Lecture 10

Midterm 1 on Mon, May 6th

Def: For any $a \geq 1$, $x \in \mathbb{R}$, define
$$a^x = \lim_{n\to\infty} a^{r_n}$$
where $r_n : \mathbb{N} \to \mathbb{Q}$ satisfies $r_n \nearrow x$.

For any $0 < a < 1$, $x \in \mathbb{R}$, define
$$a^x = \left(\frac{1}{a}\right)^{-x}.$$

Thm:

For all $a, b \in \mathbb{R}$, $x > 0$

(i) $x^{a+b} = x^a x^b$

(ii) $x^a = \left(\frac{1}{x}\right)^{-a}$

(iii) $(xy)^a = x^a y^a$

(iv) $(x^a)^b = x^{ab}$

(vi) If $0 < x < y$, $a > 0$, then $x^a < y^a$

(v) If $x > 1$ and $a < b$, then $x^a < x^b$.

§18 The Bolzano-Weierstrass Thm

Recall: For $s_n: \mathbb{N} \to \mathbb{R}$,
convergent $\Rightarrow$ bounded

bounded and monotone $\Rightarrow$ convergent

the limit of a sequence $a_n: \mathbb{N} \to \overline{\mathbb{R}}$ exists

$$\Longleftrightarrow \begin{cases} \lim_{n\to\infty} a_n = L \in \mathbb{R} \\ \lim_{n\to\infty} a_n = +\infty \\ \lim_{n\to\infty} a_n = -\infty \end{cases}$$

"the sequence converges to L"

Thm: Every sequence $s_n: \mathbb{N} \to \overline{\mathbb{R}}$ has a monotone subsequence.

Thm (Bolzano-Weierstrass): Every bounded sequence has a cnvgt subseq.

19 The Cauchy Criterion

Def: $a_n : \mathbb{N} \to \mathbb{R}$ is a Cauchy sequence if, $\forall \varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t. $m, n \geq N$ ensures $|a_n - a_m| < \varepsilon$.

A convergent sequence "bunches up" around its limit. → need to know what limit is

A Cauchy sequence "bunches up" around itself. → don't need to know limit

Thm: All convergent real valued sequences are Cauchy.

Thm: All real-valued Cauchy sequences are convergent.

30 Definition of the Limit of a Function

Def: Given $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, a is an accumulation point of X if $\forall \delta > 0$, $\exists x \in X$ s.t.

$$0 < |x - a| < \delta$$

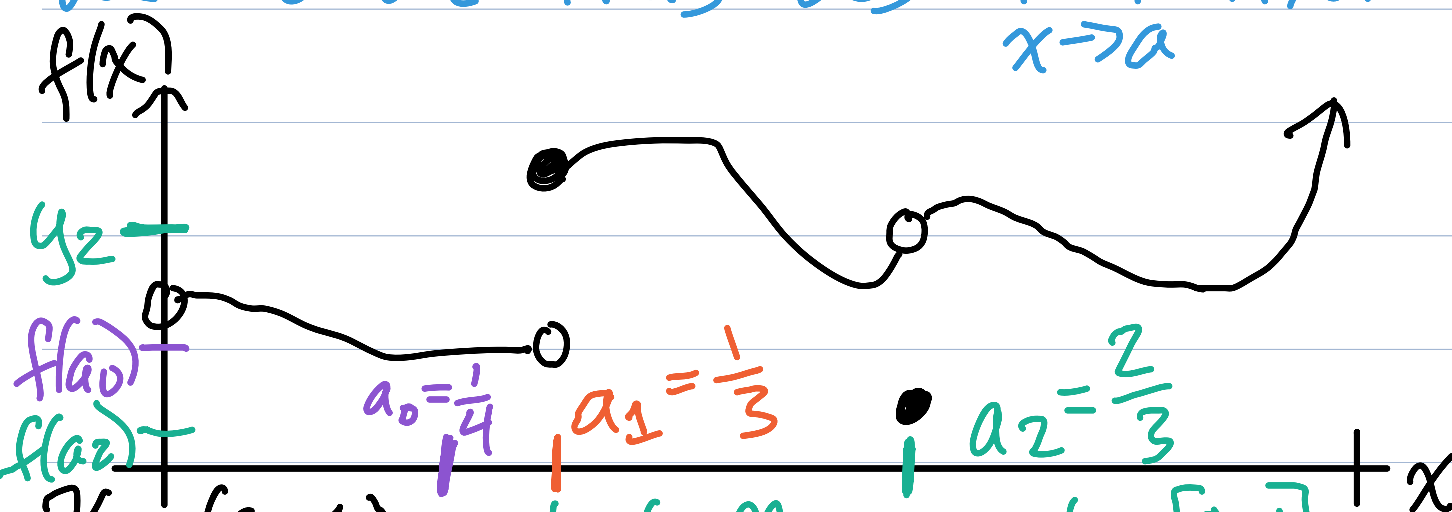
Lemma: a is an accumulation point of $X \subseteq \mathbb{R}$ $\iff \exists x_n : \mathbb{N} \to \mathbb{R}$ s.t. $x_n \in X \setminus \{a\}$ $\forall n \in \mathbb{N}$ and $x_n \to a$.

Def: Given $X \subseteq \mathbb{R}$ nonempty, $f: X \to \overline{\mathbb{R}}$, a an accumulation point of X, $L \in \overline{\mathbb{R}}$, the <u>limit of $f(x)$ as x approaches a is L</u> if, for all sequences $x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $x_n \to a$, we have

$$\lim_{n\to\infty} f(x_n) = L.$$

← "sequential defn of $\lim_{x \to a} f(x)$"

We denote this as $\lim_{x\to a} f(x) = L$.



$X = (0,1)$, set of all acc points $= [0,1]$

Ex: $\lim_{x\to a_0} f(x) = f(a_0)$

$\lim_{x\to a_1} f(x)$ D.N.E.

$\lim_{x\to a_2} f(x) = y_2 \neq f(a_2)$

Ex: $X = (0,1) \cup \{2\}$

$\lim_{x\to 2} f(x) = ?$

Note that there does not exist $x_n : \mathbb{N} \to X \setminus \{2\}$ s.t. $x_n \to 2$.

This is why a must be an acc point for the defn to make sense.

Prop: Given $X \subseteq \mathbb{R}$ nonempty, $f: X \to \mathbb{R}$, a an acc point of X, and $L \in \mathbb{R}$, then

$$\lim_{x \to a} f(x) = L$$

$$\Updownarrow$$

(*) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in X$ with $0 < |x-a| < \delta$, we have $|f(x) - L| < \varepsilon$.

Note that

$\neg(*) \iff \exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x \in X$ with $0 < |x-a| < \delta$ and $|f(x) - L| \geq \varepsilon$

Pf:

First, we will show $(*) \Rightarrow \lim_{x \to a} f(x) = L$.

Fix $x_n : \mathbb{N} \to X \setminus \{a\}$ s.t. $x_n \to a$.

We must show $\lim_{n \to \infty} f(x_n) = L$.

Fix $\varepsilon > 0$. By $(*)$, $\exists\, \delta > 0$ s.t. $\forall x \in X$ with $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.

Since $x_n \to a$, $\exists\, N$ s.t. $n \geq N$ ensures $0 < |x_n - a| < \delta$. Thus $n \geq N$ ensures $|f(x_n) - L| < \varepsilon$.

This shows $\lim_{n \to \infty} f(x_n) = L$.

Now, we will show $\neg(*) \Rightarrow \lim_{x \to a} f(x) = L$ is not true.

By $\neg(*)$, $\exists\ \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$ $\exists\ x_n \in X$ with $0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon$.

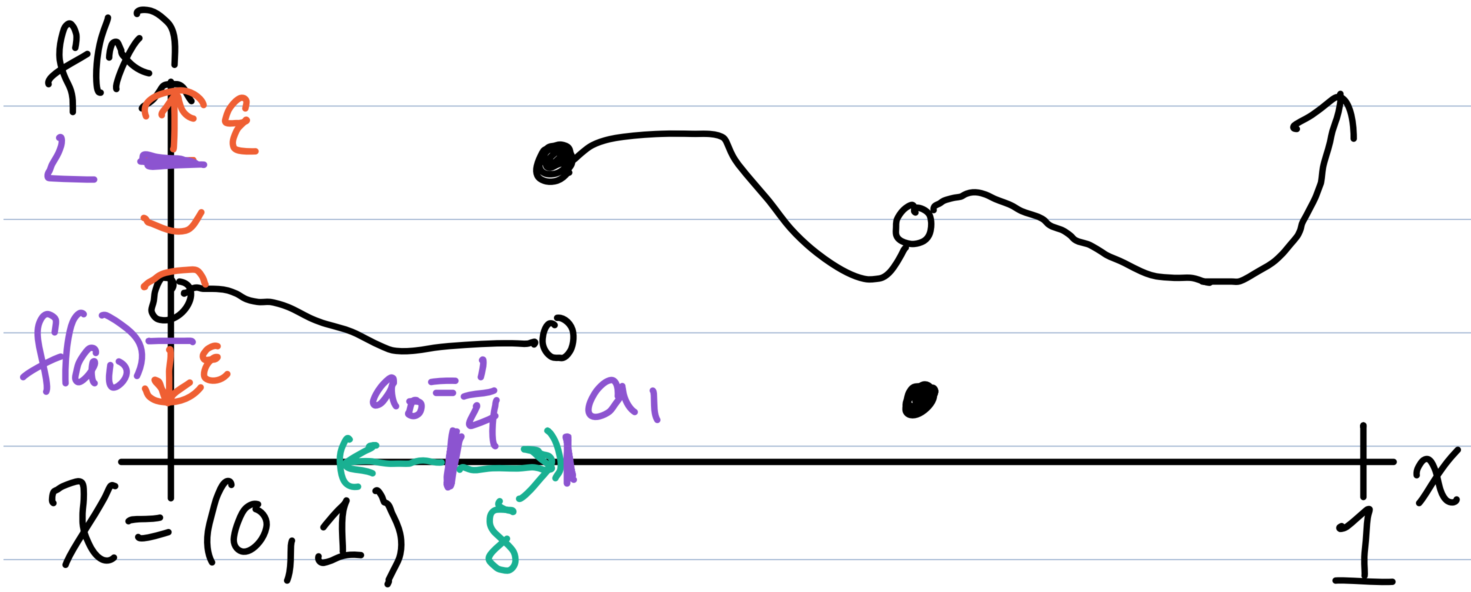
Note that $x_n : \mathbb{N} \to X \setminus \{a\}$. Furthermore, $\forall\ \varepsilon' > 0$, if $N > \frac{1}{\varepsilon'}$ then $n \geq N$ ensures

$$|x_n - a| < \frac{1}{n} < \varepsilon'.$$

Thus $x_n \to a$.

Since $|f(x_n) - L| \geq \varepsilon\ \forall n \in \mathbb{N}$,

$f(x_n)$ does not converge to L. $\square$



$0 < |x - a_0| < \delta$ are shaded in green

Ex: Consider $X = \mathbb{R}$, $f(x) = 3x - 1$.
What is $\lim_{x \to 1} f(x)$? Guess: 2

First, we will prove via sequences defn. Fix $x_n : \mathbb{N} \to \mathbb{R} \setminus \{1\}$ s.t. $x_n \to 1$. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 3x_n - 1 = 3 \cdot 1 - 1 = 2$$

limit of sum is sum of limits

Aside: Suppose $a_m = m^2 - m$

$$\lim_{m\to\infty} m^2 + \lim_{m\to\infty} -m = +\infty - \infty = ?$$

$$\lim_{m\to\infty} m^2 - m = +\infty$$