Radical Endomorphisms of Decomposable Modules ⋆

Julius M. Zelmanowitz

University of California, Oakland, CA 94607 USA

Abstract

An element of the Jacobson radical of the endomorphism ring of a decomposable module is characterized in terms of its action on the components of the decomposition. This extends to arbitrary decomposable modules a result previously known only for the special case of free modules.

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1 Introduction

The historical motivation for this study may be said to begin with a question raised by N. Jacobson in "Structure of Rings", first published in 1956 [4]. On page 23 of that seminal text, Jacobson asked for a characterization of the elements in the radical of the ring $M_I(R)$ of $I \times I$ row-finite matrices over an arbitrary ring R. In general, it is not true that a matrix whose entries lie in the radical of R is in the radical of $M_I(R)$, as will be demonstrated below.

For a ring R we let $J(R)$ denote the Jacobson radical of R, and we consider the free R-module of infinite rank consisting of all row vectors $(r_1, r_2, \ldots)$ such that each $r_i \in R$ and $r_i = 0$ for almost all i. Let $a_1, a_2, \ldots$ be a sequence of elements from $J(R)$ and consider the row-finite matrix

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Email address: julius.zelmanowitz@ucop.edu (Julius M. Zelmanowitz).

$$\alpha = \begin{pmatrix} 0 & a_2 & 0 & 0 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ 0 & 0 & 0 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If $1 - \alpha$ has a left inverse, then there will be a row vector $(b_1, b_2, \ldots)$ with $(b_1, b_2, \ldots)(1 - \alpha) = (a_1, 0, 0, \ldots)$. This leads to the system of equations:

$$\begin{aligned} &b_1 = a_1 \\ &b_i - b_{i-1}a_i = 0, \text{ for all } i \geq 2. \end{aligned}$$

Solving recursively, we get $b_i = a_1a_2 \ldots a_i$ for all $i \geq 1$. Since $b_n = 0$ for some $n \geq 1$ we conclude that $a_1a_2 \ldots a_n = 0$. Thus, a necessary condition for $J(M_I(R)) = M_I(J(R))$ to hold for I an infinite set is that $J(R)$ be left T-nilpotent. In 1961, E.M. Paterson showed conversely that left T-nilpotence of $J(R)$ is sufficient for $J(M_I(R)) = M_I(J(R))$ to hold [6]. In particular then, any local integral domain which is not a field is an example of a ring for which $J(M_I(R)) \subset M_I(J(R))$ whenever $|I| = \infty$.

Jacobson's question was settled in 1969 by N. Sexauer and J. Warnock who employed a daunting calculation to establish the following characterization of the radical elements in a ring of row-finite matrices.

Theorem. (N. Sexauer & J. Warnock [7]) *For an arbitrary ring R and a matrix $\alpha \in M_I(R)$, $\alpha \in J(M_I(R))$ if and only if $\alpha \in M_I(J(R))$ and the column left ideals of α are right vanishing.*

We explain the preceding terminology. The *column left ideals* of $\alpha = (a_{ij})_{i,j \in I} \in M_I(R)$ are the left ideals of the form

$$A_j(\alpha) = \left\{ \sum_{i \in I} r_i a_{ij} \mid r_i \in R \text{ and } r_i = 0, \text{ for almost all } i \in I \right\}$$

for each $j \in I$. An arbitrary family of left ideals $\{A_j \mid j \in I\}$ of R is called *right vanishing* if for every sequence of elements $a_{i_k} \in A_{i_k}$ with $i_1, i_2, \ldots$ a sequence of distinct elements of I, there exists an integer $n \geq 1$ with $a_{i_1}a_{i_2} \ldots a_{i_n} = 0$.

A ring of row-finite matrices over a ring with identity element is isomorphic to the endomorphism ring of a free module. Accordingly, this theorem was generalized to the endomorphism ring of an arbitrary projective module in [8], where a more conceptual proof of the Sexauer & Warnock theorem was also presented. The key ideas in [8] were the use of the characterization of the radical of a ring as the sum of the small (superfluous) one-sided ideals as well as an often-mimicked calculation contained in the work of H. Bass [1].

Another direction for generalization is to determine the elements of the radical of the endomorphism ring of a decomposable module $M = \oplus_{i\in I} M_i$ for some suitable family of modules $\{M_i \mid i \in I\}$ in terms of their action on the components of the decomposition; the test criterion for suitability being that the case where each $M_i = R$ is subsumed and the Sexauer & Warnock theorem captured as a special case. This can lead to consideration of a number of possible extensions: to the case when all M_i are isomorphic; to the case when all M_i are cyclic; to the case when all M_i are finitely generated, and so on. Before we describe our principal result, it is worth considering motivation for this line of research coming from another source.

In a successful search for extensions of the classical refinement theorems for direct sum decompositions of modules, Crawley and Jónsson introduced the concept of an exchange property for a module. An R-module M is said to have the *(finite) exchange property* if whenever M occurs as a direct summand of a (finite) direct sum $N = \oplus_{i\in I} N_i$, then $N = M \oplus (\oplus_{i\in I} N_i')$ for some submodules $N_i' \subseteq N_i$ [2]. For modules that are direct sums of indecomposable modules, the (finite) exchange property completely characterizes when such decompositions complement direct summands. So does a local vanishing condition on endomorphisms of the module, as we now explain.

First, a definition: a family of modules $\{M_i \mid i \in I\}$ is called *locally semi-T-nilpotent* if for each sequence $M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} M_{i_3} \xrightarrow{f_3} \dots$ of non-isomorphisms with pairwise distinct indices $i_k \in I$, and for each $x \in M_{i_1}$, there exists an integer $n \geq 1$ such that $xf_1f_2\dots f_n = 0$.

Theorem. *Suppose that $M = \oplus_{i\in I} M_i$ where each M_i is indecomposable and let $S = End_R M$. Then the following conditions are equivalent.*

(1) M has the exchange property.
(2) M has the finite exchange property.
(3) The decomposition $M = \oplus_{i\in I} M_i$ complements direct summands.
(4) Each M_i has a local endomorphism ring and the family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent.
(5) Each M_i has a local endomorphism ring, $S/J(S)$ is von Neumann regular and idempotents lift modulo $J(S)$.

The equivalence of (1), (2) and (4) is due to Zimmerman-Huisgen and Zimmerman [10], and the equivalence of the last three conditions to Harada and collaborators (see [3] for more complete references). From our perspective, condition (5) indicates that knowledge of the structure of $J(S)$ determines exchange properties for a completely decomposable module, and condition (4) suggests a link with the Sexauer & Warnock theorem since the criterion in (4) is expressed by a vanishing condition. Motivated by these observations, we develop a characterization of the elements in the Jacobson radical of the endomorphism ring of an arbitrary decomposable module. Somewhat surprisingly, the characterization in Theorem 1 does not require any restriction on the summands of the decomposition. In Theorem 2 we also describe the radical elements in the important subring of those endomorphisms which are "locally of finite rank."

2 Radical elements of $End_R(\oplus_{i\in I} M_i)$.

We begin by introducing some convenient notation. Let R be an arbitrary ring (not necessarily containing an identity element), let $M = \oplus_{i\in I} M_i$ be a decomposition of left R-modules, and set $S = End_R M$ acting as right operators on M. For $i \in I$, we let $e_i \in S$ denote the projection homomorphism of M onto M_i across $\oplus_{j\in I\setminus\{i\}} M_j$, and we set $S_i = End_R M_i$. Similarly, for any subset $F \subseteq I$, set $M_F = \oplus_{i\in F} M_i$, $S_F = End_R(M_F)$, and let e_F denote the projection homomorphism of M onto M_F across $\oplus_{j\in I\setminus F} M_j$. We will consistently regard S_i and S_F, respectively, as subrings of S by extending homomorphisms trivially across the complementary summands $\oplus_{j\in I\setminus\{i\}} M_j$ and $\oplus_{i\in I\setminus F} M_i$, respectively; that is, we identify $S_i = e_i S e_i$ and $S_F = e_F S e_F$. For any $\alpha \in S$, $i, j \in I$, and $F, G \subseteq I$, we abbreviate $\alpha_{ij} = e_i \alpha e_j$, $\alpha_{FG} = e_F \alpha e_G$, and $\alpha_F = \alpha_{FF}$. (This notation is consistent with the definitions of e_i and e_F as projection endomorphisms if we take $e = 1_S$, the identity endomorphism in S.) Finally, observe that for an infinite family $\beta_j \in S$, $j \in J$, the sum $\beta = \sum_{j\in J} \beta_j$ defines an element of S provided that for every $x \in M, x\beta_j = 0$ for almost all $j \in J$.

The principal result of this paper is the following characterization of the radical elements in $S = End_R M$.

Theorem 1. *For $M = \oplus_{i\in I} M_i$ with I an infinite index set and $\alpha \in S = End_R M$, either one of the following conditions are necessary and sufficient for $\alpha \in J(S)$ to hold.*

(1) For every $\gamma \in S$ and every $i \in I$, $(\gamma\alpha)_{ii} \in J(S_i)$ and, for every sequence $i_1, i_2, \ldots$ of distinct elements of I and every $x \in M$, there exists a positive integer n with $x(\gamma\alpha)_{i_1 i_2}(\gamma\alpha)_{i_2 i_3} \ldots (\gamma\alpha)_{i_n i_{n+1}} = 0$.
(2) For every $i \in I$, $(S\alpha)_{ii} \subseteq J(S_i)$ and, for every sequence $i_1, i_2, \ldots$ of distinct elements of I, every sequence $\gamma_1, \gamma_2, \ldots \in S$, and every $x \in M$, there exists a positive integer n with $x(\gamma\alpha)_{i_1 i_2}(\gamma\alpha)_{i_2 i_3} \ldots (\gamma\alpha)_{i_n i_{n+1}} = 0$.

The proof that condition (2) is sufficient to imply that $\alpha \in J(S)$ relies on a lemma from [9], which is itself an adaptation of an argument presented in [5] in the special case when the summands of the decomposition have local endomorphism rings. Its statement is as follows.

Lemma 1. *Suppose that $M = \oplus_{i\in I} M_i$ with I an infinite set and that $\beta \in S = End_R M$ is such that $\beta_F = e_F \beta e_F$ has a left inverse in $S_F = End_R M_F$ for every finite subset $F \subset I$. Then for any $x \in M$ there exists a sequence of finite subsets $\emptyset = F_0 \subset F_1 \subset F_2 \subset \dots$ of I and endomorphisms $\mu_1, \mu_2, \dots \in S$, $\delta_1, \delta_2, \dots \in S$ such that:*

(i) for each $n \geq 1$, $x\mu_n\beta = x(1 + \delta_1\delta_2 \dots \delta_n)$; and
(ii) each $\delta_k \in S_{F_k}\beta_{F_k, F_{k+1}\setminus F_k}$.

This lemma will be applied to establish the invertibility of $\beta = 1 + \alpha$ when condition (2) of the theorem holds. For, under those circumstances, condition (ii) allows one to conclude that the product $\delta_1\delta_2 \dots \delta_n$ is eventually zero, and condition (i) then implies that $\beta = 1 + \alpha$ is an epimorphism. We include a brief proof of the lemma in order to keep this exposition self-contained.

Proof of the lemma. We proceed by induction on n. For $F \subseteq I$, set $F' = I\setminus F$. Choose F_1 a finite subset of I with $x \in M_{F_1}$. Set $\beta_1 = \beta_{F_1} = e_{F_1}\beta e_{F_1} \in S_{F_1}$, $\beta_1' = e_{F_1}\beta e_{F_1'}$, and $\beta_1'' = e_{F_1'}\beta$; then $\beta = \beta_1 + \beta_1' + \beta_1''$. By hypothesis, there exists $\mu_1 \in S_{F_1}$ with $x = x\mu_1\beta_1 = x\mu_1(\beta - \beta_1' - \beta_1'') = x\mu_1(\beta - \beta_1')$ because $M_{F_1}\beta_1'' = 0$. Hence $x\mu_1\beta = x(1 + \mu_1\beta_1')$. Choose F_2 to be a finite subset of I which properly contains F_1 and with $x\mu_1\beta \in M_{F_2}$. Then $x\mu_1\beta_1' = x\mu_1\beta - x \in M_{F_2} \cap M_{F_1'} = M_{F_2\setminus F_1}$ so, taking $\delta_1 = \mu_1\beta_1' e_{F_2\setminus F_1} \in S_{F_1}\beta_{F_1, F_2\setminus F_1}$, we have $x\mu_1\beta = x(1 + \mu_1\beta_1') = x(1 + \mu_1\beta_1' e_{F_2\setminus F_1}) = x(1 + \delta_1)$, which establishes the case $n = 1$.

Now assume that $n \geq 2$ and that the $(n-1)^{\text{st}}$ case has been established, so that $x\mu_{n-1}\beta = x(1 + \delta_1 \dots \delta_{n-1})$ with each $\delta_k \in S_{F_k}\beta_{F_k, F_{k+1}\setminus F_k}$. As above, write $\beta = \beta_n + \beta_n' + \beta_n''$ where $\beta_n = \beta_{F_n} = e_{F_n}\beta e_{F_n} \in S_{F_n}$, $\beta_n' = e_{F_n}\beta e_{F_n'}$, and $\beta_n'' = e_{F_n'}\beta$. Then $x\delta_1 \dots \delta_{n-1} \in M_{F_n}$, so by hypothesis, there exists $\nu_n \in S_{F_n}$ with

$$\begin{aligned} x\delta_1 \dots \delta_{n-1} &= x\delta_1 \dots \delta_{n-1}\nu_n\beta_n \\ &= x\delta_1 \dots \delta_{n-1}\nu_n(\beta - \beta_n' - \beta_n'') \\ &= x\delta_1 \dots \delta_{n-1}\nu_n(\beta - \beta_n') \end{aligned}$$

because $M_{F_n}\beta_n'' = 0$. From the induction hypothesis,

$$x\mu_{n-1}\beta = x + x\delta_1 \dots \delta_{n-1} = x + x\delta_1 \dots \delta_{n-1}\nu_n(\beta - \beta_n'),$$

so that

$$x(\mu_{n-1} - \delta_1 \dots \delta_{n-1}\nu_n)\beta = x(1 - \delta_1 \dots \delta_{n-1}\nu_n\beta_n'). \tag{1}$$

Set $\mu_n = \mu_{n-1} - \delta_1 \dots \delta_{n-1}\nu_n$ and choose F_{n+1} to be a finite subset of I

which properly contains F_n and $x\mu_n\beta$. Then $x\delta_1\ldots\delta_{n-1}\nu_n\beta'_n = x - x\mu_n\beta \in M_{F_{n+1}} \cap M_{F'_n} = M_{F_{n+1}\setminus F_n}$ so, taking

$$\begin{aligned} \delta_n &= -\nu_n\beta'_n e_{F_{n+1}\setminus F_n} = -\nu_n e_{F_n}\beta e_{F'_n} e_{F_{n+1}\setminus F_n} &(2)\\ &= -\nu_n\beta_{F_n,F_{n+1}\setminus F_n} \in S_{F_n}\beta_{F_n,F_{n+1}\setminus F_n}, &(3)\end{aligned}$$

we have

$$x\mu_n\beta = x - x\delta_1\ldots\delta_{n-1}\nu_n\beta'_n = x - x\delta_1\ldots\delta_{n-1}\nu_n\beta'_n e_{F_{n+1}\setminus F_n} = x(1+\delta_1\ldots\delta_{n-1}\delta_n).$$

This establishes the lemma.

Proof that condition (2) implies that $\alpha \in J(S)$. Since condition (2) is also satisfied by $\nu\alpha$ for every $\nu \in S$, it suffices to prove that $\beta = 1+\alpha$ is a unit in S.

Let F be any finite subset of I, and for each $j \in F$, set $C_j = \{\sum_{i\in F}\mu_{ij} \mid \mu \in S$ and $e_jS\mu_{ij} \subseteq J(S_j)$ for every $i \in F\}$. Then each C_j is a left ideal of S_F, in fact a quasi-regular left ideal of S_F because, as is easily checked, $\mu_{jj} \in J(S_j)$ and $(e_F - \sum_{i\in F}\mu_{ij})^{-1} = (e_j - \mu_{jj})^{-1} + \sum_{i\in F\setminus\{j\}}(e_i + \mu_{ij}(e_j-\mu_{jj})^{-1})$, with $(e_j-\mu_{jj})^{-1}$ the inverse of $e_j - \mu_{jj}$ in S_j. Hence $\mu_F = \sum_{i,j\in F}\mu_{ij} \in \sum_{j\in F} C_j \subseteq J(S_F)$ for every finite subset F of I.

In particular, since, by hypothesis, $e_jS\alpha_{ij} \subseteq (S\alpha)_{jj} \subseteq J(S_j)$ for every $j \in F$, $\alpha_F \in J(S_F)$ and $\beta_F = (1+\alpha)_F = e_F + \alpha_F$ is a unit in S_F for every finite subset F of I. We may therefore apply the Lemma to learn that for each $x \in M$ there exists a sequence of finite subsets $\emptyset = F_0 \subset F_1 \subset F_2 \subset \ldots$ of I and homomorphisms $\mu_1, \mu_2, \ldots \in S$, $\delta_1, \delta_2, \ldots \in S$ such that:

(i) for each $n \geq 1$, $x\mu_n\beta = x(1+\delta_1\delta_2\ldots\delta_n)$; and
(ii) each $\delta_k \in S_{F_k}\beta_{F_k,F_{k+1}\setminus F_k}$.

We first establish that $x\delta_1\delta_2\ldots\delta_n = 0$ for some $n \geq 1$. By an application of the König Graph Theorem, it suffices to show that for every choice of $i_k \in F_k\setminus F_{k-1}$ with $k \geq 1$, there exists $m \geq 1$ (depending on the choice of the sequence $\{i_k\}$) with $x(e_{i_1}\delta_1 e_{i_2})(e_{i_2}\delta_2 e_{i_3})\ldots(e_{i_m}\delta_m e_{i_{m+1}}) = 0$. Using (ii), for each $k \geq 1$ we may write each $\delta_k = \gamma_k\beta_{F_k,F_{k+1}\setminus F_k}$ with $\gamma_k \in S_{F_k}$. Then for each $k \geq 1$,

$$\begin{aligned} e_{i_k}\delta_k e_{i_{k+1}} &= e_{i_k}\gamma_k\beta_{F_k,F_{k+1}\setminus F_k}e_{i_{k+1}}\\ &= e_{i_k}\gamma_k e_{F_k}(1+\alpha)e_{F_{k+1}\setminus F_k}e_{i_{k+1}}\\ &= e_{i_k}\gamma_k\alpha e_{i_{k+1}} = (\gamma_k\alpha)_{i_k i_{k+1}}.\end{aligned}$$

Hence, from condition (2), it follows that there exists $m \geq 1$ with

$$x(e_{i_1}\delta_1 e_{i_2})(e_{i_2}\delta_2 e_{i_3})\dots(e_{i_m}\delta_m e_{i_{m+1}}) = x(\gamma_1\alpha)_{i_1 i_2}(\gamma_2\alpha)_{i_2 i_3}\dots(\gamma_m\alpha)_{i_m i_{m+1}} = 0,$$

and thus $x\delta_1\delta_2\dots\delta_n = 0$ for some $n \geq 1$.

From (i) we know that $x\mu_n\beta = x$ and, since $x \in M$ was arbitrary, this proves that β is an epimorphism. Since β_F is a unit for each finite subset F of I, β is also a monomorphism, and this completes the proof that condition (2) is sufficient for $\alpha \in J(S)$ to hold.

The fact that (1) *implies* (2) follows from the observation that for every sequence $i_1, i_2, \dots$ of distinct elements of I and every sequence $\gamma_1, \gamma_2, \dots \in S$,

$$\begin{aligned}(\gamma_k\alpha)_{i_k i_{k+1}} &= e_{i_k}\gamma_k\alpha e_{i_{k+1}}\\ &= e_{i_k}\left(\sum_{j\geq 1} e_{i_j}\gamma_j\right)\alpha e_{i_{k+1}}\\ &= \left(\left(\sum_{j\geq 1} e_{i_j}\gamma_j\right)\alpha\right)_{i_k i_{k+1}} = (\gamma\alpha)_{i_k i_{k+1}}\end{aligned}$$

where $\gamma = \sum_{j\geq 1} e_{i_j}\gamma_j \in S$.

In order to prove the *necessity of (1)*, let $\alpha \in J(S)$ be given. Since $\gamma\alpha \in J(S)$ for every $\gamma \in S$, it suffices to show that $\alpha_{ii} \in J(S_i)$ for every $i \in I$ (which is clear since $\alpha_{ii} = e_i\alpha e_i \in e_i J(S) e_i = J(S_i)$), and that for every sequence $i_1, i_2, \dots$ of distinct elements of I and every $x \in M$, there exists a positive integer $n \geq 1$ with $x\alpha_{i_1 i_2}\alpha_{i_2 i_3}\dots\alpha_{i_n i_{n+1}} = 0$.

Without loss of generality, we may assume that $\{1, 2, \dots\} \subseteq I$ and we replace each i_j by j. To further simplify this presentation, we introduce some additional notation. Let $1 = k_1 < k_2 < \dots$ be an arbitrary increasing sequence of positive integers; later, we will specify a particular sequence for the purpose of this proof. Set $x_1 = xe_1$, and for each $i \geq 1$, set $x_{i+1} = x_i\alpha_{k_i,k_i+1}\alpha_{k_i+1,k_i+2}\dots\alpha_{k_{i+1}-1,k_{i+1}} = x_i\alpha_{k_i,k_i+1}\beta_{k_i+1,k_{i+1}}$, where $\beta_{k_i+1,k_{i+1}} = \alpha_{k_i+1,k_i+2}\dots\alpha_{k_{i+1}-1,k_{i+1}} \in Hom_R(M_{k_i+1}, M_{k_{i+1}})$. Since $x_{n+1} = x\alpha_{12}\alpha_{23}\dots\alpha_{k_{n+1}-1,k_{n+1}}$ we have to show that $x_{n+1} = 0$ for some $n \geq 1$.

Set $\beta = \sum_{i\geq 1}\beta_{k_i+1,k_{i+1}}$; β is a well-defined element of S, and the notations are compatible in the sense that for each $i \geq 1$, $e_{k_i+1}\beta e_{k_{i+1}} = \beta_{k_i+1,k_{i+1}}$. Also, observe that $e_h\beta e_\ell = 0$ whenever $(h,\ell) \neq (k_i+1, k_{i+1})$ for some $i \geq 1$. Further, note that for each $i \geq 1$, $e_{k_i}\alpha\beta = \sum_{j\geq 1}\alpha_{k_i,k_j+1}\beta_{k_j+1,k_{j+1}}$, which we rewrite as $e_{k_i} + \sum_{j=1}^{i-1}\alpha_{k_i,k_j+1}\beta_{k_j+1,k_{j+1}} = (e_{k_i} - \sum_{j\geq i}\alpha_{k_i,k_j+1}\beta_{k_j+1,k_{j+1}}) + e_{k_i}\alpha\beta$.

We introduce the abbreviations $u_i = e_{k_i} - \sum_{j\geq i}\alpha_{k_i,k_j+1}\beta_{k_j+1,k_{j+1}}$ and $v_i =$

$e_{k_i} + \sum_{j=1}^{i-1} \alpha_{k_i,k_j+1}\beta_{k_j+1,k_{j+1}}$ for $i \geq 1$ (by convention, $v_1 = e_{k_1}$), and we let $U = \{s \in S | s = \sum_{i\geq 1} s_i u_i \textit{ for some sequence } s_i \in S\}$. Then each $u_i, v_i \in S$, $v_i = u_i + e_{k_i}\alpha\beta$ and U is a left ideal of S. Furthermore, $U \subseteq Hom_R(M, M_0)$ where $M_0 = \oplus_{i\geq 1} M_{k_i}$. Finally, put $v = \sum_{i\geq 1} v_i$; v is a well-defined element of S and $v \in Hom_R(M, M_0)$.

We begin the proof proper by first showing that $v|_{M_0}$ is an automorphism of M_0. To see this, note that $v|_{M_0}$ is the ascending union of $\{v_{(m)} | m \geq 1\}$ where $v_{(m)} = \sum_{i=1}^m v_i = \sum_{i=1}^m (e_{k_i} + \sum_{j=1}^{i-1} \alpha_{k_i,k_j+1}\beta_{k_j+1,k_{j+1}}) = \sum_{i=1}^m e_{k_i} + \sum_{i=1}^m \sum_{j=1}^{i-1} e_{k_i}\alpha e_{k_j+1}\beta e_{k_{j+1}} \in End_R(M_{k_1} \oplus \ldots \oplus M_{k_m})$. Furthermore, $\sum_{i=1}^m \sum_{j=1}^{i-1} e_{k_i}\alpha e_{k_j+1}\beta e_{k_{j+1}} \in J(End_R(M_{k_1} \oplus \ldots \oplus M_{k_m}))$ because $\alpha \in J(S)$. Since $\sum_{i=1}^m e_{k_i}$ is the identity automorphism of $M_{k_1} \oplus \ldots \oplus M_{k_m}$, it follows that $v_{(m)}$ is an automorphism of $M_{k_1} \oplus \ldots \oplus M_{k_m}$ for each $m \geq 1$. Hence $v|_{M_0}$ is an automorphism of M_0.

Next, $v = \sum_{i\geq 1} v_i = \sum_{i\geq 1} u_i + \sum_{i\geq 1} e_{k_i}\alpha\beta = \left(\sum_{i\geq 1} u_i\right) + e_0\alpha\beta \in U + S\alpha\beta$, where $e_0 = \sum_{i\geq 1} e_{k_i}$ is the identity element of M_0. Then $e_0 = (v|_{M_0})^{-1}(v|_{M_0}) = (v|_{M_0})^{-1}v \in U + S\alpha\beta \subseteq Hom_R(M, M_0) = Se_0$, and therefore $U + S\alpha\beta = Se_0$. Since $\alpha \in J(S)$, $S\alpha\beta$ is a small submodule of ${}_SS$, hence of Se_0, and therefore $U = Se_0$. In particular, $e_1 = e_{k_1} = e_{k_1}e_0 \in Se_0 = U$, so we may write $e_1 = \sum_{i\geq 1} s_i u_i \in U$ for some choice of $s_i \in S$. Since $x_1 = xe_1 = \sum_{i\geq 1} xs_i u_i$, there must exist an integer $n \geq 1$ with $xs_k u_k = 0$ for all $k > n$. Hence

$$
\begin{aligned}
x_1 &= \sum_{i=1}^n xs_i u_i = \sum_{i=1}^n xs_i(e_{k_i} - \sum_{j\geq i} \alpha_{k_i,k_j+1}\beta_{k_j+1,k_{j+1}}) \\
&= xs_1 e_{k_1} + \sum_{i=2}^n (xs_i e_{k_i} - \sum_{j=1}^{i-1} xs_j \alpha_{k_j,k_{i-1}+1}\beta_{k_{i-1}+1,k_i}) - \\
&\quad \sum_{i>n} \left(\sum_{j=1}^{i-1} xs_j \alpha_{k_j,k_{i-1}+1}\beta_{k_{i-1}+1,k_i} \right).
\end{aligned}
$$

Examining the first $n+1$ components of this equation in $M_0 = \oplus_{i\geq 1} M_{k_i}$ yields the following system of equations:

$$
\begin{aligned}
&x_1 = xs_1 e_{k_1} \\
&xs_i e_{k_i} = \textstyle\sum_{j=1}^{i-1} xs_j \alpha_{k_j,k_{i-1}+1}\beta_{k_{i-1}+1,k_i} \text{ for } i = 2, 3, \ldots, n \\
&\textstyle\sum_{j=1}^{n} xs_j \alpha_{k_j,k_n+1}\beta_{k_n+1,k_{n+1}} = 0.
\end{aligned}
$$

We now fix a particular choice for the sequence $1 = k_1 < k_2 < \ldots$ recursively as follows: $k_1 = 1$ and k_i having been chosen, $k_{i+1} > k_i$ is chosen so that $x_i\alpha_{k_i,h} =$

0 for every $h > k_{i+1}$. This choice is possible because $x_i\alpha_{k_i,h} = x_ie_{k_i}\alpha e_h = 0$ for almost all h.

With this choice for the sequence, we can solve the preceding system of equations. Substituting the first equation in the second gives

$$xs_2e_{k_2} = xs_1\alpha_{k_1,k_1+1}\beta_{k_1+1,k_2} = xs_1e_{k_1}\alpha_{k_1,k_1+1}\beta_{k_1+1,k_2} = x_1\alpha_{k_1,k_1+1}\beta_{k_1+1,k_2} = x_2.$$

Substituting this into the third equation gives

$$\begin{aligned} xs_3e_{k_3} &= xs_1\alpha_{k_1,k_2+1}\beta_{k_2+1,k_3} + xs_2\alpha_{k_2,k_2+1}\beta_{k_2+1,k_3} \\ &= xs_1e_{k_1}\alpha_{k_1,k_2+1}\beta_{k_2+1,k_3} + xs_2e_{k_2}\alpha_{k_2,k_2+1}\beta_{k_2+1,k_3} \\ &= x_1\alpha_{k_1,k_2+1}\beta_{k_2+1,k_3} + x_2\alpha_{k_2,k_2+1}\beta_{k_2+1,k_3} \\ &= x_2\alpha_{k_2,k_2+1}\beta_{k_2+1,k_3} = x_3 \end{aligned}$$

because $x_1\alpha_{k_1,h} = 0$ for all $h > k_2$. Continuing in this manner through the n$^{\text{th}}$ equation, we have that $xs_ie_{k_i} = x_i$ for $i = 2, 3, \ldots, n$. Inserting this into the final equation yields $\sum_{j=1}^{n} x_j\alpha_{k_j,k_n+1}\beta_{k_n+1,k_{n+1}} = 0$. Again, since $x_j\alpha_{k_j,h} = 0$ for every $h > k_{j+1}$, we conclude that $x_n\alpha_{k_n,k_n+1}\beta_{k_n+1,k_{n+1}} = 0$. That is, $x_{n+1} = 0$, and with this the proof of the theorem is concluded.

3 Endomorphisms locally of finite rank.

In this section, with all notation as in the previous section, we consider an important subring of $S = End_RM$ relative to the decomposition $M = \oplus_{i\in I}M_i$, namely $S^0 = \{\alpha \in S|\ \textit{for each}\ i \in I,\ M_i\alpha e_j = 0\ \textit{for almost all}\ j \in I\}$. Very loosely speaking, S^0 consists of the endomorphisms of M which are "locally of finite rank relative to the decomposition $M = \oplus_{i\in I}M_i$." Other descriptions are $S^0 = \{\alpha \in S|\ \textit{for each}\ i \in I,\ e_i\alpha = e_i\alpha e_G\ \textit{for some finite subset}\ G \subseteq I\}$, and $S^0 = \{\alpha \in S|\ \textit{for each finite subset}\ F \subseteq I,\ e_F\alpha = e_F\alpha e_G\ \textit{for some finite subset}\ G \subseteq I\}$. Observe that $e_G \in S^0$ for every subset $G \subseteq I$ and that $S_F \subseteq S^0$ for every finite subset $F \subseteq I$.

A relatively straightforward adaptation of the proofs of Lemma 1 and Theorem 1 provides a characterization of the elements in the Jacobson radical of S^0. The description is a simplification of that given in Theorem 1 and proofs will therefore be omitted.

We first recall some information about the Jacobson radical in *R-Mod*. For $K, L \in$ *R-Mod*, $J(Hom_R(K, L)) = \{f \in Hom_R(K, L)|\ \textit{for all}\ g \in Hom_R(L, K),\ 1_L + gf\ \textit{is an automorphism of}\ L\} = \{f \in Hom_R(K, L)|\ \textit{for all}\ h \in Hom_R(L, K),\ 1_K + fh\ \textit{is an automorphism of}\ K\}$. Also, $J(Hom_R(K, L))$ is an ideal in the category R-Mod; *that is*, it is closed under addition and for any $X, Y \in R$-Mod, $Hom_R(X, K)J(Hom_R(K, L))Hom_R(L, Y) \subseteq J(Hom_R(X, Y))$.

Lemma 2. *Suppose that $M = \oplus_{i \in I} M_i$ with I an infinite set and that $\beta \in S^0 \subseteq End_R M$ is such that $\beta_F = e_F \beta e_F$ has a left inverse in $S_F = End_R M_F$ for every finite subset $F \subset I$. Then for any finite subset $F_1 \subset I$ there exists a sequence of finite subsets $F_1 \subset F_2 \subset F_3 \subset \ldots$ of I and endomorphisms $\mu_1, \mu_2, \ldots \in S^0$, $\delta_1, \delta_2, \ldots \in S^0$ such that:*

(i) for each $n \geq 1$, $\mu_n \beta = e_{F_1}(1 + \delta_1 \delta_2 \ldots \delta_n)$; and
(ii) each $\delta_k \in S_{F_k} \beta_{F_k, F_{k+1} \setminus F_k}$.

Theorem 2. *For $M = \oplus_{i \in I} M_i$ with I an infinite index set and $\alpha \in S^0 \subseteq End_R M$, either one of the following conditions are necessary and sufficient for $\alpha \in J(S^0)$ to hold.*

(1) For every $\gamma \in S^0$ and every $i \in I$, $(\gamma\alpha)_{ii} \in J(S_i)$ and, for every sequence $i_1, i_2, \ldots$ of distinct elements of I, there exists a positive integer n with $(\gamma\alpha)_{i_1 i_2}(\gamma\alpha)_{i_2 i_3} \ldots (\gamma\alpha)_{i_n i_{n+1}} = 0$.
(2) For every $i, j \in I$, $\alpha_{ij} \in J(Hom_R(M_i, M_j))$ and, for every sequence $i_1, i_2, \ldots$ of distinct elements of I, and every sequence $\gamma_1, \gamma_2, \ldots \in S^0$, there exists a positive integer n with $(\gamma_1\alpha)_{i_1 i_2}(\gamma_2\alpha)_{i_2 i_3} \ldots (\gamma_n\alpha)_{i_n i_{n+1}} = 0$.

Since $S = S^0$ whenever each module M_i in the decomposition $M = \oplus_{i \in I} M_i$ is finitely generated, we have the following immediate consequence of Theorem 2.

Corollary 1. *Suppose that $M = \oplus_{i \in I} M_i$ with each M_i a finitely generated R-module and let $\alpha \in S = End_R M$. Then $\alpha \in J(S)$ if and only if the following two conditions hold:*

(i) for every $i, j \in I$, $\alpha_{ij} \in J(Hom_R(M_i, M_j))$; and
(ii) for every sequence $\gamma_1, \gamma_2, \ldots \in S$, and for every sequence $i_1, i_2, \ldots$ of distinct elements of I, there exists a positive integer n with $(\gamma_1\alpha)_{i_1 i_2}(\gamma_2\alpha)_{i_2 i_3} \ldots (\gamma_n\alpha)_{i_n i_{n+1}} = 0$.

Corollary 2. (N. Sexauer & J. Warnock [7]) *For an arbitrary ring R and a matrix $\alpha \in M_I(R)$, $\alpha \in J(M_I(R))$ if and only if $\alpha \in M_I(J(R))$ and the column left ideals of α are right vanishing.*

Proof. For a ring R with identity element, this follows immediately from the preceding corollary. For an arbitrary ring R, we can regard R as an ideal of an overring R^1 which contains an identity element. The result then follows immediately from the fact that $M_I(R)$ is an ideal of $M_I(R^1)$ and $J(M_I(R)) = J(M_I(R^1)) \cap M_I(R)$.

In certain circumstances the Jacobson radical of S^0 has a particularly simple structure. For example, consider E.M. Patterson's result, cited in the introduction to this paper, that $J(M_I(R)) = M_I(J(R))$ for I an infinite set if and only if $J(R)$ is left T-nilpotent. This is extended to arbitrary module decompositions in Corollary 3 below.

First, a definition: a family of modules $\{M_i | i \in I\}$ is called *semi-T-nilpotent* if for each sequence $M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} M_{i_3} \xrightarrow{f_3} \cdots$ with each $f_k \in J(Hom_R(M_{i_k}, M_{i_{k+1}}))$ and with pairwise distinct indices $i_k \in I$, there exists an integer $n \geq 1$ such that $f_1 f_2 \dots f_n = 0$. (This definition is compatible with the earlier concept of *locally* semi-T-nilpotent families for modules with local endomorphism rings because the requirement that $f_k \in J(Hom_R(M_{i_k}, M_{i_{k+1}}))$ is equivalent to f_k being a non-isomorphism in that case.)

Corollary 3. *For $M = \oplus_{i \in I} M_i$ with I an infinite index set, $J(S^0) = \{\alpha \in S^0 | \alpha_{ij} \in J(Hom_R(M_i, M_j))$ for all $i, j \in I\}$ if and only if $\{M_i | i \in I\}$ is semi-T-nilpotent.*

Proof. Suppose that $\{M_i | i \in I\}$ is semi-T-nilpotent. It is always true that $J(S^0) \subseteq \{\alpha \in S^0 | \alpha_{ij} \in J(Hom_R(M_i, M_j))$ for all $i, j \in I\}$ so suppose that $\alpha \in S^0$ is such that $\alpha_{ij} \in J(Hom_R(M_i, M_j))$ for all $i, j \in I$. Then given distinct elements $i_1, i_2, \dots$ of I and a sequence $\gamma_1, \gamma_2, \dots \in S$, there exists a positive integer n with $(\gamma_1 \alpha)_{i_1 i_2} (\gamma_2 \alpha)_{i_2 i_3} \dots (\gamma_n \alpha)_{i_n i_{n+1}} = 0$ because $\{M_i | i \in I\}$ is semi-T-nilpotent. Hence, by Theorem 2*(1)*, we have that $\alpha \in J(S^0)$.

Conversely, suppose that $J(S^0) = \{\alpha \in S^0 | \alpha_{ij} \in J(Hom_R(M_i, M_j))$ for all $i, j \in I\}$ and let the sequence $M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} M_{i_3} \xrightarrow{f_3} \cdots$ be given with each $f_k \in J(Hom_R(M_{i_k}, M_{i_{k+1}}))$ and with pairwise distinct indices $i_k \in I$. Set $\alpha = \sum_{k \geq 1} e_{i_k} f_k e_{i_{k+1}}$. Then $\alpha \in S^0$ and

$$\alpha_{ij} = \begin{cases} 0, & \text{if } (i,j) \neq (i_k, i_{k+1}) \text{ for any } k \geq 1 \\ f_k, & \text{if } (i,j) = (i_k, i_{k+1}) \text{ for some } k \geq 1, \end{cases}$$

so $\alpha_{ij} \in J(Hom_R(M_i, M_j))$ for each $i, j \in I$, and therefore $\alpha \in J(S^0)$. Hence from Theorem 2, there exists a positive integer n with $\alpha_{i_1 i_2} \alpha_{i_2 i_3} \dots \alpha_{i_n i_{n+1}} = 0$; that is, $f_1 f_2 \dots f_n = 0$, proving that $\{M_i | i \in I\}$ is semi-T-nilpotent.

Question. *Is an analogous result true for $S = End_R M$, when $M = \oplus_{i \in I} M_i$? That is, is it true that $J(S) = \{\alpha \in S | \alpha_{ij} \in J(Hom_R(M_i, M_j))$ for all $i, j \in I\}$ if and only if $\{M_i | i \in I\}$ is "locally semi-T-nilpotent" (with respect to sequences of radical homomorphisms)?* An affirmative answer would shed additional light on the structure of completely decomposable exchange modules.

As another application, in [7] it was shown that when $J(R)$ is a prime ring then $J(M_I(R)) = \{$*column-bounded matrices in* $M_I(J(R))\} = \{(a_{ij})_{i,j \in I} | a_{ij} \in$

$J(R)$, *and* $a_{ij} = 0$ *for all* $j \notin K$, K *a finite subset of* $I\}$. We conclude by exhibiting a module-theoretic generalization of this result.

First, with $M = \oplus_{i\in I} M_i$ and notation as above, set $C_J(\oplus_{i\in I} M_i) = \{f \in S^0 | f_{ij} \in J(Hom_R(M_i, M_j))$ *for all* $i, j \in I$, *and* $fe_j = 0$ *for almost all* $j \in I\}$. When each $M_i = R$, a ring with identity element, then $C_J(\oplus_{i\in I} M_i)$ can be identified with the ring of column-bounded matrices in $M_I(R)$. From Theorem 2, we know that $C_J(\oplus_{i\in I} M_i) \subseteq J(S^0)$. The following definition provides a sufficient condition for equality to hold: call the decomposition $M = \oplus_{i\in I} M_i$ *radical-prime* if given $0 \neq f \in J(Hom_R(M_i, M_j))$ and $0 \neq g \in J(Hom_R(M_k, M_\ell))$ with $j \neq \ell$ there exists $h \in Hom_R(M_j, M_k)$ with $fhg \neq 0$.

Corollary 4. *If the decomposition* $M = \oplus_{i\in I} M_i$ *is radical-prime then* $J(S^0) = C_J(\oplus_{i\in I} M_i)$. *In particular, if each* $M_i \cong N$ *for some module* N *with* $J(End_R N)$ *a prime ring, then* $J(S^0) \cong \{$column-bounded matrices in $M_I(J(End_R N))\}$.

Proof. It suffices to show that if $f \in S^0 \backslash C_J(\oplus_{i\in I} M_i)$ then $f \notin J(S^0)$. We can assume that each $f_{ij} \in J(Hom_R(M_i, M_j))$; otherwise, from Theorem 2, $f \notin J(S^0)$. Since $f \notin C_J(\oplus_{i\in I} M_i)$ we can also assume without loss of generality that $\{1, 2, \ldots\} \subseteq I$ and that $fe_i \neq 0$ for all $i \geq 1$.

For each $i \geq 1$, choose $k_i \in I$ with $e_{k_i} f e_i \neq 0$, and note that $e_{k_i} f e_i \in J(Hom_R(M_{k_i}, M_i))$ for each $i \in I$. Since the decomposition is radical-prime there exists $\gamma_1 \in Hom_R(M_1, M_{k_2})$ with $f_{k_1 1}\gamma_1 f_{k_2 2} = e_{k_1} f e_1 \gamma_1 e_{k_2} f e_2 \neq 0$. Set $f_1 = e_1 \gamma_1 e_{k_2} f e_2 = (\gamma_1 f)_{12} \neq 0$ and note that $f_1 \in J(Hom_R(M_1, M_2))$. Next use the radical-prime property to choose
$\gamma_2 \in Hom_R(M_2, M_{k_3})$ with $f_1 \gamma_2 f_{k_3 3} = f_1 e_2 \gamma_2 e_{k3} f e_3 \neq 0$ and set
$f_2 = e_2 \gamma_2 e_{k_3} f e_3 = (\gamma_2 f)_{23} \neq 0$. Then $(\gamma_1 f)_{12} (\gamma_2 f)_{23} = f_1 f_2 \neq 0$. Continuing in this manner for each $n \geq 1$ we can find elements $\gamma_n \in S^0$ with $(\gamma_1 f)_{12} (\gamma_2 f)_{23} \ldots (\gamma_n f)_{n,n+1} \neq 0$. Hence, from Theorem 2, $f \notin J(S^0)$.

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