Lecture 18

Announcement: Revision due tomorrow

Recall:

<u>Def</u>: Given a metric space $(\mathcal{X}, d)$, $x : [0,1] \to \mathcal{X}$ is a <u>(constant speed) geodesic</u> if

$$d(x(t), x(s)) = |t-s| \, d(x(0), x(1))$$

<u>Prop</u>: Given $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, $\mu_0, \mu_1 \ll \mathcal{L}^d$, there exists a constant speed geodesic between them.

Furthermore, there exists a velocity v s.t. (μ, v) is a soln (CE) and

$$\left(\int |v(x,t)|^2 \, d\mu_t(x) \right)^{1/2} \equiv W_2(\mu_0, \mu_1) \quad \forall t \in [0,1]$$

Def: If $\mu \in \mathcal{P}(\mathbb{R}^d)$, $m \in \mathcal{M}_s^d(\mathbb{R}^d)$, $\mu, m \ll \mathcal{L}^d$, the <u>kinetic energy</u> is

$$\mathbb{B}(\mu, m) = \frac{1}{2} \int \frac{|m(x)|^2}{\mu(x)}\, dx$$

How to extend to general $\mu \in \mathcal{P}(\mathbb{R}^d)$?

Def: For $r \in \mathbb{R}$, $x \in \mathbb{R}^d$,

$$f_{\mathbb{B}}(r, x) := \begin{cases} \frac{1}{2}|x|^2/r & \text{if } r > 0 \\ 0 & \text{if } r = x = 0 \\ +\infty & \text{if } r = 0, x \neq 0 \text{ or } r < 0. \end{cases}$$

Fact: $f_{\mathbb{B}}$ is proper, lsc, convex.

Compute convex conjugate:

$$f_{\mathbb{B}}^*(s, y) = \sup_{(r,x) \in \mathbb{R} \times \mathbb{R}^d} \{ sr + y \cdot x - f_{\mathbb{B}}(r, x) \}$$

$$= \sup_{\substack{r > 0, \cancel{x \in \mathbb{R}^d} \\ c > 0}} \{ sr + y \cdot x - \tfrac{1}{2} |x|^2 / r \} \vee 0$$

$x = cy$

$$= \sup_{r,c>0} \Big\{ sr + |y|^2 \underbrace{\Big[c - \frac{c^2}{2r} \Big]}_{\text{maximum attained at } c=r} \Big\} \vee 0$$

$$= \sup_{r>0} \Big\{ sr + |y|^2 \frac{r}{2} \Big\}$$

$$= \sup_{r>0} \Big\{ r\Big(s + \frac{|y|^2}{2}\Big) \Big\}$$

$$= \chi_{\{(s,y):\, s + \frac{|y|^2}{2} \le 0\}}(s,y)$$

$$= \begin{cases} 0 & \text{if } \quad s + \frac{|y|^2}{2} \le 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$f_{\mathcal{B}}^{**}(r,x) = \sup_{(s,y) \in \mathbb{R}\times\mathbb{R}^d} \Big\{ rs + x\cdot y - \chi_{\{s + \frac{|y|^2}{2} \le 0\}} \Big\}$$

$$= \sup_{\substack{(s,y)\in\mathbb{R}\times\mathbb{R}^d \\ s + \frac{|y|^2}{2} \le 0}} \{ rs + x\cdot y \}$$

Fenchel Moreau ensures $f_{\mathcal{B}} = f_{\mathcal{B}}^{**}$.

Prop: Given $\mu \in \mathcal{P}(\mathbb{R}^d)$, $m \in \mathcal{M}_s^d(\mathbb{R}^d)$, define

$$\mathcal{B}(\mu, m) := \sup_{\substack{f \in C_b(\mathbb{R},\mathbb{R}),\ g \in C_b(\mathbb{R}^d, \mathbb{R}^d) \\ f + \frac{1}{2}|g|^2 \leq 0}} \left\{ \int f \, d\mu + \int g \cdot dm \right\}$$

Then,

(i) $\mathcal{B}(\mu, m)$ is convex, lsc wrt narrow convergence

(ii) if $\mu, m \ll \omega$, ω Borel meas on $\mathbb{R}^d$, $\omega = \mathcal{L}^d$

$$\mathcal{B}(\mu, m) = \int f_{\mathcal{B}}(\mu(x), m(x)) \, d\omega(x),$$

where $d\mu(x) = \mu(x) \, d\omega(x)$,
$dm(x) = m(x) \, d\omega(x)$

(iii) $\mathcal{B}(\mu, m) = \begin{cases} \frac{1}{2}\int |v|^2 d\mu & \text{if } m \ll \mu, \ dm = v \, d\mu \\ +\infty & \text{otherwise} \end{cases}$

Pf:

(i) Note that

$C := \{ f, g \in C_b : f + \frac{\gamma}{2}|g|^2 \leq 0 \}$

is a nonempty, closed, convex set.

Thus χ_C is proper, lsc, convex.

Furthermore,

$\mathcal{B}(\mu, m) = \chi_C^*(\mu, m)$

(That is, $\mathcal{B}$ is the restriction of χ_C^* to $\mathcal{P}(\mathbb{R}^d) \times \mathcal{M}_s^d(\mathbb{R}^d)$.)

Thus $\mathcal{B}$ is convex, lsc wrt narrow convergence.

(ii) We will first show $\mathcal{B}(\mu, m) \leq \int f_{\mathcal{B}}(\mu, m) d\omega$

$$\mathcal{B}(\mu, m)$$

$$= \sup_{\substack{f \in C_b(\mathbb{R},\mathbb{R}),\ g \in C_b(\mathbb{R}^d,\mathbb{R}^d) \\ f + \frac{1}{2}|g|^2 \leq 0}} \left\{ \int f \, d\mu + \int g \cdot dm \right\}$$

$$= \sup_{\substack{f \in C_b(\mathbb{R},\mathbb{R}),\ g \in C_b(\mathbb{R}^d,\mathbb{R}^d) \\ f + \frac{1}{2}|g|^2 \leq 0}} \left\{ \int (f \mu + g \cdot m) \, d\omega \right\}$$

$$\leq \int \sup_{s + \frac{|y|^2}{2} \leq 0} \{ s \mu(x) + y \cdot m(x) \} \, d\omega(x)$$

$$= \int f_{\mathcal{B}}(\mu(x), m(x)) \, d\omega(x).$$

We will now show the reverse inequality.

Since $\mu(x) \geq 0$ ω-a.e.,

$$f_{\mathcal{B}}(\mu(x), m(x))$$

$$= \sup_{s + |y|^2/2 \leq 0} \{ s \mu(x) + y \cdot m(x) \}$$

$$= \sup_{y \in \mathbb{R}^d} \left\{ -\frac{|y|^2}{2} \mu(x) + y \cdot m(x) \right\}$$

$$= \sup_{n \in \mathbb{N}} \sup_{|y| \leq n} \left\{ -\frac{|y|^2}{2} \mu(x) + y \cdot m(x) \right\}$$

Thus, for ω-a.e. x, $\exists$ $g_n(x)$ s.t. $|g_n(x)| \leq n$ and

$$-\frac{|g_n(x)|^2}{2} \mu(x) + g_n(x) \cdot m(x) \nearrow f_{\mathcal{B}}(\mu(x), m(x))$$

So, by MCT,

$$\lim_{n \to \infty} \int \left(-\frac{|g_n|^2}{2} \mu + g_n \cdot m \right) d\omega = \int f_{\mathcal{B}}(\mu, m) \, d\omega$$

This shows

$$\sup_{\substack{f \in L^\infty(\mathbb{R}, \mathbb{R}),\ g \in L^\infty(\mathbb{R}^d, \mathbb{R}^d) \\ f + \frac{1}{2}|g|^2 \leq 0}} \left\{ \int f \, d\mu + \int g \cdot dm \right\} \geq \int f_{\mathcal{B}}(\mu, m) \, d\omega$$

To complete proof of (ii), it suffices to show...

CLAIM: For $\mu \in \mathcal{P}(\mathbb{R}^d)$, $m \in \mathcal{M}_s^d(\mathbb{R}^d)$

$$\left.\begin{array}{l}\sup\limits_{\substack{f \in L^\infty(\mathbb{R},\mathbb{R}),\ g \in L^\infty(\mathbb{R}^d,\mathbb{R}^d) \\ f + \frac{1}{2}|g|^2 \leq 0}} \left\{ \int f \, d\mu + \int g \cdot dm \right\}\end{array}\right\} := (*)$$

$$\leq \sup_{\substack{f \in C_b(\mathbb{R},\mathbb{R}),\ g \in C_b(\mathbb{R}^d,\mathbb{R}^d) \\ f + \frac{1}{2}|g|^2 \leq 0}} \left\{ \int f \, d\mu + \int g \cdot dm \right\}$$

First, note that

$$(*) = \sup_{n \in \mathbb{N}} \sup_{\substack{g \in L^\infty(\mathbb{R}^d,\mathbb{R}^d) \\ |g| \leq n}} \left\{ \int \frac{-|g|^2}{2} d\mu + \int g \cdot dm \right\}$$

By defn of sup, $\exists$ g_n, $\|g_n\|_\infty \leq n$ s.t.

$$\int \frac{-|g_n|^2}{2} d\mu + \int g_n \cdot dm \nearrow (*)$$

By Lusin's theorem, $\forall$ $\varepsilon > 0$, $\exists$ $\tilde{g}_{n,\varepsilon} \in C_b(\mathbb{R}^d)$ s.t. $\|\tilde{g}_{n,\varepsilon}\|_\infty \leq \|g_n\|_\infty$

and $\tilde{g}_{n,\varepsilon} = g_n$ except on a set $E_{n,\varepsilon}$, with $(\mu + |m|)(E_{n,\varepsilon}) < \varepsilon$.

$$\int -\frac{|g_n|^2}{2}\,d\mu + \int g_n\,dm$$

$$= \int_{(E_{n,\varepsilon})^c} -\frac{|\tilde{g}_{n,\varepsilon}|^2}{2}\,d\mu + \int_{(E_{n,\varepsilon})^c} \tilde{g}_{n,\varepsilon}\,dm$$

$$+ \int_{E_{n,\varepsilon}} -\frac{|g_n|^2}{2}\,d\mu + \int_{E_{n,\varepsilon}} g_n\,dm$$

$$= \int_{\mathbb{R}^d} -\frac{|\tilde{g}_{n,\varepsilon}|^2}{2}\,d\mu + \int_{\mathbb{R}^d} \tilde{g}_{n,\varepsilon}\,dm$$

$$+ \int_{E_{n,\varepsilon}} -\frac{|g_n|^2}{2}\,d\mu + \int_{E_{n,\varepsilon}} g_n\,dm$$

$$+ \int_{E_{n,\varepsilon}} -\frac{|\tilde{g}_{n,\varepsilon}|^2}{2}\,d\mu + \int_{E_{n,\varepsilon}} \tilde{g}_{n,\varepsilon}\,dm$$

$$\leq \int_{\mathbb{R}^d} -\frac{|\tilde{g}_{n,\varepsilon}|^2}{2}\,d\mu + \int_{\mathbb{R}^d} \tilde{g}_{n,\varepsilon}\,dm$$

$$+ 2n^2\left(\int_{E_{n,\varepsilon}} d\mu + \int_{E_{n,\varepsilon}} d|m|\right)$$

This proves the claim.

It remains to show (iii).

First, suppose $\exists$ Borel set A s.t. $\mu(A)=0$, but $m(A)\neq 0$.

By CLAIM, it suffices to show $\exists$ $f_n, g_n \in L^\infty$, $f_n + \frac{1}{2}|g_n|^2 \leq 0$, s.t.

$$\int f_n d\mu + \int g_n \cdot dm \nearrow +\infty.$$

Define $f_n = \frac{-n^2}{2} 1_A$, $g_n = n \frac{\overline{m(A)}}{|m(A)|} 1_A$.

Then,

$$\int f_n d\mu + \int g_n \cdot dm = 0 + n|m(A)| \nearrow +\infty.$$

Now, suppose $m \ll \mu$, $dm = v\, d\mu$.

Applying part (ii),

$$B(\mu, m) = \int f_B(1, v(x))\, d\mu(x) = \int |v|^2 d\mu.$$