MATH 164: HOMEWORK 8.5

(Not to be turned in–extra practice for final)

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Consider the linear inequality constrained problem

$$\begin{aligned} \text{minimize} \quad & f(x) \ , \\ \text{subject to} \quad & Ax \geq b \ , \end{aligned}$$

where A has full row rank. Fix a point x_*. The necessary and sufficient conditions for x_* to be a local minimizer are listed in the following table. The first four conditions are the same in both columns. The fifth conditions are different.

Necessary	**Sufficient**
$Ax_* \geq b$ $\exists \lambda_* \in \mathbb{R}^m$ so $\nabla f(x_*) = A^T \lambda_*$ $\lambda_* \geq 0$ $\lambda_*^T (Ax_* - b) = 0$	$Ax_* \geq b$ $\exists \lambda_* \in \mathbb{R}^m$ so $\nabla f(x_*) = A^T \lambda_*$ $\lambda_* \geq 0$ $\lambda_*^T (Ax_* - b) = 0$
$Z^t D^2 f(x_*) Z$ is positive semidefinite where $\hat{A}$ is the submatrix of active constraints at x_* and Z is a basis matrix for $\mathcal{N}(\hat{A})$	$Z_+^t D^2 f(x_*) Z_+$ is positive definite where $\hat{A}_+$ be the submatrix of $\hat{A}$ for which the corresponding elements of λ_* are strictly positive and Z_+ is a basis matrix for $\mathcal{N}(\hat{A}_+)$.

Based on our experience with linear **equality** constrained problems, we would have hoped that the fifth sufficient condition would have been

(**) " $Z^t D^2 f(x_*) Z$ is positive definite, where $\hat{A}$ is the submatrix of active constraints at x_* and Z is a basis matrix for $\mathcal{N}(\hat{A})$"

The following example shows that is is possible for x_* to satisfy the first four sufficiency conditions and **(**)** and still not be a local minimum. This is why, instead of **(**)**, we need the stronger condition listed in the "sufficient" column.

Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = 2x_1^2 + 4x_2^3 \ , \\ \text{subject to} \quad & x_2 \leq 0 \ . \end{aligned}$$

(a) Show that $x_* = (0, 0)^T$ satisfies the first four sufficient conditions.

(b) Show that x_* satisfies condition **(**)**, where Z is a basis matrix for $\mathcal{N}(\hat{A})$.

(c) Show that x_* is **not** a local minimizer. (Hint: consider the value of f at the point $(x_1, x_2) = (0, -\epsilon)$ for $\epsilon > 0$ small.)

(d) Since x_* is **not** a local minimizer, the fifth sufficient condition must not hold. Verify that this is true by computing $Z_+^t D^2 f(x_*) Z_+$ and showing it is not positive definite. (Hint: a submatrix with no rows is the zero matrix. A basis matrix for the null space of the zero matrix is the identity matrix.)

Question 2 (Textbook Problem 14.2.7)

Let A be a matrix of full row rank. Find the point in the set $Ax = b$ which minimizes $f(x) = \frac{1}{2}x^T x$. (Hint: You may use the fact that AA^t is invertible. Use Lagrange multipliers. This problem illustrates how Lagrange multipliers can make a problem much easier to solve!)

Question 3 (Textbook Problem 14.4.1)

Find a local minimizer of the following problem.

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x_1^2 + x_2^2 \ , \\ \text{subject to} \quad & 2x_1 + x_2 \geq 2 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0 \ . \end{aligned}$$

Here is the approach for how to find local minimizers of a linear inequality constrained problem. Break the problem into cases. In each case, you will consider a different possible choices for the submatrix of active constraints $\hat{A}$ and find all points that satisfy the sufficient conditions for a local minimizer. These points must be local minimizers. Any points which fail the necessary conditions cannot be local minimizers. For this problem, there is only one local minimizer, and you can use convexity to conclude it is a global minimizer.

Question 4 (Similar to Textbook Problem 14.4.2)

Find a local minimizer of the following problem.

$$\begin{aligned} \text{minimize} \quad & f(x) = -x_1^2 + x_2^2 - x_1 x_2 \\ \text{subject to} \quad & 2x_1 - x_2 \geq 2 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \ . \end{aligned}$$

Follow the same approach as in the previous problem. Again, there will only one point that satisfies the sufficient conditions to be a local minimizer. All other points will fail the necessary conditions to be a local minimizer.

Question 5 (Textbook Problem 14.4.6)

Let A be an $m \times n$ matrix whose rows are linearly independent. Prove that there exists a vector $p \in \mathbb{R}^n$ so that $Ap = e_1$ where $e_1 = (1, 0, \ldots, 0)^T$.