Lecture 9

Solution to HW3, Q7 posted

Midterm 1 on Mon, May 6th

Cor: For $a_n : \mathbb{N} \to \overline{\mathbb{R}}$ monotone, $\lim_{n\to\infty} a_n$ exists.

17 Real Exponents

Idea: For any $a > 0$, define a^x as $\lim_{n\to\infty} a^{r_n}$ where $r_n : \mathbb{N} \to \mathbb{Q}$ satisfying $r_n \nearrow x$.

Thm: $\forall x \in \mathbb{R}$, $\exists\, r_n : \mathbb{N} \to \mathbb{Q}$ s.t. $r_n \nearrow x$.

Lemma: Suppose $a > 1$, $x \in \mathbb{R}$ and $r_n, s_n : \mathbb{N} \to \mathbb{Q}$ s.t. $r_n \nearrow x$, $s_n \nearrow x$. Then
$$\lim_{n\to\infty} a^{r_n} = \lim_{n\to\infty} a^{s_n}.$$

note: result clearly holds when $a=1$.

Def: For any $a \geq 1$, $x \in \mathbb{R}$, define
$$a^x = \lim_{n\to\infty} a^{r_n}$$
where $r_n : \mathbb{N} \to \mathbb{Q}$ satisfies $r_n \nearrow x$.
For any $0 < a < 1$, $x \in \mathbb{R}$, define
$$a^x = \left(\frac{1}{a}\right)^{-x}.$$

Thm:
For all $a, b \in \mathbb{R}$, $x > 0$
(i) $x^{a+b} = x^a x^b$
(ii) $x^a = \left(\frac{1}{x}\right)^{-a}$

(iii) $(xy)^a = x^a y^a$

(iv) $(x^a)^b = x^{ab}$

(vi) If $0<x<y$, $a>0$, then $x^a<y^a$

(v) If $x>1$ and $a<b$, then $x^a<x^b$.

Rmk:

Suppose (v) holds for $0<a<b$.

So $x>1 \Rightarrow x^a<x^b \Rightarrow x^{-b}<x^{-a}$

Thus $-b<-a$ ensure $x^{-b}<x^{-a}$.

Pf: We will show (i). Recall that we already have shown the result for $a,b\in\mathbb{Q}$. Now, suppose $a,b\in\mathbb{R}$. Choose $r_n, s_n : \mathbb{N}\to\mathbb{Q}$ s.t. $r_n \nearrow a$, $s_n \nearrow b$.

Hence $r_n + s_n \nearrow a+b$.
By definition of real exponents, for $x \geq 1$,

$$x^{a+b} = \lim_{n\to\infty} x^{r_n+s_n} = \lim_{n\to\infty} x^{r_n} x^{s_n}$$
$$= x^a x^b$$

For $0 < x < 1$, we have $\frac{1}{x} > 1$, so
$$x^{a+b} = \left(\frac{1}{x}\right)^{-a-b} = \left(\frac{1}{x}\right)^{-a}\left(\frac{1}{x}\right)^{-b} = x^a x^b.$$

by previous case

18 The Bolzano-Weierstrass Thm

Recall: For $s_n : \mathbb{N} \to \mathbb{R}$,
convergent $\Rightarrow$ bounded

not nec. true for $s_n : \mathbb{N} \to \overline{\mathbb{R}}$; consider $s_n = (-\infty, 1, 1, 1, \ldots)$

bounded and monotone $\Rightarrow$ convergent

the limit of a sequence $a_n : \mathbb{N} \to \overline{\mathbb{R}}$ exists

$$\Longleftrightarrow \begin{cases} \lim_{n\to\infty} a_n = L \in \mathbb{R} \\ \lim_{n\to\infty} a_n = +\infty \\ \lim_{n\to\infty} a_n = -\infty \end{cases}$$

"the sequence converges to L"

Thm: Every sequence $s_n : \mathbb{N} \to \overline{\mathbb{R}}$ has a monotone subsequence.

Pf: We will say that the n^{th} element of a sequence is dominant if it is greater than every element that follows, that is

s_n is dominant if $s_n > s_m$ $\forall\, m > n$.

$\boxed{\text{Case 1}}$ Suppose s_n has infinitely many dominant elements.

Define s_{n_k} to be the subsequence of dominant elements.
Then $s_{n_k} > s_{n_{k+1}}$ $\forall\, k \in \mathbb{N}$,
so s_{n_k} is a decreasing subsequence, hence monotone.

$\boxed{\text{Case 2}}$ Suppose s_n has finitely many dominant elements.

- Choose n_1 so that s_{n_1} is beyond all dominants elts
- Since s_{n_1} is not dominant, $\exists\ n_2$ s.t. $s_{n_2} \geq s_{n_1}$.
- Assume we have chosen s_{n_k} not dominant with $s_{n_k} \geq s_{n_{k-1}}$.
- Since s_{n_k} not dominant, so n_{k+1} so that $s_{n_{k+1}} \geq s_{n_k}$ and $s_{n_{k+1}}$ not dominant.

Thus, we have found a subseq. that is increasing, hence monotone.

Thm (Bolzano-Weierstrass): Every bounded sequence has a cnvgt subseq.

Pf: This follows immediately from prev thm.

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Last important type of sequence...

19 The Cauchy Criterion

Def: $a_n : \mathbb{N} \to \mathbb{R}$ is a <u>Cauchy sequence</u> if, $\forall \varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t. $m, n \geq N$ ensures $|a_n - a_m| < \varepsilon$.

A convergent sequence "bunches up" around its limit. → need to know what limit is

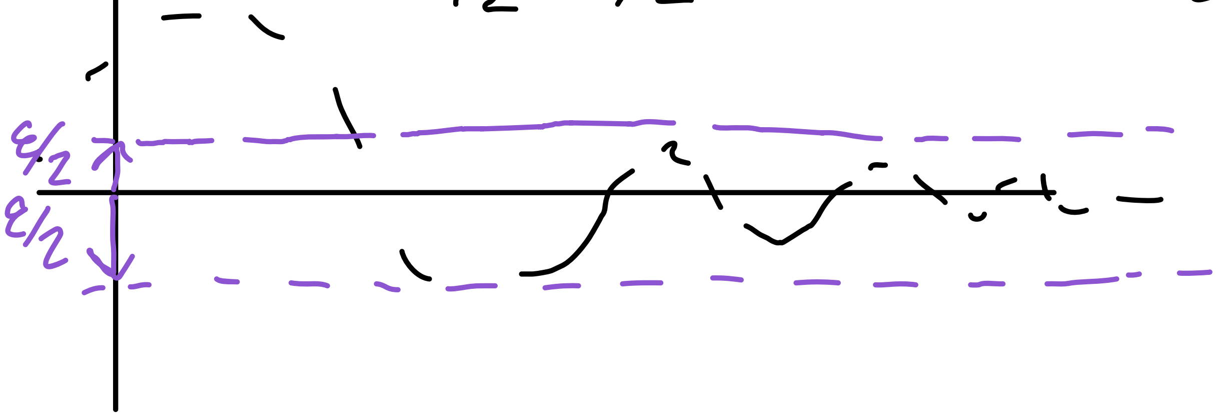
A Cauchy sequence "bunches up" around itself. → don't need to know limit

Thm: All convergent real valued sequences are Cauchy.

Pf: Assume $a_n : \mathbb{N} \to \mathbb{R}$ converges to some $L \in \mathbb{R}$. Fix $\varepsilon > 0$ arbitrary. $\exists N$ s.t. $n \geq N$ ensures $|a_n - L| < \frac{\varepsilon}{2}$.

Then $m, n \geq N$,

$$\begin{aligned} |a_m - a_n| &= |a_m - L + L - a_n| \\ &\leq |a_m - L| + |L - a_n| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned} \qquad \square$$



Suprisingly, the converse is true.

This is another way to express the fact that $\mathbb{R}$ is a "continuum" with no "gaps."

<u>Thm</u>: All real-valued Cauchy sequences are convergent.

<u>Pf</u>: Let $a_n : \mathbb{N} \to \mathbb{R}$ be Cauchy.

<u>Claim</u>: a_n is bounded.
Given $\varepsilon = 1$, $\exists$ N s.t. $n, m \geq N$ ensures $|a_m - a_n| < 1$. Thus $n \geq N$ ensures $|a_n| = |a_n - a_N + a_N| < 1 + |a_N|$.

Hence $\forall n \in \mathbb{N}$,
$|a_n| \leq \max\{|a_1|, \ldots, |a_{N-1}|, 1 + |a_N|\}$.

By Bolzano-Weierstrass, there is a subseq a_{n_k} that converges to some $L \in \mathbb{R}$.

Fix $\varepsilon > 0$. Since a_n is Cauchy, $\exists N$ s.t. $m, n \geq N$ ensures

$$|a_m - a_n| < \varepsilon/2.$$

Since a_{n_k} converges to L, $\exists N'$ s.t. $k \geq N'$ ensures

$$|a_{n_k} - L| < \varepsilon/2.$$

Thus, choose K suff large s.t. $K \geq N'$ and $\underbrace{n_K}_{\text{Strictly increasing sequence of natural \#s.}} \geq N$, we have, $\forall n \geq N$,

$$\begin{aligned}|a_n - L| &= |a_n - a_{n_k} + a_{n_k} - L| \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

$\square$

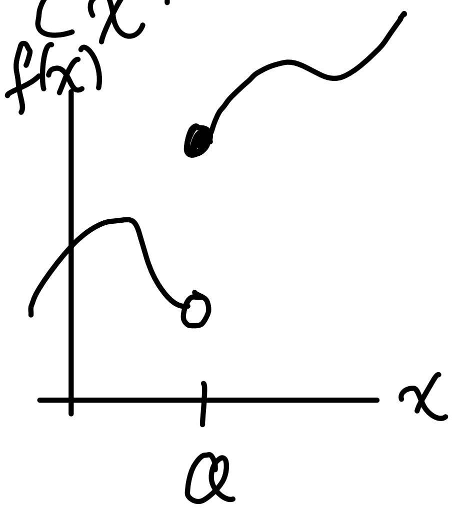
Rmk: Why don't we define what it means for an extended real-valued seq. to be Cauchy?

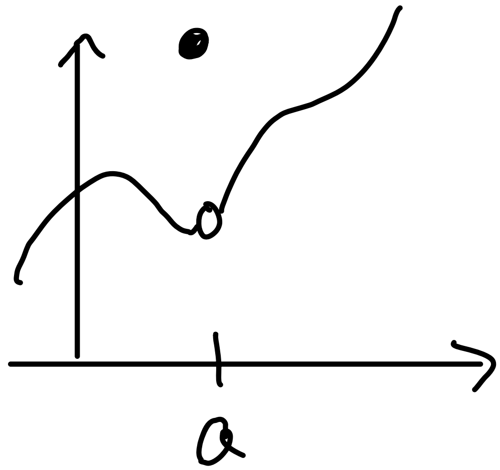
We've just shown that, for a real valued sequence, convergent $\overset{?}{\Longleftrightarrow}$ Cauchy.

Issues: $+\infty - (+\infty) = ?$, Cauchy $\not\Rightarrow$ cnvgt

30 Definition of the Limit of a Function

Ex:





Def: Given $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, a is an accumulation point of X if $\forall\, \delta > 0$, $\exists\, x \in X$ s.t.

$$0 < |x - a| < \delta$$

Ex: $X = \{q \in \mathbb{Q} : q > 0\}$, $\overbrace{a}^{\text{or any } a \geq 0} = 0$

$X = \{\frac{1}{n} : n \in \mathbb{N}\}$, $a = 0$

Lemma: a is an accumulation point of $X \in \mathbb{R}$

$\Longleftrightarrow$ $\exists\, x_n : \mathbb{N} \to \mathbb{R}$ s.t. $x_n \in X \setminus \{a\}$ $\forall n \in \mathbb{N}$ and $x_n \to a$.

(*)

Pf: Suppose a is an acc point. Then $\forall n \in \mathbb{N}$, $\exists\, x_n \in X \setminus \{a\}$ s.t. $|x_n - a| < \frac{1}{n}$. Thus $x_n \to a$.

Suppose (*) holds. Fix $\delta > 0$. By defn of x_n, $\exists\, N$ s.t. $0 < |x_N - a| < \delta$. $\square$