) & -\frac{1}{\sqrt{12}}(1-1/v) \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_v(y) = \begin{pmatrix} v+\frac{1}{2} & -v+\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{12}}(7-4v) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{12}}(4/v-1) \\ \frac{1}{2}v & -\frac{1}{2}v & 1 & \frac{1}{\sqrt{3}}(1-v) \\ \frac{\sqrt{3}}{2}v & -\frac{\sqrt{3}}{2}v & 0 & 2-v \end{pmatrix}$$

The discrete faithful representation corresponding to the hyperbolic structure occurs at $t = 1/2$.

$$\phi_t(x) = \begin{pmatrix} 1 & 0 & 1 & t-1 \\ 0 & 1 & 1 & t \\ 0 & 0 & 1 & t+\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\phi_t(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2+1/t & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

The representations become conjugate for $v = 2t$.

B An alternate proof of Zariski denseness

In this appendix we provide an alternate proof of Zariski denseness. This was our original proof of Zariski denseness and we believe it provides a technique for analyzing the Zariski closure of specific examples when more general theorems do not apply.

Let $\mathfrak{sl}_4$ denote the Lie algebra of $SL(4,\mathbf{R})$, let G_v denote the Zariski closure of $\rho_v(\Gamma)$ in $SL(4,\mathbf{R})$, and let $\mathfrak{g}_v$ denote the Lie algebra of G_v. The goal of this section will be to complete the proof of Theorem 1.2 by proving the following lemma.

Lemma B.1. *For all but finitely many values of v, $G_v = SL(4,\mathbf{R})$.*

The idea of the proof is as follows: the group G_v acts on $\mathfrak{g}_v$ via the adjoint action. By restricting this action to $\rho_v(\Gamma)$ we get an action of Γ on $\mathfrak{g}_v$. We will find an element $w \in \mathfrak{g}_v$ such that the span of the orbit $\Gamma \cdot w$ is 15-dimensional. This implies that $\mathfrak{g}_v$ is at least 15 dimensional and since $SL(4,\mathbf{R})$ is a 15-dimensional Lie group we conclude that $G_v = SL(4,\mathbf{R})$.

Proof of Lemma B.1. Let x be one of the meridional generators of Γ. Then

$$\rho_v(x) = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{2}(1+1/v) & \frac{1}{\sqrt{12}}(1-1/v) \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}(1+1/v) & -\frac{1}{\sqrt{12}}(1-1/v) \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix $\rho_v(x)$ can be written as $\exp(w)$, where

$$w = \begin{pmatrix} \frac{v-1}{4v} & \frac{v-1}{4v} & \frac{v+1}{2v} & -\frac{v-1}{2\sqrt{3}v} \\ \frac{1}{4}\left(\frac{1}{v}-1\right) & \frac{1}{4}\left(\frac{1}{v}-1\right) & -\frac{v+1}{2v} & \frac{v-1}{2\sqrt{3}v} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Furthermore, because of the size and frequency of the Jordan blocks of x work of Culver [13] shows that w is the unique element of $\mathfrak{sl}_4$ the exponentiates to $\rho_v(x)$. We claim that $w \in \mathfrak{g}_v$.

By the same arguements from §4 we see that G_v is a semi-simple subgroup of $SL(4, \mathbf{R})$. Since ρ_v is irreducible we see that any element contained in the center of G_v must be contained in the center of $SL(4, \mathbf{R})$, and so the center of G_v is finite. By applying the main result of Lai [17], we see that there exists an integer p and $u \in \mathfrak{g}_v$ such that $\exp(u) = \rho_v(x)^p$. The matrix $\rho_v(x)^p$ has the same Jordan shape as $\rho_v(x)$ and so u is the only element of $\mathfrak{sl}_4$ that exponentiates to $\rho_v(x)^p$. On the other hand $\exp(pw) = \rho_v(x)^p$ and so we see that $u = pw$ and thus $pw \in \mathfrak{g}_v$. Since $\mathfrak{g}_v$ is closed under scaling this proves the claim.

In the Mathematica notebook [26] we exhibit an explicit set of 15 elements in the orbit of w under the adjoint action of Γ. We subsequently show that with finitely many exceptions this set is linearly independent. This shows that apart from these finitely many exceptions that $\mathfrak{g}_v = \mathfrak{sl}_4$ in which case we conclude that Γ_v is Zariski dense in $SL(4, \mathbf{R})$. $\square$

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Department of Mathematics,
University of California,
Santa Barbara, CA 93106.
Email: long@math.ucsb.edu

Department of Mathematics,
University of California,
Santa Barbara, CA 93106.
Email: sballas@math.ucsb.edu