Math CCS 117: Practice Midterm 1

(Not to be turned in)

Question 1

Define a sequence s_n as follows: $s_1 = 1$ and, for $n \geq 1$, $s_{n+1} = \frac{1}{3}(s_n + 1)$. Find $\lim_{n\to+\infty} s_n$.

Question 2

(a) Suppose $\lim_{n\to+\infty} s_n = +\infty$ and s_{n_k} is a subsequence of s_n. Prove that $\lim_{k\to+\infty} s_{n_k} = +\infty$.

(b) Suppose s_n is a sequence for which the limit does not exist—that is s_n doesn't converge or diverge to $\pm\infty$—and s_{n_k} is a subsequence of s_n. Does the limit of s_{n_k} not exist? Justify your answer with a proof or counterexample.

Question 3

Suppose A and B are nonempty subsets of $\mathbb{R}$. (Note that we do not assume that either A or B is bounded above.) Define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Prove $\sup(A + B) = \sup A + \sup B$.

Question 4 - Extra Credit

Given a sequence s_n of real numbers, define its arithmetic mean by

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

(a) If s_n converges, prove that σ_n converges.

(b) Give an example to show that the converse of part (a) is not true.

(c) Let $a_n = s_{n+1} - s_n$. Assume that $\lim_{k\to+\infty} k a_k = 0$ and σ_n converges. Prove that s_n converges.

Hint: First, show that

$$s_{n+1} - \sigma_{n+1} = \frac{1}{n+1}\sum_{k=1}^{n} k a_k.$$

Moral of the problem: while the convergence of σ_n is not, in general, sufficient to imply the convergence of s_n, if we also know that the increments of s_n converge to zero sufficiently quickly, it is sufficient.