Lecture 4

Recall: Properties of Absolute Value

Def: For $x \in \mathbb{R}$, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & x < 0 \end{cases}$

Thm: For all $x, y, a \in \mathbb{R}$, $a > 0$

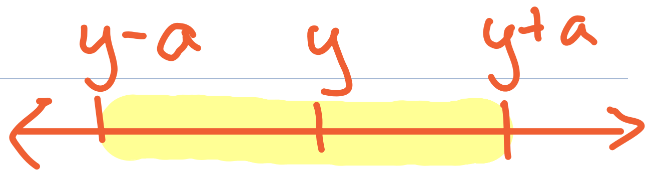
(i) $|x| < a \iff -a < x < a$

(ii) $x \leq |x|$

(iii) $|xy| = |x||y|$

(iv) $|x+y| \leq |x| + |y|$ Triangle ineq

$|x-y| \leq |x| + |y|$



Lemma: $\forall x, y, a \in \mathbb{R}$, $a > 0$,

$|x-y| < a \iff y-a < x < y+a$

Facts (HW3): Given $x, y \in \mathbb{R}$,

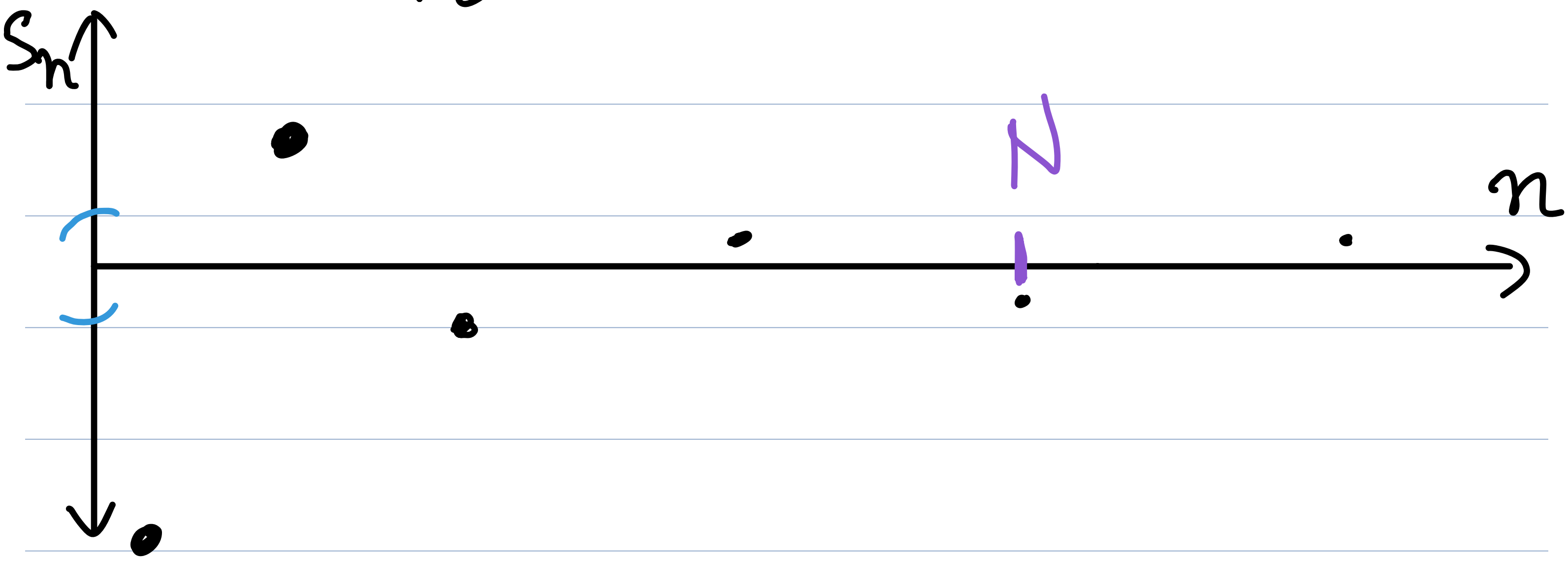
- $x \le y + \varepsilon \quad \forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \Rightarrow x \le y$
- $||x| - |y|| \le |x - y|$ Reverse triangle ineq

Ch 4: Sequences of Real Numbers

Section 10: Sequences

Def: A sequence a_n converges to $L \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \ge N$ ensures $|a_n - L| < \varepsilon$.
We call L the limit of a_n and write $\lim_{n \to \infty} a_n = L$.

Ex: $s_n = \frac{(-1)^n}{n}$



Def: A sequence that does not converge to any $L \in \mathbb{R}$ diverges.

Ex: $b_n = (-1)^n$, $(-1, 1, -1, 1, \ldots)$

Pf: Assume, for the sake of contradiction, that b_n converges to some $L \in \mathbb{R}$.

Then, for $\varepsilon = \frac{1}{2}$, $\exists\ N \in \mathbb{N}$ s.t. $n \geq N$ ensures $|b_n - L| < \frac{1}{2}$
$\iff b_n - \frac{1}{2} < L < b_n + \frac{1}{2}$.

Since $\exists\ n$ even with $n \geq N$, $1 - \frac{1}{2} = \frac{1}{2} < L$. Since $\exists\ n$ odd with $n \geq N$, $L < -\frac{1}{2} = (-1) + \frac{1}{2}$.

This is a contradiction.

Thm: The limit of a sequence is unique.

Pf: Suppose a_n converges to both $L, L' \in \mathbb{R}$. Fix $\varepsilon > 0$ arbitrary.

Then $\exists\, N, N'$ s.t. "$\varepsilon/2$ style argument"

- $n \geq N \Rightarrow |a_n - L| < \varepsilon/2$
- $n \geq N' \Rightarrow |a_n - L'| < \varepsilon/2$

Let $\tilde{N} = \max\{N, N'\}$. Then $\forall n > \tilde{N}$,

"add and substract"

$$\begin{aligned} |L - L'| &= |L - a_n + a_n - L'| \\ &= |(L - a_n) - (L' - a_n)| \\ &\leq |L - a_n| + |L' - a_n| \quad \text{Δ ineq} \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, by Fact, $|L - L'| = 0 \Rightarrow L = L'$. $\square$

An equivalent defn of convergence...

Def: A sequence a_n converges to $L \in \mathbb{R}$ if, $\forall\ \varepsilon > 0$, $\exists\ N$ s.t. $n \geq N$, ensures $|a_n - L| < \varepsilon$.
We call L the limit of a_n and write $\lim\limits_{n\to\infty} a_n = L$.

Alt Def: A sequence a_n converges to $L \in \mathbb{R}$ if, $\forall\ \varepsilon > 0$, $|a_n - L| \geq \varepsilon$ holds for at most finitely many $n \in \mathbb{N}$.

Remark:

- Sometimes we will consider s_n that are only defined for $\underbrace{n \text{ sufficiently large}}_{\exists N \text{ s.t. } n \geq N}$

Its limit is still well-defined.

Ex: $s_n = \frac{1}{n-5}$

- We may modify finitely many elements in a sequence, and the $\underbrace{\text{limiting behavior}}_{\text{whether it converges or diverges}}$ does not change.

11 Subsequences

Def: Given a sequence s_n, it is...

- <u>increasing</u>, in case $n \leq m \Rightarrow s_n \leq s_m$
- <u>strictly increasing</u>, in case $n < m \Rightarrow s_n < s_m$
- <u>decreasing</u>, in case $n \leq m \Rightarrow s_n \geq s_m$
- strictly decreasing, in case $n < m \Rightarrow s_n < s_m$

Ex: $\frac{1}{n}$, n^2, $\cancel{(-1)^n}$

Def: Given a sequence s_n, for any strictly increasing sequence n_k of natural numbers, a sequence of the form s_{n_k} is a <u>subsequence</u> of s_n.

Remark: We could write s_n as $s(n)$, n_k as $n(k)$, and s_{n_k} as $s(n(k))$.

Informally, a subsequence is any infinite collection of elements from the original sequence, listed in order

Ex: $s_n = (\overset{n=1}{-1}, \overset{n=2}{2}, \overset{n=3}{-3}, \overset{n=4}{4}, -5, \ldots)$

$$s_{n_k} = (\overset{k=1}{-1}, \overset{k=2}{-3}, \overset{k=3}{-5}, \ldots)$$

$$n_k = (\overset{k=1}{1}, \overset{k=2}{3}, \overset{k=3}{5}, \overset{k=4}{7}, \ldots)$$

Lemma: If n_k is a strictly increasing sequence of natural numbers, then $n_k \geq k$ $\forall k \in \mathbb{N}$.

Pf: We proceed by induction.
Base case: $k=1$, since $n_1 \in \mathbb{N}$, $n_1 \geq 1$.
Inductive step: Suppose $\underbrace{n_{k-1} \geq k-1}_{(*)}$.
then $n_k > n_{k-1}$. Since
$n_k, n_{k-1} \in \mathbb{N}$, $n_k \geq n_{k-1}+1 \overset{(*)}{\geq} k$.

Thm: If a sequence s_n converges to a limit $L \in \mathbb{R}$, then every subsequence also converges to L.

Pf: Fix an arbitrary subsequence s_{n_k} of s_n. Fix arbitrary $\varepsilon > 0$. Since $\lim_{n \to \infty} s_n = L$, $\exists$ N s.t. $n \geq N$ ensures $|s_n - L| < \varepsilon$. If $k \geq N$, then by the previous lemma, $n_k \geq N$, so $|s_{n_k} - L| < \varepsilon$. This shows $\lim_{k \to \infty} s_{n_k} = L$.

Ex: $(-1)^n$ diverges, since subsequences of even and odd elements have different limits.

Ex: the constant sequence $a_n = (L, L, L, \ldots)$ converges to L.

12 The Algebra of Limits

<u>Thm</u> (Limit of Sum is Sum of Limit):
If a_n and b_n are convergent sequences, so is a_n+b_n and
$$\lim_{n\to\infty}(a_n+b_n)=\underbrace{\lim_{n\to\infty}a_n}_{L}+\underbrace{\lim_{n\to\infty}b_n}_{M}.$$

Pf: Fix $\varepsilon>0$. Since a_n and b_n converge, $\exists\ N_a, N_b \in \mathbb{N}$ s.t.
$$n\ge N_a \Rightarrow |a_n-L|<\frac{\varepsilon}{2}$$
$$n\ge N_b \Rightarrow |b_n-M|<\frac{\varepsilon}{2}.$$

Let $N=\max\{N_a,N_b\}$. Then $n\ge N \Rightarrow$
$$\begin{aligned}|(a_n+b_n)-(L+M)| &= |(a_n-L)+(b_n-M)| \\ &\le |a_n-L|+|b_n-M| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

$\square$