Math 164: Homework 7

Due Friday, May 22nd

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1 (Textbook Problem 6.2.8)

Solve the following linear program using duality theory.

$$\begin{aligned} \text{minimize } & z = x_1 + 2x_2 + \cdots + nx_n, \\ \text{subject to } & x_1 \geq 1 \ , \\ & x_1 + x_2 \geq 2 \ , \\ & \ldots \\ & x_1 + x_2 + \cdots + x_n \geq n \ , \\ & x_1, x_2, \ldots, x_n \geq 0 \ . \end{aligned}$$

(Hint: show that $x_1 = n$ and $x_2, \ldots, x_n = 0$ is feasible for the primal and find a feasible solution for the dual for which the values of the objective functions are equal. Then explain why this ensures you have found the optimal solution.)

Question 2* (Textbook Problem 6.2.16)

Consider the linear program

$$\begin{aligned} \text{minimize } & z = 2x_1 + 9x_2 + 3x_3 \ , \\ \text{subject to } & -3x_1 + 2x_2 + x_3 \geq 1, \\ & x_1 + 4x_2 - x_3 \geq 1 \ , \\ & x_1, x_2, x_3 \geq 0 \ . \end{aligned}$$

(i) Find the dual to this problem and solve it graphically.

(ii) Use complementary slackness to obtain the solution to the primal. **See page 184 in the textbook for the notion of complementary slackness for linear programs in canonical form.**

Question 3*

Recall that a symmetric $n \times n$ matrix A is **positive semi-definite** if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$. Show that a matrix A is positive semi-definite if and only if all eigenvalues of A are nonnegative. (Hint: Use the Spectral Theorem from linear algebra. This theorem tells you that if a matrix is symmetric, then there exists an orthonormal basis of eigenvectors.)

Question 4* (Similar to Textbook Problem 2.3.20)

Determine if the following functions are convex, concave, both, or neither.

(a) $f(x_1, x_2) = 2x_1 - 4x_2$

(b) $f(x_1, x_2) = x_1^2 + x_2^2$

(c) $f(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 5x_2^2 + 31x_1 - 70x_2$

Question 5* (Similar to Textbook Problem 2.6.4)

Consider the function

$$f(x_1, x_2) = 2x_1^3 + 2x_1x_2^2 + x_1 + 2x_2^3 \; .$$

(a) Find the first three terms of the Taylor series for f centered at $x_0 = (2, 1)$. By "first three terms of the Taylor series" I mean the following expression:

$$f(x_0) + \nabla f(x_0)^t(x - x_0) + (x - x_0)^t D^2 f(x_0)(x - x_0).$$

Note that this is identical to the Taylor expansion we considered in class, except the argument of the Hessian is x_0 instead of ξ.

(b) Following the notation of the book, let $p = x - x_0$. Evaluate the expression you found in part (a) for $p = (-0.1, 0.1)^T$ and compare with the value of $f(x_0 + p)$. Would we expect these values to become more similar or more different if we took $p = (-1, 1)^T$ instead?

Question 6 (Textbook Problem 2.6.4)

Find the first three terms of the Taylor series for

$$f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$$

about the point $x_0 = (3, 4)^T$.

Question 7* (Textbook Problem 2.6.6)

Prove that if $p^T \nabla f(x_k) < 0$, then $f(x_k + \epsilon p) < f(x_k)$ for $\epsilon > 0$ sufficiently small. (Hint: Consider the Taylor expansion of f with $x = x_k + \epsilon p$ and $x_0 = x_k$. Then look at $f(x_k + \epsilon p) - f(x_k)$. Pretend that $p^T D^2(\xi)p$ is a constant independent of ϵ and use the fact from class that $C_1\epsilon^2 < C_2\epsilon$ for ϵ sufficiently small.)

(While is not technically true that $p^T D^2(\xi)p$ is a constant independent of ϵ, those of you who took Math 131A could probably show that $p^T D^2(\xi)p$ is bounded above by a constant M, which is all you actually need for this problem.)

Question 8* (Textbook Problem 3.2.2)

Suppose A is an $m \times n$ matrix with full row rank. We say that a matrix Z of dimension $n \times r$, $r \geq n - m$, and rank $n - m$ is a *null-space matrix* for A if it satisfies $AZ = 0$. If $r = n - m$ (i.e. the columns of Z are linearly independent), then Z is a *basis matrix* for the null space of A.

Let Z be an $n \times r$ null-space matrix for the matrix A. If Y is any invertible $r \times r$ matrix, prove that $\hat{Z} = ZY$ is also a null-space matrix for A. Clearly explain how you use that Y is invertible.