Lecture 10

<u>Def</u>: Given normed vector spaces $\mathcal{X}$ and $\mathcal{U}$ and a convex function $F: \mathcal{X} \times \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$.

<u>primal problem</u>: $P_0 := \inf_{x \in \mathcal{X}} f(x)$, $f(x) = F(x,0)$

<u>dual problem</u>: $D_0 := \sup_{v \in \mathcal{U}^*} g(v)$, $g(v) = -F^*(0,v)$.

Note that

$$D_0 = \sup_{v \in \mathcal{U}^*} \inf_{(x,u) \in \mathcal{X} \times \mathcal{U}} F(x,u) - \langle v, u \rangle$$

<u>Thm</u> (Equivalence of Primal and Dual Problems): Given $F: \mathcal{X} \times \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$ convex, suppose $P_0 < +\infty$.

Define the inf-projection $P(u) := \inf_{x \in \mathcal{X}} F(x,u)$. Then

(i) $P_0 = D_0 \iff P$ is lsc at $u = 0$.

(ii) $P_0 = D_0$ and a maximizer of dual problem exists

$\Longleftrightarrow \partial P(0) \neq \emptyset.$

Kantorovich Duality

$(\mathcal{X}, d)$ Polish space

$\mu, \nu \in \mathcal{P}(\mathcal{X})$

$c: \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ lower semicontinuous

$$\min_{\gamma : \gamma \in \Gamma(\mu,\nu)} \overbrace{\int_{\mathcal{X}\times\mathcal{X}} c(x^1, x^2)\, d\gamma(x^1, x^2)}^{\mathbb{K}(\gamma)} \qquad (KP)$$

Given $\varphi \in C_b(\mathcal{X})$, $\mu \in \mathcal{M}^s(\mathcal{X})$, let

$$\langle \mu, \varphi \rangle = \int_{\mathcal{X}} \varphi(x)\, d\mu(x).$$

Fact: $\mu = \nu \Longleftrightarrow \langle \mu, \varphi \rangle = \langle \nu, \varphi \rangle \quad \forall \varphi \in C_b(\mathcal{X}).$

Prop: For any $f : \mathcal{X} \to \mathbb{R}$ lsc and bdd below, $\mu \in \mathcal{P}(\mathcal{X})$,

$$\int f\, d\mu = \sup \left\{ \int g\, d\mu : g \in C_b(\mathcal{X}),\ g \leq f \right\}$$

Pf: Note that "≥" follows quickly, since $g \leq f \Rightarrow \int g\,d\mu \leq \int f\,d\mu$.

Now we consider "≤". Recall from Lec5, $\exists \{g_k\}_{k=1}^{\infty} \subseteq C_b(X)$ s.t. $g_k \nearrow f$. Then, by MCT, $c_0 := \inf g_1 > -\infty$.

$$\lim_{k\to\infty} \int (g_k - c_0)\,d\mu = \int (f - c_0)\,d\mu$$

Thus $\int g_k\,d\mu \nearrow \int f\,d\mu$.

nvs X, $C \subseteq X$ convex

$$\chi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise} \end{cases}$$

<u>Lemma</u>: Given $\mu, \nu \in \mathcal{P}(X)$, $\gamma \in \mathcal{M}(X \times X)$

$$\sup_{\varphi, \psi \in C_b(X)} \langle \mu - \pi^1 \# \gamma, \varphi \rangle + \langle \nu - \pi^2 \# \gamma, \psi \rangle = \chi_{\Gamma(\mu,\nu)}(\gamma).$$

Applying this to Kantorovich's problem, we obtain

$$\inf_{\gamma \in \Gamma(\mu,\nu)} \mathbb{K}(\gamma) = \inf_{\gamma \in \mathcal{M}(X \times X)} \mathbb{K}(\gamma) + \chi_{\Gamma(\mu,\nu)}(\gamma)$$

$$= \inf_{\gamma \in \mathcal{M}(X \times X)} \sup_{\varphi, \psi \in C_b(X)} \mathbb{K}(\gamma) + \langle \mu - \pi^1 \# \gamma, \varphi \rangle + \langle \nu - \pi^2 \# \gamma, \psi \rangle$$

$$= -\sup_{\gamma \in \mathcal{M}(X\times X)} \ \inf_{\varphi,\psi \in C_b(X)} -\mathbb{K}(\gamma) - \langle \mu - \pi^1\#\gamma, \varphi\rangle - \langle \nu - \pi^2\#\gamma, \psi\rangle$$

How should we choose $\mathcal{X}, \mathcal{U}, F(x,u)$ so that this coincides with

$$D_0 = \sup_{v \in \mathcal{U}^*} \ \inf_{(x,u)\in \mathcal{X}\times\mathcal{U}} F(\overset{(\varphi,\psi)}{x}, u) - \langle \overset{\gamma}{v}, u\rangle \ ?$$

Suppose X cpt, so $(C(X))^* = \mathcal{M}^s(X)$.

$\mathcal{U} = C(X\times X)$ $\qquad$ $\mathcal{X} = C(X)\times C(X)$

$\mathcal{U}^* = \mathcal{M}^s(X\times X)$ $\qquad$ $\mathcal{X}^* = \mathcal{M}^s(X)\times\mathcal{M}^s(X)$

Gathering the γ's...

$$-\mathbb{K}(\gamma) - \langle \mu - \pi^1\#\gamma, \varphi\rangle - \langle \nu - \pi^2\#\gamma, \psi\rangle$$
$$= -\int \varphi \, d\mu - \int \psi \, d\nu - \int c(x^1,x^2) - \varphi\circ\pi^1 - \psi\circ\pi^2 \, d\gamma(x^1,x^2)$$

$$= -\int \varphi \, d\mu - \int \psi \, d\nu - \sup_{\substack{u \in C(X\times X) \\ u \le c - \varphi\circ\pi^1 - \psi\circ\pi^2}} \int u(x^1,x^2)\, d\gamma(x^1,x^2)$$

$$= \inf_{\substack{u \in C(X\times X) \\ u \le c - \varphi\circ\pi^1 - \psi\circ\pi^2}} -\int \varphi \, d\mu - \int \psi \, d\nu - \langle \gamma, u \rangle$$

$$= \inf_{u \in C(X\times X)} \underbrace{-\int \varphi \, d\mu - \int \psi \, d\nu + \chi_{\{(\varphi,\psi),u \,:\, u \le c - \varphi\circ\pi^1 - \psi\circ\pi^2\}}((\varphi,\psi),u)}_{F(x,u)} - \langle \gamma, u \rangle$$

Therefore, we may rewrite (KP) as the following saddle point problem:

$$D_0 = \sup_{v \in \mathcal{U}^*} \inf_{(x,u) \in \mathcal{X}\times\mathcal{U}} F(x,u) - \langle v, u \rangle$$

$$= \sup_{\gamma \in \mathcal{M}(X\times X)} \inf_{((\varphi,\psi),u) \in (C(X)\times C(X))\times C(X\times X)} F((\varphi,\psi),u) - \langle \gamma, u \rangle$$

$$= -\inf_{\gamma \in P(\mu,\nu)} \mathbb{K}(\gamma)$$

What is the corresponding primal problem?

primal problem: $P_0 := \inf_{x \in X} f(x)$, $f(x) = F(x,0)$

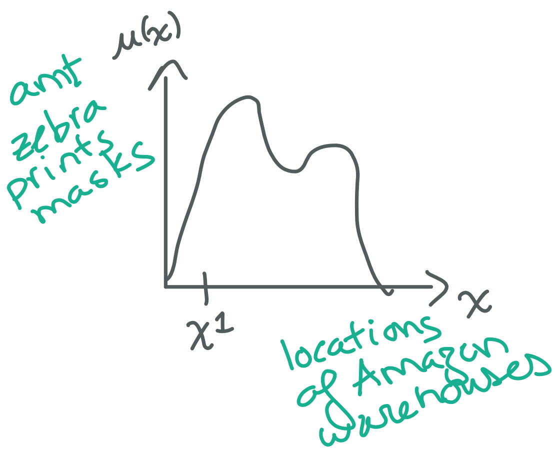
By defn of F above,

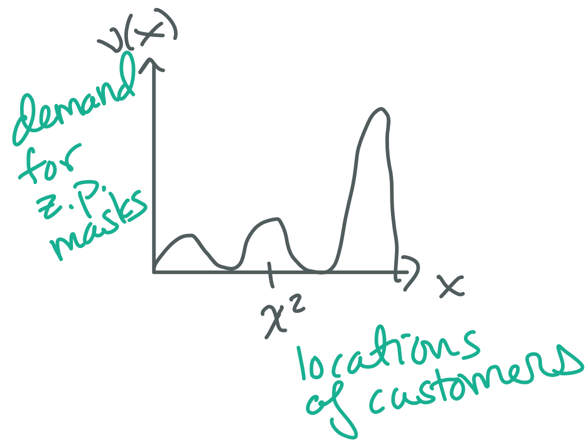
$$P_0 = \inf_{\substack{(\varphi,\psi) \in C(X)\times C(X) \\ 0 \le c - \varphi\circ\pi^1 - \psi\circ\pi^2}} -\int \varphi \, d\mu - \int \psi \, d\nu$$

$$= -\sup_{\substack{(\varphi,\psi) \in C(X)\times C(X) \\ \varphi\circ\pi^1 + \psi\circ\pi^2 \le c}} \int \varphi \, d\mu + \int \psi \, d\nu$$

$\varphi\circ\pi^1 + \psi\circ\pi^2 \le c$ → $\varphi(x^1) + \psi(x^2) \le c(x^1, x^2)$ $\Updownarrow$ $\varphi \oplus \psi \le c$

The Shipper's Problem (Caffarelli)





- It costs Amazon $c(x^1, x^2)$ dollars to move one zebra print mask from x^1 to x^2.

- You want to make extra $$$ to support <u>fancy spiked coffee.</u>

- You charge <u>Amazon</u> $\varphi(x^1)$ dollars to pick up one <u>3. p. mask</u> from location x^1 and $\psi(x^2)$ dollars to deliver to x^2.

Obviously, if <u>Amazon</u> will let you ship, ~~the~~ the following must be true:

$$\varphi(x^1) + \psi(x^2) \leq c(x^1, x^2).$$

- $-P_0 = \sup_{(\varphi,\psi) \in C(X) \times C(X)} \left\{ \int \varphi \, d\mu + \int \psi \, d\nu : \varphi(x_1) + \psi(x_2) \leq c(x_1, x_2) \right\}$

  = largest amount of money you can make

- $-D_0 = \inf_{\gamma \in \Gamma(\mu,\nu)} \int c(x^1, x^2) \, d\gamma(x^1, x^2)$

  = least amt it would cost <u>Amazon</u> to do it themselves

- We always have $P_0 \geq D_0 \Leftrightarrow -D_0 \geq -P_0$. If there is no duality gap, $P_0 = D_0$.

Now: prove $\mathcal{P}_0 = \mathcal{D}_0$ for (KP)

Suppose $\mathcal{X}$ cpt Polish space.

Thm: For all $\mu, \nu \in \mathcal{P}(\mathcal{X})$,

$$\underbrace{\inf_{\gamma \in \Gamma(\mu,\nu)} \mathbb{K}(\gamma)}_{= -D_0} = \underbrace{\sup_{\substack{(\varphi,\psi) \in C(\mathcal{X}) \times C(\mathcal{X}) \\ \varphi \oplus \psi \leq c}} \int \varphi \, d\mu + \int \psi \, d\nu}_{= -P_0}$$

Exercise: Given a compact metric space $(\mathcal{X}, d)$, $\mathcal{F}$ a collection of functions $\mathcal{X} \times \mathcal{X} \to \mathbb{R}$. If

$$\{ x^1 \mapsto f(x^1, x^2) : f \in \mathcal{F}, x^2 \in \mathcal{X} \}$$

is equicontinuous, then

$$\{ x^1 \mapsto \inf_{x^2 \in \mathcal{X}} f(x^1, x^2) : f \in \mathcal{F} \}$$

is equicontinuous.

"Double convexification trick"

Prop: Suppose $c : X \times X \to [0,+\infty)$ is continuous. Given

$$\{(\varphi_i, \psi_i)\}_{i \in I} \subseteq C(X) \times C(X), \quad \psi_i \geq 0,$$
$$\{u_i\}_{i \in I} \subseteq C(X \times X) \text{ unif bdd, e-cts,}$$
$$F((\varphi_i, \psi_i), u_i) < +\infty \quad \forall i \in I.$$

define

$$\tilde{\varphi}_i(x^1) = \inf_{x^2 \in X} c(x^1, x^2) - u_i(x^1, x^2) - \psi_i(x^2),$$

$$\tilde{\psi}_i(x^2) = \inf_{x^1 \in X} c(x^1, x^2) - u_i(x^1, x^2) - \tilde{\varphi}_i(x^1).$$

Then $\{\tilde{\varphi}\}_{i \in I}$, $\{\tilde{\psi}\}_{i \in I}$ are unif. bdd, e-cts, and

$$F((\tilde{\varphi}_i, \tilde{\psi}_i), u_i) \leq F((\varphi_i, \psi_i), u_i).$$

Pf: Since $F((\varphi_i, \psi_i), u_i) < +\infty$,

$$u_i(x^1, x^2) + \varphi_i(x^1) + \psi_i(x^2) \leq c(x^1, x^2), \quad \forall x^1, x^2 \in X.$$

This ensures $\sup_{\substack{i \in I \\ x^2 \in X}} \psi_i(x^2) < +\infty$. ($C_\psi$)

$u_i(x^1, x^2) + \tilde{\varphi}_i(x^1) + \psi_i(x^2) \leq c(x^1, x^2)$

By defn of $\tilde{\varphi}_i$

- $\varphi_i \leq \tilde{\varphi}_i$, $\quad u_i + \tilde{\varphi}_i \oplus \psi_i \leq c$
- By lemma, $\{\tilde{\varphi}_i\}_{i \in I}$ e-cts.

- $\tilde{\varphi}_i$ bdd above, bdd below, unif ini
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\begin{aligned}
F((\tilde{\varphi}_i, \psi_i), u_i) &= -\int \tilde{\varphi}_i \, d\mu - \int \psi_i \, d\nu \\
&\leq -\int \varphi_i \, d\mu - \int \psi_i \, d\nu \\
&= F((\varphi_i, \psi_i), u_i)
\end{aligned}
$$

By defn of $\tilde{\psi}_i$

- $\psi_i \leq \tilde{\psi}_i$, $\quad u_i + \tilde{\varphi}_i \oplus \tilde{\psi}_i \in C$
- $\{\psi_i\}_{i \in I}$ unif bdd, e-cts
- $F((\tilde{\varphi}_i, \tilde{\psi}_i), u) \leq F((\tilde{\varphi}_i, \psi_i), u)$. $\square$