Math 117: Homework 2

Due Sunday, April 14th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Consider an ordered field F and suppose $S \subseteq F$ is nonempty and bounded above. Prove that a is the supremum of S if and only if a is an upper bound for S and, for all $\epsilon > 0$, there exists $s \in S$ so that $s > a - \epsilon$.

Question 2*

Consider an ordered field F and a nonempty subset $X \subseteq F$

DEFINITION 1 (maximum/minimum)**.** An element $a \in F$ is the *maximum* of X if $a \in X$ and a is an upper bound for X. Similarly, b is the *minimum* of X if $b \in X$ and b is a lower bound of X.

(a) If the maximum of X exists, prove that the maximum is the supremum of X.

(b) If $\sup X \in X$, prove that $\max(X) = \sup(X)$.

Question 3*

Suppose A and B are nonempty subsets of $\mathbb{R}$ that are bounded above.

(a) If $A \subset B$, prove that $\sup A \leq \sup B$.

(b) For any A and B, prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Question 4

Suppose A and B are nonempty subsets of $\mathbb{R}$ that are bounded above. Define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Prove $\sup(A + B) = \sup A + \sup B$.

Question 5*

Consider the following proposition:

PROPOSITION 1. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum.*

This question will lead you through the proof of the proposition.

(a) Suppose S is as in the proposition above. Define the set $-S = \{-s : s \in S\}$. Show that $-S$ is bounded above.

(b) Use the definition of $\mathbb{R}$ to prove that $-S$ has a supremum, $\sup(-S)$.

(c) Prove that $-\sup(-S)$ is the infimum of S.

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Question 6*

Solve question 6.5 from the textbook.

Question 7

Suppose $X \subseteq \mathbb{N}$ is nonempty and bounded above. Prove that $\max(X)$ exists.

Question 8*

Solve question 7.9 in the textbook. This shows our definition of $\mathbb{R}$ specifies $\mathbb{R}$ up to isomorphism.

Question 9

Prove that a set X is infinite if and only if X has the same cardinality as a proper subset of itself.

Question 10*

The *extended real numbers* is the set $\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. We may extend the binary operations of addition $+$ and multiplication $\cdot$ and the ordering $\leq$ from $\mathbb{R}$ to $\overline{\mathbb{R}}$ via the following rules:

(a) $\pm\infty + x = \pm\infty$ for all $x \in \mathbb{R}$.
$+\infty + (+\infty) = +\infty$ and $-\infty + (-\infty) = -\infty$.
$\pm\infty + (\mp\infty)$ is not defined.

(b) $\pm\infty \cdot x = \pm\infty$ for $x \in (0, +\infty]$ and $\pm\infty \cdot x = \mp\infty$ for $x \in [-\infty, 0)$.
$(\pm\infty) \cdot (\pm\infty) = +\infty$ and $(\pm\infty) \cdot (\mp\infty) = -\infty$.
$\pm\infty \cdot x$ is not defined for $x = 0$.

(c) $-\infty < x < +\infty$ for all $x \in \mathbb{R}$.

We now recall the definition of *convex* and *non-decreasing* functions.

DEFINITION 2. A function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is *convex* if

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y) \ , \text{ for all } \alpha \in [0,1].$$

DEFINITION 3. A function $f : \mathbb{R} \to \mathbb{R}$ is *non-decreasing* if

$$x \leq y \implies f(x) \leq f(y).$$

Using these definitions, we will now establish a few basic properties of convex functions.

(i) Suppose f is a convex function satisfying $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Is $-f$ a convex function? Justify your answer.

(ii) Suppose f and g are convex functions and $c_1, c_2 \in \mathbb{R}$. Under what conditions on c_1 and c_2 is $c_1 f(x) + c_2 g(x)$ always a convex function?

(iii) Suppose f is convex and g is convex and increasing. Is $g \circ f$ convex? Is $f \circ g$ convex? Justify your answer.

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