Math 117: Homework 5

Due Sunday, May 12th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Consider a sequence s_n satisfying $s_n \neq 0$ for all $n \in \mathbb{N}$ and for which $\left|\frac{s_{n+1}}{s_n}\right|$ converges to L.

(a) If $L < 1$, show that $\lim_{n\to+\infty} s_n = 0$.

Hint: Explain why you can select a so that $L < a < 1$ and prove that there exists N so that $n > N$ ensures $|s_{n+1}| < a|s_n|$. Then use induction to show $|s_n| \leq a^{n-N-1}|s_{N+1}|$ for all $n > N$.

(b) Now prove that if $L > 1$ (which includes the possibility $L = +\infty$), then $\lim |s_n| = +\infty$.

Hint: Explain why you can apply the previous part to the sequence $t_n = \frac{1}{|s_n|}$.

Question 2*

Consider a nonempty set $X \subseteq \mathbb{R}$. Prove that there exists a sequence $x_n : \mathbb{N} \to X$ so that $\lim_{n\to+\infty} x_n = \sup(X)$.

Question 3

Solve 14.1 from the textbook.

Question 4

Solve 15.3 from the textbook.

Question 5*

Solve 15.4 from the textbook.

Question 6*

Solve 16.1 from the textbook.

Question 7*

Solve 16.8 from the textbook.

Question 8*

In this problem, we will consider sequences s_n satisfying the following property:

$$\exists s \in \mathbb{R} \text{ s.t. every subsequence } s_{n_k} \text{ of } s_n \text{ has a further subsequence } s_{n_{k_l}} \text{ satisfying } \lim_{l \to +\infty} s_{n_{k_l}} = s. \tag{*}$$

(a) Prove that if $\lim s_n = s$, then property (*) holds.

(b) Prove that if property (*) holds, then $\lim s_n = s$.

Question 9

Solve 17.3 from the textbook.

Question 10

Solve 19.3 from the textbook.

Background on Infinite Series

In calculus, you encountered infinite series of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots, \quad \text{for } a_k : \mathbb{N} \to \mathbb{R}.$$

In fact, these are just limits of sequences. In particular, if we define the sequence

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

to be the sum of the first n terms of the series, then we define the value of the infinite series to be

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to +\infty} s_n.$$

DEFINITION 1. Given a series $\sum_{k=1}^{\infty} a_k$, define the sequence $s_n = \sum_{k=1}^{n} a_k$. Then the series $\sum_{k=1}^{\infty} a_k$ *converges* to a number L if and only if the sequence s_n converges to L. Likewise, the series *diverges to* $+\infty$ *or* $-\infty$ if and only if the sequence s_n diverges to $+\infty$ or $-\infty$.

Question 11* (Cauchy criterion)

(a) Prove that the following is an equivalent definition of a Cauchy sequence:

s_n is a Cauchy sequence if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $n > m \geq N$ ensures $|s_n - s_m| < \epsilon$.

(b) Prove the following theorem about series, known as the Cauchy criterion.

THEOREM 1 (Cauchy Criterion)**.** *A series* $\sum_{k=1}^{\infty} a_k$ *is convergent if and only if*

$$\text{for all } \epsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ so that } n > m \geq N \text{ ensures } \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon.$$

(c) Now use the Cauchy Criterion to prove the following corollary:

COROLLARY 2. *If a series* $\sum_{k=1}^{\infty} a_k$ *is convergent, then* $\lim_{k \to +\infty} a_k = 0$.

Question 12

(a) Prove the following by induction: for $a \neq 1$,

$$\sum_{i=0}^{m-1} a^i = 1 + a + a^2 + \cdots + a^{m-1} = \frac{1 - a^m}{1 - a}.$$

(b) Use part (a) to show that

$$\sum_{i=n}^{m-1} a^i = a^n + a^{n+1} + \cdots + a^{m-2} + a^{m-1} = \frac{a^n - a^m}{1 - a}.$$

(Hint: $\sum_{i=n}^{m-1} a^i = \sum_{i=0}^{m-1} a^i - \sum_{i=0}^{n-1} a^i$.)

(c) Recall that, by the triangle inequality, we may estimate

$$\left| \sum_{i=1}^{n} a_i \right| = |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| = \sum_{i=1}^{n} |a_i|.$$

Let s_n be a sequence such that $|s_{n+1} - s_n| \leq 4^{-n}$ for all $n \in \mathbb{N}$. Use part (b) and the above inequality to prove s_n is a Cauchy sequence.

(d) Does the sequence from part (c) converge? Justify your answer.

Question 13*

Recall the following facts:

$$\lim_{n \to +\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } |r| = 1 \\ +\infty & \text{if } r > 1 \\ \text{does not exist} & \text{if } r \leq -1, \end{cases}$$

and

$$\text{for } r \neq 1, \ \sum_{k=1}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}.$$

(a) Prove that for $|r| < 1$, $\sum_{k=1}^{\infty} r^k = \frac{1}{1-r}$.

(b) Prove that for $|r| \geq 1$, $\sum_{k=1}^{\infty} r^k$ does not converge. (**Hint**: Use Corollary 2 above.)