Math 117: Homework 3

Due Sunday, April 21st at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Use the properties of an ordered field F to prove the following facts, for any $x, y, z \in F$:

(i) $0 < 1$

(ii) If $x > y$ and $y > z$, then $x > z$.

(iii) If $x > y$, then $x + z > y + z$.

(iv) If $x > y$ and $z > 0$, then $xz > yz$.

(v) If $x > y$ and $z < 0$, then $xz < yz$.

Question 2*

Use the definition of the absolute value, the definition of an ordered field F, and the results from Q1 to prove the following results for all $a, b \in \mathbb{F}$.

(i) $|a| \geq 0$

(ii) $|ab| = |a|\ |b|$

(iii) $|a| \geq a$ and $|a| \geq -a$

(iv) $|a + b| \leq |a| + |b|$. This is known as the *triangle inequality.*

Question 3

Use the definition of the absolute value, the definition of an ordered field F, and the results of Q1 and Q2 to prove the following facts for all $a, b, c \in F$.

(a) $|b| \leq a$ if and only if $-a \leq b \leq a$.

(b) $|a - b| \leq c$ if and only if $c - a \leq b \leq c + a$

(c) $||a| - |b|| \leq |a - b|$. This is known as the *reverse triangle inequality.*

Question 4*

Use the definition of an ordered field to proved that, for all $a, b, \epsilon \in F$, if $a \leq b + \epsilon$ for all $\epsilon \in F$ with $\epsilon > 0$, then $a \leq b$.

Question 5*

This question will guide you through how to use the Archimedean property and the definition of the real numbers to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Suppose $a, b \in \mathbb{R}$ with $a < b$.

(i) Suppose $b > 0$.

  (a) Prove that there exists $n \in \mathbb{N}$ so that $b - a > \frac{1}{n}$.
  (b) Prove that there exists $m \in \mathbb{N}$ so that $\frac{m-1}{n} < b \leq \frac{m}{n}$.
  (c) Prove that these choices of n, m satisfy $a < \frac{m-1}{n}$.
  (d) Combine the previous steps to prove the following claim: there exists $r \in \mathbb{Q}$ with $a < r < b$.

(ii) Now, suppose $b \in \mathbb{R}$.

  (a) Prove that there exists $n' \in \mathbb{N}$ so that $b + n' > 0$.
  (b) Use (ii)(a) and (i)(d) to show that there exists $r \in \mathbb{Q}$ with $a < r < b$.

Question 6

Prove that the two definitions of convergence of a sequence stated in class are equivalent.

Question 7*

Justify the following facts, using the definition of convergence.

(i) $\lim_{n\to+\infty} \frac{1}{n-2} = 0$

(ii) $\lim_{n\to+\infty} \frac{2n}{n+2} = 2$.

(iii) $\lim_{n\to+\infty} \frac{(-1)^n}{n} = 0$.

Question 8*

Prove that $(n + \frac{1}{n})_{n=1}^{\infty}$ diverges.

Question 9*

Solve question 10.11 from the textbook.

Question 10

Solve question 10.12 from the textbook.

Question 11*

Solve question 11.8 from the textbook.

Question 12

Solve question 11.9 from the textbook.