Math 164: Homework 6

Due Friday, May 15th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1 (Textbook Problem 6.2.1)

Consider the following linear program:

$$\begin{aligned} \text{maximize } & z = -x_1 - x_2 \ , \\ \text{subject to } & -x_1 + x_2 \geq 1, \\ & 2x_1 - x_2 \leq 2 \ , \\ & x_1, x_2 \geq 0 \ . \end{aligned}$$

Find the dual to the problem. Solve the primal and the dual graphically, and verify that the results of the strong duality theorem hold.

Question 2*

Prove the following corollary of the weak duality theorem.
Suppose x is feasible for the primal LP (in standard form) and y feasible for the dual LP. If $c^t x = b^t y$, then x is optimal for the primal and y is optimal for the dual.

Question 3*

Find the dual of a linear program in standard form.

Question 4 (Textbook Problem 6.2.3)

Prove the following corollary of the weak duality theorem.
If the primal is unbounded, then the dual is infeasible. If the dual is unbounded, then the primal is infeasible.

Question 5* (Textbook Problem 5.2.4)

Final all the values of the parameter a such that the following linear program has a finite optimal solution:

$$\begin{aligned} \text{minimize } & z = -ax_1 + 4x_2 + 5x_3 - 3x_4 \ , \\ \text{subject to } & 2x_1 + x_2 - 7x_3 - x_4 = 2, \\ & x_1, x_2, x_3, x_4 \geq 0 \ . \end{aligned}$$

(Hint: use duality.)

Question 6* (Textbook Problem 6.2.19)

Consider the linear programming problem

$$\begin{aligned} \text{minimize } z &= c^T x \ , \\ \text{subject to } Ax &\le b, \\ x &\ge 0 \ . \end{aligned}$$

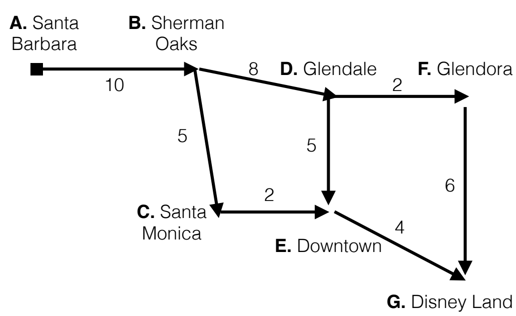
Assume that this problem and its dual are both feasible. Let x_* be optimal for the primal, let z_* be its associated objective value, and let y_* be optimal for the dual problem. Show that either

$$z_* = y_*^T A x_* \text{ or } z_* = -y_*^T A x_*.$$

(The answer you get will depend on which equivalent representation you choose for the dual problem.)

Question 7*

Suppose you *still* want to find the maximum number of cars per hour that can get from Santa Barbara to Disney Land. The traffic network is represented in the figure below. The numbers on each arrow represent the capacity of that road in thousands of cars per hour. Assume that all cars that enter an intersection also leave the intersection.



You showed on the first homework assignment that this problem can be reformulated as the following linear program (where x_{ij} represents the amount of water flowing from i to j):

$$\begin{aligned} &\text{maximize } z = x_{AB}, \\ &\text{subject to } x_{BD} + x_{BC} = x_{AB} && 0 \le x_{AB} \le 10,\ 0 \le x_{BC} \le 5 \\ &\qquad x_{BC} - x_{CE} = 0 && 0 \le x_{BD} \le 8,\ 0 \le x_{DE} \le 5 \\ &\qquad x_{BD} - x_{DF} - x_{DE} = 0 && 0 \le x_{CE} \le 2,\ 0 \le x_{DF} \le 2 \\ &\qquad x_{EG} - x_{DE} - x_{CE} = 0 && 0 \le x_{DE} \le 5,\ 0 \le x_{EG} \le 4 \\ &\qquad x_{FG} - x_{DF} = 0 && 0 \le x_{FG} \le 6. \end{aligned}$$

(a) Find the dual linear program. Then show it can be rewritten in the following form:

$$\begin{aligned} &\text{minimize } w = 10y_{AB} + 5y_{BC} + 8y_{BD} + 2y_{CE} + 5y_{DE} + 2y_{DF} + 4y_{EG} + 6y_{FG}, \\ &\text{subject to } y_G - y_A = 1 \\ &\qquad y_A - y_B + y_{AB} \ge 0, \quad y_C - y_B + y_{BC} \ge 0, \\ &\qquad y_B - y_D + y_{BD} \ge 0, \quad y_C - y_E + y_{CE} \ge 0, \\ &\qquad y_D - y_E + y_{DE} \ge 0, \quad y_D - y_F + y_{DF} \ge 0, \\ &\qquad y_E - y_G + y_{EG} \ge 0, \quad y_F - y_G + y_{FG} \ge 0, \\ &\qquad y_{AB}, y_{BC}, y_{BD}, y_{CE}, y_{DE}, y_{DF}, y_{EG}, y_{FG} \ge 0 \end{aligned}$$

Now assume you are filled with an evil desire to stop all traffic to the happiest place on earth, and suppose that the numbers on each arrow represent the cost of demolishing that road. It turns out that the dual linear program can be interpreted as the cheapest way to demolish roads so that no traffic can get through to Disney Land!

We will call each way of demolishing roads such that any traffic starting at Santa Barbara cannot get to Disney Land a *cut*. (For example, demolishing BD and BC is a cut, but demolishing BD and DF is not a cut.) Cuts can be represented in your dual linear program in the following way:

- If node i is on the Santa Barbara side of your cut, let $y_i = 0$.
- If note j is on the Disney Land side of your cut, let $y_j = 1$.
- If you cut the arc connecting i to j, let $y_{ij} = 1$. Otherwise, let $y_{ij} = 0$.

(b) There are four possible cuts that only involve demolishing two of the roads. Show that all of them are feasible for the dual linear program. How much does each cut cost? Explain why the objective function measures the cost of a given cut.

Note that there are feasible points for the dual linear program that don't correspond to cuts. For example, you could choose a feasible point where the dual variables were something aside from zero or one. However, one can prove that there will always be an optimal solution to this problem that *does* correspond to a cut.

(c) Use the following approach to solve both the primal and dual linear programs. First, choose a feasible solution for the max flow problem and write down the value of the objective function. Then, choose a feasible solution for the min cut problem and write down the value of the objective function. Repeat until you find a flow amount that equals the cost of your cut. Name the theorem that ensures these must be optimal solutions. (You may use any duality theorem here, even if we only proved it in class for linear programs in standard form.)

Max Flow/Min Cut: Application to Big Data

A natural question you can ask if you have a large data set is how to classify elements of that data set into groups. For example, one popular data set is voting records of Congress, and the classification problem is to see if you can guess the political party of a member based on their voting record. (This problem is more interesting if you look at historical voting records, before most members of Congress always voted along party lines.)

One way to solve this classification problem is to represent your data set as a similarity network. The nodes represent members of congress and the numbers on the arcs connecting the nodes represent how similar their voting record was. Then, to classify members to either the Republican or Democratic Party, you try to find a minimum cut – i.e. a way to break the network into two pieces, so that the cost of breaking connected nodes apart is "cheapest". In this way, you assign connected nodes to different groups if the corresponding congressional members didn't have very similar voting records.

This connection to big data is one of the reasons the max flow/min cut problem is a hot topic nowadays. It turns out it is useful for things aside from water and traffic flow!