⑧ Let $0'$ and $1'$ denote the additive and multiplicative identities of $\mathbb{R}'$. Let $\mathbb{N}'$ denote the set where all elements may be expressed as a finite sum of the form

$$\underbrace{1' + \cdots + 1'}_{n \text{ times}} = \sum_{i=1}^{n} 1'.$$

As shown in class, $\mathbb{N}'$ is the smallest set satisfying $1' \in \mathbb{N}'$ and $n' \in \mathbb{N}' \Rightarrow n'+1 \in \mathbb{N}'$.

Note that if $\sum_{i=1}^{n} 1' < \sum_{i=1}^{m} 1'$, since (WLOG $n \geq m$), by the properties of an ordered field,

$$0 = \sum_{i=1}^{m} 1' - \sum_{i=1}^{m} 1' > \sum_{i=1}^{n} 1' - \sum_{i=1}^{m} 1'$$

Thm 4.2

Since $1' > 0'$, if $n \geq m$, the sum of positive numbers is positive, so the RHS is nonneg, which is a contradiction. Thus, we must have $n < m$.

Conversely, we also see that, since $1' > 0'$, $n < m$ ensures $\sum_{i=1}^{n} 1' < \sum_{i=1}^{m} 1'$.

Define $f: \mathbb{N} \to \mathbb{N}'$ by $f(n) = \sum_{i=1}^{n} 1'$.
By definition, f is surjective (its range is all finite sums of $1'$) and f is injective (since f is strictly increasing).

By definition, $\forall\, n, m \in \mathbb{N}$, we have

$\boxed{A}$ $$f(n+m) = \sum_{i=1}^{n+m} 1' = \sum_{i=1}^{n} 1' + \sum_{i=1}^{m} 1' = f(n) + f(m)$$

$\boxed{B}$ $$f(n) < f(m) \iff n < m$$

$\boxed{C}$

Next we show $f(nm) = f(n)f(m)$.
Base case: $m = 1$
$$f(nm) = f(n) = f(n) \cdot 1' = f(n)f(m).$$
Assume $f(nm) = f(n)f(m)$ for $m \in \mathbb{N}$.
Then, $f(n(m+1)) = f(nm+n) \overset{\boxed{A}}{=} f(nm) + f(n)$
$= f(n)f(m) + f(n) = f(n)(f(m) + 1')$
$= f(n)(f(m) + f(1)) \overset{\boxed{A}}{=} f(n)f(m+1)$.
This shows $\boxed{C}$.

Let $\mathbb{Z}' = \mathbb{N}' \cup \{0'\} \cup -\mathbb{N}'$. Extend $f : \mathbb{Z} \to \mathbb{Z}'$ by defining

$$f(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{N} \\ -f(-x) & \text{if } x \in -\mathbb{N} \\ 0' & \text{if } x = 0 \end{cases}$$

by defn of ours ordering, exactly one of these is true

By defn, f is surjective. Likewise,

$$f(x) <' f(y) \iff \begin{cases} x < y, \text{ by above, if } x \in \mathbb{N} \\ x < y, \text{ by above, if } \underbrace{x \in -\mathbb{N}} \\ 0 < y, \text{ if } x = 0. \end{cases}$$

Since $-x \in \mathbb{N}$

$-f(-x) < -f(-y)$

$f(-x) > f(-y)$

$-x > -y$

$\iff x < y$.

Thus, f is strictly increasing.

We check that this extension still satisfies properties $\boxed{A}$ and $\boxed{C}$. Fix $x, y \in \mathbb{Z}$. If both $x, y \in \mathbb{N}$, the result has already been shown. If $x \in \mathbb{Z}$ and $y = 0$, then

$$f(x+y) = f(x) = f(x) + 0' = f(x) + f(0)$$

$$f(xy) = f(0) = 0' = f(x) \cdot 0' = f(x) f(y)$$

↑ Thm 3.4 ↑ Thm 3.4

If $x \in \mathbb{N}$, $y \in -\mathbb{N}$, then

$$f(x+y) = \begin{cases} \sum_{i=1}^{x+y} 1' & \text{if } x+y \geq 0 \\ -\sum_{i=1}^{-x-y} 1' & \text{if } x+y < 0 \end{cases}$$

$$= \begin{cases} \sum_{i=1}^{x} 1' - \sum_{i=1}^{-y} 1' & \text{if } x+y \geq 0 \\ -\left(\sum_{i=1}^{-y} 1' - \sum_{i=1}^{x} 1'\right) & \text{if } x+y < 0 \end{cases}$$

$$= f(x) - f(-y)$$

$$= f(x) + f(y)$$

Likewise,

$-y \in \mathbb{N}$

$$f(xy) = -f(-xy) = -f(x(-y)) = -f(x)f(-y)$$
$$= f(x)f(y).$$

Finally, if both $x, y \in -\mathbb{N}$,

$-x, -y \in \mathbb{N}$

$$f(x+y) = -f(-x-y) = -[f(-x)+f(-y)] = f(x)+f(y)$$

$$f(xy) = f((-x)(-y)) = f(-x)f(-y) = (-f(-x))(-f(-y))$$
$$\ldots = f(x)f(y).$$

exercise 3.8 gives $(-1)(-1)=1$

Thus, properties $\boxed{A}$, $\boxed{B}$, and $\boxed{C}$ continue to hold for $f : \mathbb{Z} \to \mathbb{Z}'$.

Let $\mathbb{Q}' = \{ p'/q' : q' \in \mathbb{Z} \setminus \{0\}, p' \in \mathbb{Z} \}$

Extend $f : \mathbb{Q} \to \mathbb{Q}'$ by defining

$f\left(\frac{p}{q}\right) = \frac{f(p)}{f(q)}$, for all $q \in \mathbb{Z} \setminus \{0\}, p \in \mathbb{Z}$.

Note that this is a well-defined function, since $\frac{p}{q} = \frac{r}{s}$ for $q, s \in \mathbb{Z} \setminus \{0\}$, $p, r \in \mathbb{Z}$ implies $sp = qr$, so $f(s)f(p) = f(sp) = f(qr) = f(q)f(r)$. Likewise $f(q), f(s) \in \mathbb{Z}' \setminus \{0\}$. Thus, $\frac{f(p)}{f(q)} = \frac{f(r)}{f(s)}$.

Since $f : \mathbb{Z} \to \mathbb{Z}'$ was surjective, we likewise have $f : \mathbb{Q} \to \mathbb{Q}'$ is surjective.

We now check properties $\boxed{A}$, $\boxed{B}$, and $\boxed{C}$.

Fix $p_1, p_2 \in \mathbb{Z}$, $q_1, q_2 \in \mathbb{Z} \setminus \{0\}$.

For $\boxed{B}$,

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} \iff p_1 q_2 < q_1 p_2$$
$$\iff f(p_1 q_2) < f(q_1 p_2)$$
$$\iff f(p_1) f(q_2) < f(q_1) f(p_2)$$
$$\iff \frac{f(p_1)}{f(q_1)} < \frac{f(p_2)}{f(q_2)}$$

$q_1, q_2 \neq 0$
$\Downarrow$ II
$f(q_1), f(q_2) \neq 0'$

In particular, this shows f is injective.

For $\boxed{A}$

$$f\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) = f\left(\frac{p_1 q_2 + q_1 p_2}{q_1 q_2}\right) = \frac{f(p_1) f(q_2) + f(q_1) f(p_2)}{f(q_1) f(q_2)}$$
$$= \frac{f(p_1)}{f(q_1)} + \frac{f(p_2)}{f(q_2)} = f(p_1/q_1) + f(p_2/q_2)$$

For $\boxed{C}$,

$$f\left(\frac{p_1}{q_1} \frac{p_2}{q_2}\right) = \frac{f(p_1) f(p_2)}{f(q_1) f(q_2)} = f\left(\frac{p_1}{q_1}\right) f\left(\frac{p_2}{q_2}\right).$$

By definition $\mathbb{R}$ (resp $\mathbb{R}'$) is an ordered field containing $\mathbb{Q}$ (resp $\mathbb{Q}'$) so that every subset (that is bounded above) has a supremum.

<u>Claim</u>: For any $x \in \mathbb{R}$, $x = \sup\{q \in \mathbb{Q} : q < x\}$
($x' \in \mathbb{R}'$, $x' = \sup\{q' \in \mathbb{Q}' : q' < x'\}$)

<u>Pf</u>: We show the result for $\mathbb{R}'$. The same argument works for $\mathbb{R}$. By definition of the set $S = \{q' \in \mathbb{Q}' : q' < x'\}$, x' is an upper bound of S, so $\sup(S) \leq x'$. Assume, for the sake of contradiction, that $\sup(S) < x'$. By density of $\mathbb{Q}'$ in $\mathbb{R}'$, $\exists\, q' \in \mathbb{Q}'$ s.t. $\sup(S) < q' < x'$. But then $q' \in S$, which contradicts that $\sup(S)$ is an upper bound for S. Thus $\sup(S) = x'$.

We extend $f : \mathbb{R} \to \mathbb{R}'$ by

$$f(x) = \sup\{f(q) : q < x,\ q \in \mathbb{Q}\}$$

If $x \in \mathbb{Q}$, this agrees with our previous definition.

We now show f is surjective.
Fix $y' \in \mathbb{R}'$.

Let $x = \inf \overbrace{\{q \in \mathbb{Q} : f(q) \geq y'\}}^{S}$

Suppose $q' \in \mathbb{Q}$, $q' \leq y'$. Then, for all $q \in S$, $f(q) \geq y' \geq q' \Rightarrow q \geq f^{-1}(q')$. Thus $f^{-1}(q')$ is a lower bound for S, so $x \geq f^{-1}(q')$. This shows $q' \in T$, so $f(x) \geq q'$. Since $q' \in \mathbb{Q}$ was an arbitrary number satisfying $q' \leq y'$, this shows

$$f(x) \geq \sup\{q \in \mathbb{Q}' : y' \geq q'\} = y'.$$

On the other hand, if $f(x) > y'$, then y' is not an upper bound for T, so there exists $q \in \mathbb{Q}$ with $q < x$ and $f(q) > y'$. Thus $q \in S$ and $q < x$, which contradicts that x is a lower bound for S. Thus $f(x) = y'$, so f is surjective.

Next, we show $\boxed{A}$, $\boxed{B}$, and $\boxed{C}$. Fix $x,y \in \mathbb{R}$.

To see $\boxed{B}$, note that

$$x \le y \Rightarrow \{f(q) : q < x, q \in \mathbb{Q}\} \subseteq \{f(q) : q < y, q \in \mathbb{Q}\}$$
$$\Rightarrow f(x) \le f(y) \quad \swarrow \text{HW2, Q3 (a)}$$

Furthermore, if $x < y$, by density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, q_1, q_2 \in \mathbb{Q}$ s.t. $x < q_1 < q_2 < y$. Since f is strictly increasing on $\mathbb{Q}$, $f(x) \le f(q_1) < f(q_2) \le f(y) \Rightarrow f(x) < f(y)$.

In particular, this shows f is injective.

To see $S_1 = S_2$, note that $S_2 \subseteq S_1$ is immediate. If $f(q) \in S_1$, then $x > q - y$, so $\exists\, q_1 \in \mathbb{Q}$ s.t. $x > q_1 > q - y$. Let $q_2 = q - q_1$.

To see $\boxed{A}$, note that

$$f(x+y) = \sup \overbrace{\{f(q) : q < x+y, q \in \mathbb{Q}\}}^{S_1}$$
$$= \sup \underbrace{\{f(q) : q = q_1 + q_2,\ q_1 < x, q_2 < y,\ q_1, q_2 \in \mathbb{Q}\}}_{S_2}$$
$$= \sup \{f(q_1) + f(q_2) : q_1 < x,\ q_2 < y,\ q_1, q_2 \in \mathbb{Q}\}$$
$$= \sup(\{f(q_1) : q_1 < x, q_1 \in \mathbb{Q}\} + \{f(q_2) : q_2 < y, q_2 \in \mathbb{Q}\})$$

HW2 Q4 $\downarrow$

$$= f(x) + f(y).$$

Before turning to $\boxed{C}$, observe that, $\forall x \in \mathbb{R}$,

$$
\begin{aligned}
f(-x) &= \sup\{f(q) : q < -x,\ q \in \mathbb{Q}\} \\
&= \sup\{f(q) : -q > x,\ q \in \mathbb{Q}\} \\
&= \sup\{f(-r) : r > x,\ r \in \mathbb{Q}\} \\
&= \sup\{-f(r) : r > x,\ r \in \mathbb{Q}\} && \text{HW2, Q5} \\
&= -\inf\{f(r) : r > x,\ r \in \mathbb{Q}\} \\
&= -\inf\{r' \in \mathbb{Q}' : f^{-1}(r') > x\} \\
&= -\inf\{r' \in \mathbb{Q}' : r' > f(x)\} && \text{HW2, Q5} \\
&= \sup\{-r' \in \mathbb{Q}' : -r' < -f(x)\} \\
&= \sup\{q' \in \mathbb{Q}' : q' < -f(x)\} = -f(x)
\end{aligned}
$$

Now, we show $\boxed{C}$. Note that it is immediate if $y = 0$, so suppose $y \neq 0$.

If $y \in \mathbb{Q}$ and $y > 0$,

$$
\begin{aligned}
f(xy) &= \sup\{f(q) : q < xy,\ q \in \mathbb{Q}\} \\
&= \sup\{f(q) : \tfrac{q}{y} < x,\ q \in \mathbb{Q}\} \\
&= \sup\{f(ry) : r < x,\ q \in \mathbb{Q}\} \\
&= \sup\{f(r)f(y) : r < x,\ q \in \mathbb{Q}\} && \text{Exercise 5.7} \\
&= f(y)\sup\{f(r) : r < x,\ q \in \mathbb{Q}\} \\
&= f(y)f(x).
\end{aligned}
$$

Thus, if $y \in \mathbb{Q}$ and $y < 0$,

$$f(xy) = f((-x)(-y)) = f(-x)f(-y) = (-f(x))(-f(y))$$
$$= f(x)f(y).$$

Finally, if $y \in \mathbb{R}$ and $x > 0$,

Exercise 5.7

$$\begin{aligned} f(x)f(y) &= f(x) \sup\{f(q) : q < y,\ q \in \mathbb{Q}\} \\ &= \sup\{f(x)f(q) : q < y,\ q \in \mathbb{Q}\} \\ &= \sup\{f(xq) : q < y,\ q \in \mathbb{Q}\} \\ &= \sup\{f(xq) : xq < xy,\ q \in \mathbb{Q}\} \\ &= \sup\{f(z) : z < xy,\ \tfrac{z}{x} \in \mathbb{Q}\} \\ &= \sup\underbrace{\{z' : z' < f(xy),\ \tfrac{f^{-1}(z')}{x} \in \mathbb{Q}\}}_{\mathcal{U}} \\ &= f(xy) \end{aligned}$$

Last step

As a consequence, if $y \in \mathbb{R}$ and $x < 0$,

$$\begin{aligned} f(x)f(y) &= -f(-x)f(y) \\ &= -f(-xy) \\ &= f(xy) \end{aligned}$$

Justification of Last Step:

- Since $f(xy)$ is an upper bound for the set $\mathcal{U}$, we have $\sup(\mathcal{U}) \leq f(xy)$.

- If $f(xy) > \sup(\mathcal{U})$, then
$f(xy)/f(x) > \sup(\mathcal{U})/f(x)$, so $\exists$
$r' \in \mathbb{Q}'$ s.t. $f(xy)/f(x) > r' > \sup(\mathcal{U})/f(x)$,
hence $f(xy) > r' f(x) \overset{(***)}{>} \sup(\mathcal{U})$.
Then $f^{-1}(r' f(x)) = f^{-1}(r')x$
satisfies $f^{-1}(r')x/x \in \mathbb{Q}$ and
$r' f(x) < f(xy)$. Hence $r' f(x) \in \mathcal{U}$.
This contradicts $(***)$.