

MATH 108A MIDTERM SOLUTIONS

① Definitions!

(2A) $\underline{w} \in \text{span}(v_1, \dots, v_k)$ so
 $\underline{w} = \alpha_1 v_1 + \dots + \alpha_k v_k$ NOT all α_i 's = 0
 else $\underline{w} = \underline{0}$. Reorder so $\alpha_1 \neq 0$.

Then $v_1 = -\frac{1}{\alpha_1} w_1 - \frac{1}{\alpha_1} (\alpha_2 v_2 + \dots + \alpha_k v_k)$. \circledast
 $\Rightarrow \text{span}(\underline{w}, v_2, \dots, v_k) \subset \text{span}(v_1, v_2, \dots, v_k)$

(2B) Since $\underline{w}, v_1, \dots, v_k$ are LD. Then $\exists \lambda_0 \underline{w} + \lambda_1 v_1 + \dots + \lambda_k v_k = \underline{0}$
 not all λ 's = 0.

$\lambda_0 \neq 0$, else this means $\lambda_1 v_1 + \dots + \lambda_k v_k = \underline{0}$
 $\& \Rightarrow \lambda_1 = \dots = \lambda_k = 0$ since v_i 's are LI so all λ 's
 zero. If $\lambda_0 \neq 0$ then $\underline{w} = -\frac{1}{\lambda_0} (\lambda_1 v_1 + \dots + \lambda_k v_k)$
 $\Rightarrow \underline{w} \in \text{span}(v_1, \dots, v_k)$; it isn't.
 So $\underline{w}, v_1, \dots, v_k$ are LI

(3AB) If $A \supset B$, then $A \cup B = A$ is subspace. \square

If $A \cup B$ is subspace & $A \not\supset B$ take $b \in B \setminus A$.
 Pick $a \in A$. Then $a + b \in A \cup B$.

If $a + b \in A$, then $b \in A$; it isn't. So $a + b \in B$
 $\Rightarrow a \in B$ i.e. $A \subset B$

(4A) V is finite dim so $\exists \{v_1, \dots, v_k\}$ a finite spanning set. If L , then we're done, this is a basis. If not $\exists \lambda_1 v_1 + \dots + \lambda_k v_k = 0$ λ_i 's not all zero. Reorder, $\lambda_1 \neq 0$, then $v_1 = -\frac{1}{\lambda_1}(\lambda_2 v_2 + \dots + \lambda_k v_k)$ so $V = \text{span}(v_1, \dots, v_k) = \text{span}(v_2, \dots, v_k)$. Repeat until you get L set.

(5A) v_1, \dots, v_k spans V $\dim V = k$. If not L , $\exists \lambda_1 v_1 + \dots + \lambda_k v_k = 0$ λ_i 's not all zero & we're at 4. $\text{span}(v_1, \dots, v_k) = \text{span}(v_2, \dots, v_k)$ & $\dim V \leq k-1$. It isn't. So v_1, \dots, v_k is L .

~~5B~~ (5B) If v_1, \dots, v_k doesn't span, pick $w \notin \text{span}(v_1, \dots, v_k)$ then by ~~5B~~ Q2 $\{v_1, \dots, v_k, w\}$ is L so

$$\dim V \geq k+1. \text{ It isn't}$$

So v_1, \dots, v_k spans.