Lecture 15

Recall:

Thm: Given $\mathcal{X} \subset\subset \mathbb{R}^d$, $\mu, \nu \in \mathcal{P}(\mathcal{X})$,

$$\inf_{\gamma \in \Gamma(\mu,\nu)} \mathbb{K}_1(\gamma) = \sup_{\varphi \in C(\mathcal{X}),\, \|\varphi\|_{Lip} \leq 1} \int \varphi \, d(\mu - \nu)$$

and $\exists\, \varphi^*$ that achieves maximum.

Remark: More generally, if $\mathcal{X} = \mathbb{R}^d$,

$$\inf_{\gamma \in \Gamma(\mu,\nu)} \mathbb{K}_1(\gamma) = \sup_{\varphi \in C_b(\mathbb{R}^d),\, \|\varphi\|_{Lip} \leq 1} \int \varphi \, d(\mu - \nu)$$

Exercise: $W_p(\mu,\nu) \leq W_q(\mu,\nu)$ $\forall\, p \leq q$.
If $\operatorname{supp}\mu, \operatorname{supp}\nu$ are compact, $\exists\, C$ s.t.

$$W_q^q(\mu,\nu) \leq C\, W_p^p(\mu,\nu) \quad \forall\, p \leq q.$$

Goal: Prove W_2 is a metric on $\mathcal{P}_2(\mathbb{R}^d)$.

Our proof will use Brenier's theorem... which requires $\mu \ll \mathcal{L}^d$.

We need a way to approximate $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ by μ_ε that are abs cts.

Prop: (W_2 jointly lsc wrt narrow convergence) [Polish space: X]
Suppose $\mu_n, \nu_n \in \mathcal{P}(X)$ satisfying $\mu_n \to \mu$ and $\nu_n \to \nu$ narrowly.

Then $\liminf_{n\to\infty} W_2(\mu_n, \nu_n) \geq W_2(\mu, \nu)$.

Approximation by Convolution

Def: (mollifier) $\varphi : \mathbb{R}^d \to [0, +\infty)$, bdd, meas
$\varphi(x) = \varphi(-x)$, $\int \varphi(x)\,dx = 1$
$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right)$ ← $\int \varphi_\varepsilon(x)\,dx = 1$

Def: Given $\mu \in \mathcal{P}(\mathbb{R}^d)$, define

$$\varphi_\varepsilon * \mu(x) = \int \varphi_\varepsilon(x-y)\, d\mu(y)$$

We will often abuse notation and write

$$\underbrace{d(\varphi_\varepsilon * \mu)(x)}_{\in \mathcal{P}(\mathbb{R}^d)} = \underbrace{\varphi_\varepsilon * \mu(x)}_{\text{density}}\, dx$$

Note: $\int \varphi_\varepsilon * \mu(x)\, dx = \iint \varphi_\varepsilon(x-y)\, d\mu(y)\, dx = \iint \varphi_\varepsilon(x-y)\, dx\, d\mu(y) = \int 1\, d\mu(y) = 1$

Lemma: Suppose $\mu \in \mathcal{P}(\mathbb{R}^d)$.

(i) For any $f: \mathbb{R}^d \to \mathbb{R}$ meas, bdd below,

$$\int f\, d(\underline{\varphi_\varepsilon * \mu}) = \int (\underline{\varphi_\varepsilon * f})\, d\mu$$

"associativity of convolution"

(ii) $\varphi_\varepsilon * \mu \to \mu$ narrowly, as $\varepsilon \to 0$.

Now, we consider behavior of W_2 and convolution.

Lemma: Given $\mu \in \mathcal{P}(\mathbb{R}^d)$ and φ_ε as above,

$\leftarrow \int |x|^2 \varphi(x)\,dx$

(i) $W_2(\mu, \varphi_\varepsilon * \mu) \leq \varepsilon (m_2(\varphi))^{1/2}$

(ii) $W_2(\varphi_\varepsilon * \mu, \varphi_\varepsilon * \nu) \leq W_2(\mu, \nu)$

(iii) $\lim_{\varepsilon \to 0} W_2(\varphi_\varepsilon * \mu, \varphi_\varepsilon * \nu) = W_2(\mu, \nu)$

$m_2(\mu) := \int |x|^2 d\mu(x)$

Thm $(W_2, \mathcal{P}_2(\mathbb{R}^d))$ is a metric space.

Pf:

Note that, since $|x^1 - x^2|^2 \leq 2|x^1|^2 + 2|x^2|^2$,

γ is an OT plan from μ to ν

$$
\begin{aligned}
W_2(\mu, \nu) &= \int |x^1 - x^2|^2 d\gamma \\
&\leq 2\int |x^1|^2 d\gamma + 2\int |x^2|^2 d\gamma \\
&= 2 m_2(\mu) + 2 m_2(\nu) \\
&< +\infty
\end{aligned}
$$

Thus, $\underline{W_2 : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty)}$.

Furthermore, W_2 is symmetric since γ is an OT plan from μ to ν iff $\tilde{\gamma}$ is an OT plan from ν to μ where $\tilde{\gamma}(A \times B) = \gamma(B \times A)$.

$W_2(\mu,\nu) = \int |x^1 - x^2|^2 d\gamma = \int |x^2 - x^1|^2 d\tilde{\gamma} \leq W_2(\nu,\mu)$

Interchanging roles of μ and ν,
$W_2(\nu,\mu) \leq W_2(\mu,\nu)$.

Thus, $W_2(\mu,\nu) = W_2(\nu,\mu)$, so W_2 is <u>symmetric</u>.

To see that W_2 is <u>nondegenerate</u>, suppose $W_2(\mu,\nu) = 0$. Then, if γ is an OT plan from μ to ν,

$$\int |x^1 - x^2|^2 d\gamma = 0.$$

Thus $x^1 = x^2$ γ-a.e.

Hence, for any $f \in C_b(\mathbb{R}^d)$,

$$\begin{aligned}\int f(x^1) d\mu(x^1) &= \int f(x^1) d\gamma(x^1, x^2) \\ &= \int f(x^2) d\gamma(x^1, x^2) \\ &= \int f(x^2) d\nu(x^2)\end{aligned}$$

$\Rightarrow \mu = \nu$.

Converseley, if $\mu = \nu$, $\gamma = (id \times id)\# \mu$ is a transport plan from μ to ν and

$$0 \leq W_2(\mu, \nu) \leq \left(\int |x^1 - x^2|^2 d\gamma \right)^{1/2}$$
$$= \left(\int |x - x|^2 d\mu \right)^{1/2}$$
$$= 0$$

It remains to show the triangle inequality.

First, suppose $\mu_0, \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ are abs cts wrt $\mathcal{L}^d$.

By Brenier's Theorem, $\exists$ OT maps $t_1 \# \mu_0 = \mu_1$, $t_2 \# \mu_1 = \mu_2$.

$$(t \circ s)\# \mu = t \# (s \# \mu)$$

Note that $(t_2 \circ t_1)\# \mu_0 = \mu_2$.

$$W_2(\mu_0, \mu_2) \leq \left(\int | t_2 \circ t_1(x) \overset{+ t_1(x) - t_1(x)}{-} x|^2 d\mu_0(x) \right)^{1/2}$$
$$\leq \left(\int | t_2 \circ t_1(x) - t_1(x)|^2 d\mu_0(x) \right)^{1/2}$$
$$+ \left(\int | t_1(x) - x|^2 d\mu_0(x) \right)^{1/2}$$

$$= \left(\int |t_2(x) - x|^2 d\mu_1(x)\right)^{1/2}$$
$$+ W_2(\mu_0, \mu_1)$$
$$= W_2(\mu_1, \mu_2) + W_2(\mu_0, \mu_1).$$

Finally, for general $\mu_0, \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$, fix a mollifier φ_ε, so $\varphi_\varepsilon * \mu_i \ll \mathcal{L}^d$ for $i = 0, 1, 2$. Then

$$W_2(\mu_0, \mu_2) = \lim_{\varepsilon \to 0} W_2(\varphi_\varepsilon * \mu_0, \varphi_\varepsilon * \mu_2)$$
$$\leq \lim_{\varepsilon \to 0} W_2(\varphi_\varepsilon * \mu_0, \varphi_\varepsilon * \mu_1) + W_2(\varphi_\varepsilon * \mu_1, \varphi_\varepsilon * \mu_2)$$
$$= W_2(\mu_0, \mu_1) + W_2(\mu_1, \mu_2) \quad \square$$

Now, we will characterize the topology of $(W_2, \mathcal{P}_2(\mathbb{R}^d))$.

Thm: Given $\mu_n, \mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\lim_{n \to \infty} W_2(\mu_n, \mu) = 0 \iff \begin{array}{l} \mu_n \to \mu \text{ narrowly} \\ M_2(\mu_n) \to M_2(\mu) \end{array}$$

Recall from Lec 7, narrow conv $\mu_n \to \mu$ is equivalent to

$$\int f\,d\mu_n \to \int f\,d\mu \quad \forall f \in C_c(\mathbb{R}^d) \qquad (*)$$

added after class

Lemma: $\mu_n \to \mu$ narrowly iff $\int f\,d\mu_n \to \int f\,d\mu$ $\forall f \in C_c^\infty(\mathbb{R}^d)$.

Pf:
"$\Rightarrow$" obvious
"$\Leftarrow$"

Fix $f \in C_c(\mathbb{R}^d)$.
Fix $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$. Know $\varphi_\varepsilon * f \in C_c^\infty(\mathbb{R}^d)$, $\varphi_\varepsilon * f \to f$ in $L^\infty(\mathbb{R}^d)$.

Thus, $\forall \varepsilon > 0$, $\lim_{n\to\infty} \int \varphi_\varepsilon * f\,d\mu_n = \int \varphi_\varepsilon * f\,d\mu$.

Fix $\delta > 0$. $\exists\ \varepsilon_\delta$ s.t. $\|\varphi_{\varepsilon_\delta} * f - f\|_{L^\infty} < \frac{\delta}{4}$.
$\exists\ n_\delta$ s.t. $\forall n > n_\delta$, $|\int \varphi_{\varepsilon_\delta} * f\,d\mu_n - \varphi_{\varepsilon_\delta} * f\,d| < \frac{\delta}{2}$.

Thus, $\forall n > n_\delta$,

$$\begin{aligned}
\left|\int f\,d\mu_n - \int f\,d\mu\right| &\le \left|\int (\varphi_{\varepsilon_\delta} * f - f)\,d\mu_n\right| + \left|\int (\varphi_{\varepsilon_\delta} * f - f)\,d\mu\right| \\
&\quad + \left|\int \varphi_{\varepsilon_\delta} * f\,d\mu_n - \int \varphi_{\varepsilon_\delta} * f\,d\mu\right| \\
&\le 2\|\varphi_{\varepsilon_\delta} * f - f\|_{L^\infty} + \frac{\delta}{2} \\
&< \delta
\end{aligned}$$

$\square$.

Pf: (of Theorem)

Suppose $\lim_{n\to\infty} W_2(\mu_n, \mu) = 0$.

Note that, $\forall\, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

γ is OT plan, $x^2 = 0$ γ a.e.

$$W_2^2(\nu, \delta_0) = \int |x^1 - x^2|^2 d\gamma = \int |x^1|^2 d\gamma = m_2(\nu).$$

Thus,

$\gamma(A \times B) \leq \gamma(\mathbb{R}^d \times B) = \delta_0(B)$

$$\lim_{n\to\infty} m_2(\mu_n) = \lim_{n\to\infty} W_2^2(\mu_n, \delta_0)$$
$$= W_2^2(\mu, \delta_0)$$
$$= m_2(\mu).$$

It remains to show $\mu_n \to \mu$ narrowly. Fix $f \in C_c^\infty(\mathbb{R}^d)$. Will show (*)

By exercise,

$$W_2(\mu_n, \mu) \geq W_1(\mu_n, \mu)$$
$$= \sup \int \varphi \, d\mu_n - \int \varphi \, d\mu$$

$\psi \in C_b(\mathbb{R}^d)$, $\|\psi\|_{Lip} \leq 1$ ✓

If $\|f\|_{Lip} = 0$, $f = 0$ and the result holds trivially.

Otherwise, define $\psi = \frac{\pm f}{\|f\|_{Lip}}$.

$$W_2(\mu_n, \mu) \geq \frac{1}{\|f\|_{Lip}} \left| \int f d\mu_n - f d\mu \right|$$

This shows $\int f d\mu_n \to \int f d\mu$.
Since $f \in C_c^\infty(\mathbb{R}^d)$ was arbitrary, $\mu_n \to \mu$ narrowly.