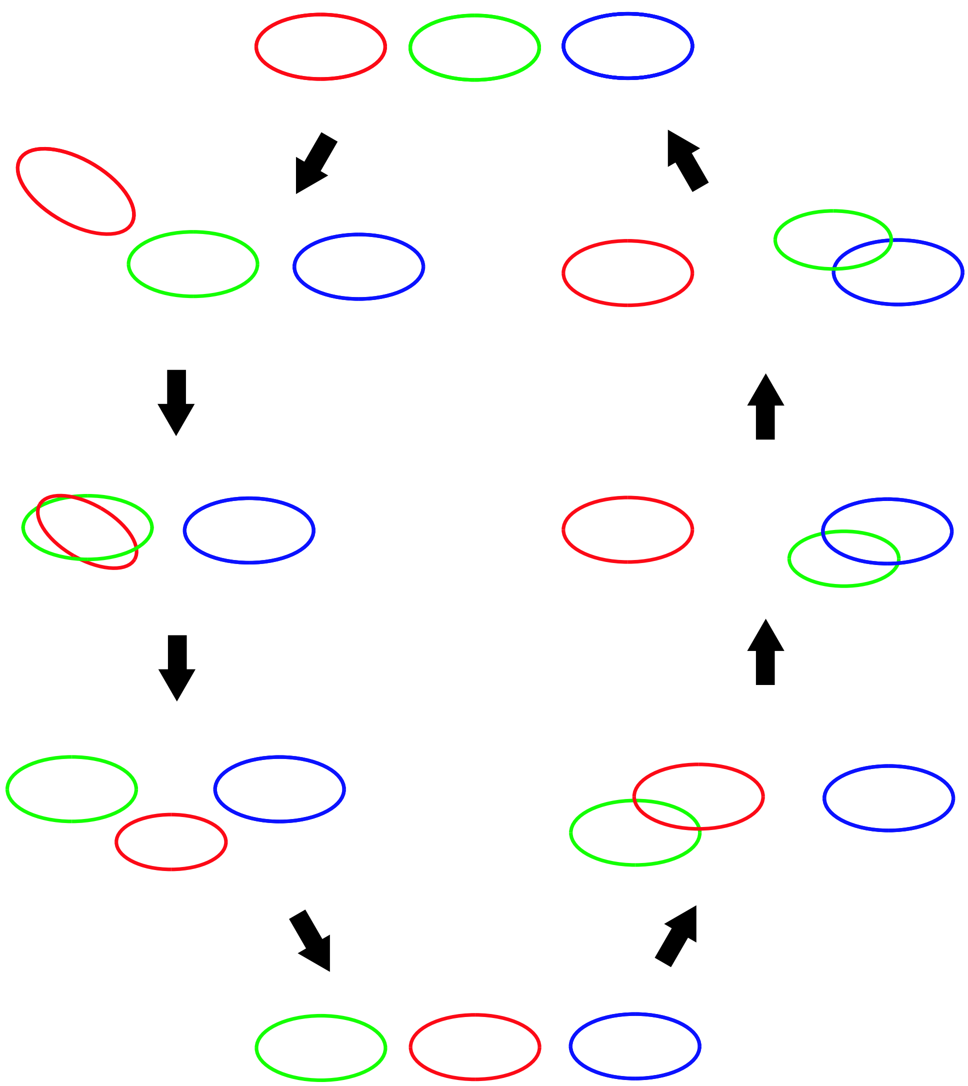
Hypertrees and the ℓ^2 Betti numbers of the pure symmetric automorphism group



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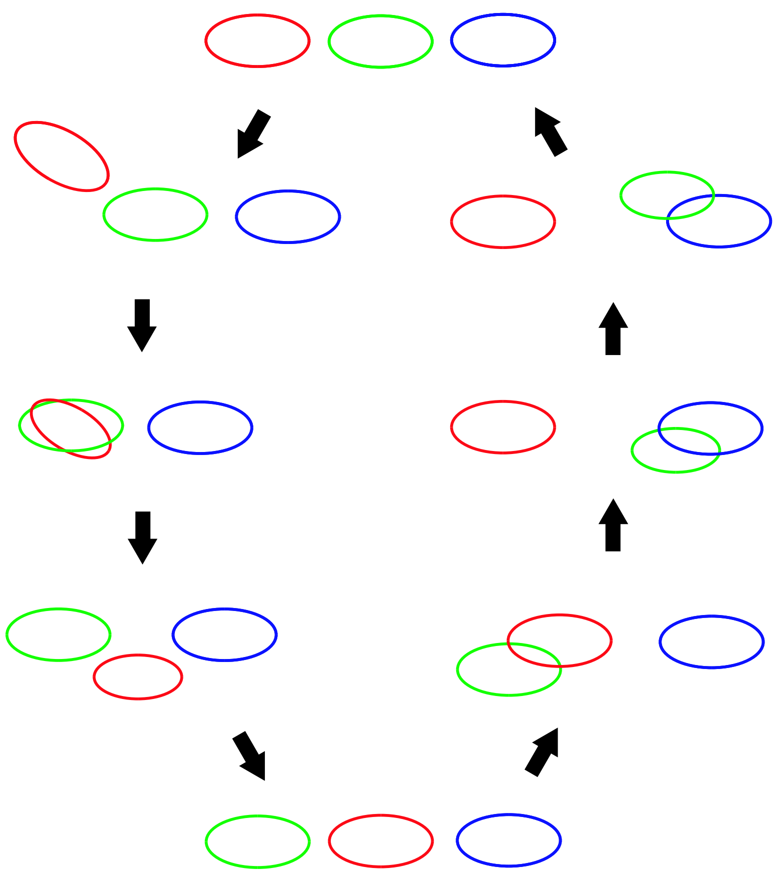
Σ_n and $P\Sigma_n$

$L_n =$ trivial n-link in S^3.

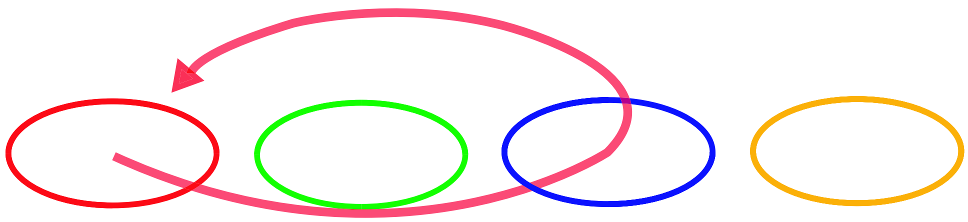
$\Sigma_n =$ the group of motions of L_n in S^3.
(Introduced by Fox $\Rightarrow$ Dahm $\Rightarrow$ Goldsmith $\cdots$)

$P\Sigma_n =$ the index $n!$ subgroup of motions where the n components of L_n return to their original positions. (This is the *pure* motion group.)



Representing $P\Sigma_n$

Thm(Goldsmith, Mich. Math. J. '81)
There is a faithful representation of $P\Sigma_n$ into Aut $(F(x_1, \ldots, x_n))$ induced by sending the generators of $P\Sigma_n$



to automorphisms

$$\alpha_{ij}(x_k) = \begin{cases} x_k & k \neq i \\ x_j^{-1} x_i x_j & k = i \end{cases} .$$

The image in Aut(F_n) is referred to as the group of *pure symmetric automorphisms* since it is the subgroup of automorphisms where each generator is sent to a conjugate of itself.

Thinking of $P\Sigma_n$ as a subgroup of Aut(F_n) we can form the image of $P\Sigma_n$ in Out(F_n), denoted $OP\Sigma_n$.

Some of What's Known

- $P\Sigma_n$ contains PB_n.
- $P\Sigma_n$ has cohomological dimension $n-1$.
  (Collins, *CMH* '89)
- $P\Sigma_n$ has a regular language of normal forms.
  (Guttiérrez and Krstić, *IJAC* '98)

Our Results

Theorem A. *$P\Sigma_{n+1}$ is an n-dim'l duality group.* (Brady-M-Meier-Miller, *J. Algebra*, '01)

Theorem B. *The ℓ^2-Betti numbers of $P\Sigma_{n+1}$ are all trivial except in top dimension, where*

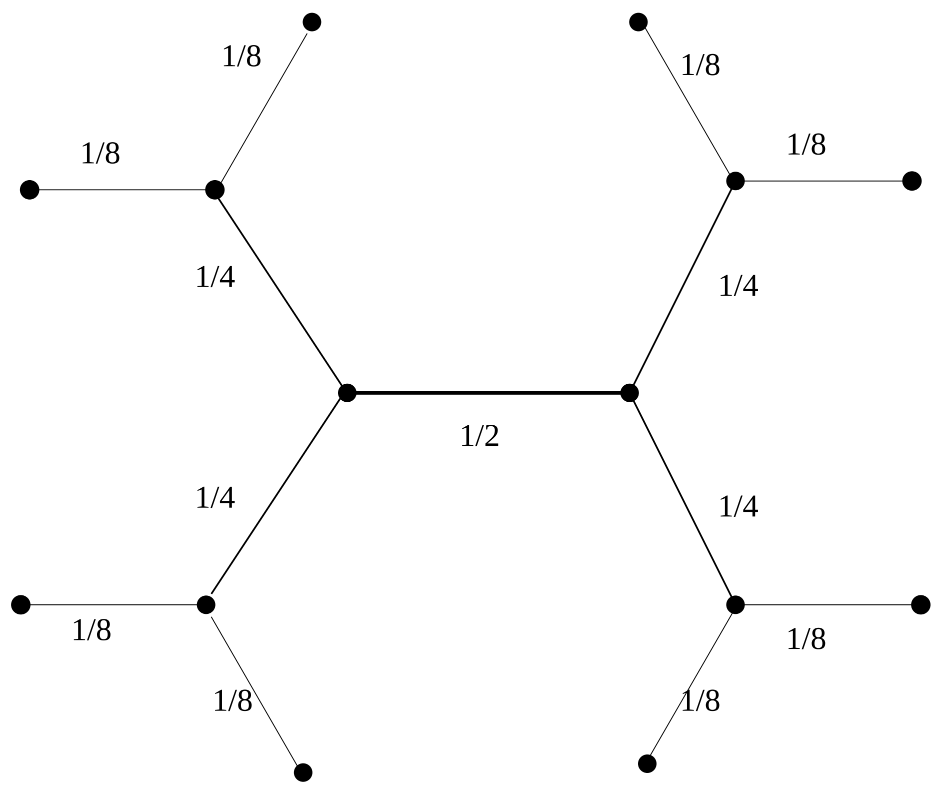
$$\chi(P\Sigma_{n+1}) = (-1)^n b_n^{(2)} = (-1)^n n^n \ .$$

(M-Meier, *Math. A.*, '04)

Both are cohomology computations that occur in the universal cover of a $K(P\Sigma_{n+1}, 1)$. While both have to do with asymptotic properties of $P\Sigma_{n+1}$, the proofs ultimately boil down to combinatorial arguments.

ℓ^2-Cohomology

For a group G (admitting a finite $K(G,1)$) let $\ell^2(G)$ be the Hilbert space of square-summable functions. The classic cocycle is:



In general, concrete computations are rare. One of the few is due to Davis and Leary who compute the ℓ^2-cohomology of arbitrary right-angled Artin groups (*Journal LMS*, '03).

McCullough-Miller Complex

The cohomology computations are done via an action of $OP\Sigma_n$ on a contractible simplicial complex MM_n, constructed by McCullough and Miller (*MAMS*, '96).

The complex MM_n is a space of F_n-actions on simplicial trees, where the actions all take the decomposition of F_n as a free product

$$F_n = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ copies}}$$

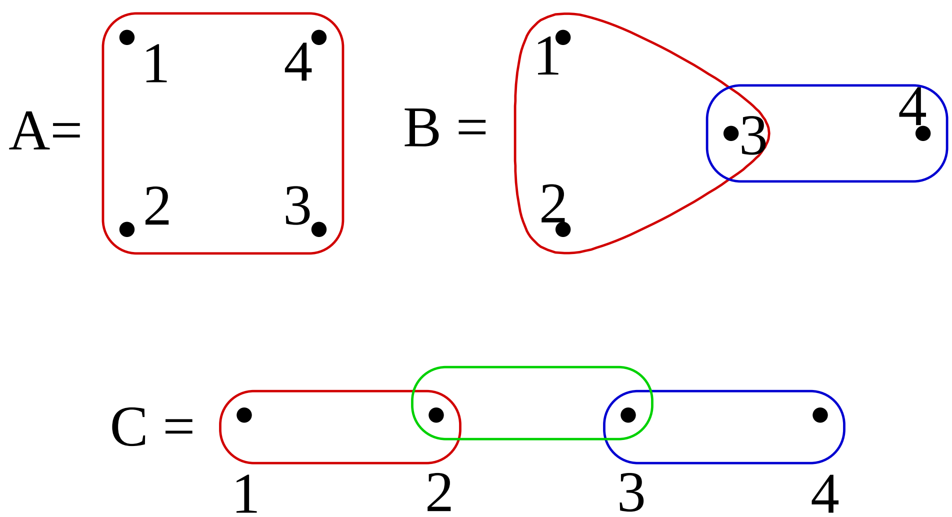
seriously.

Each action in this space can be described by a marked hypertree ...

Hypertrees

Def: A *hypertree* is a connected hypergraph with no hypercycles.

In hypergraphs, the "edges" are subsets of the vertices, not just pairs of vertices.



The growth is quite dramatic: The number of hypertrees on $[n]$, for $n \geq 3$ is =

$\{4, 29, 311, 4447, 79745, 1722681, 43578820, \ldots\}$

(Smith and Warme,Kalikow)

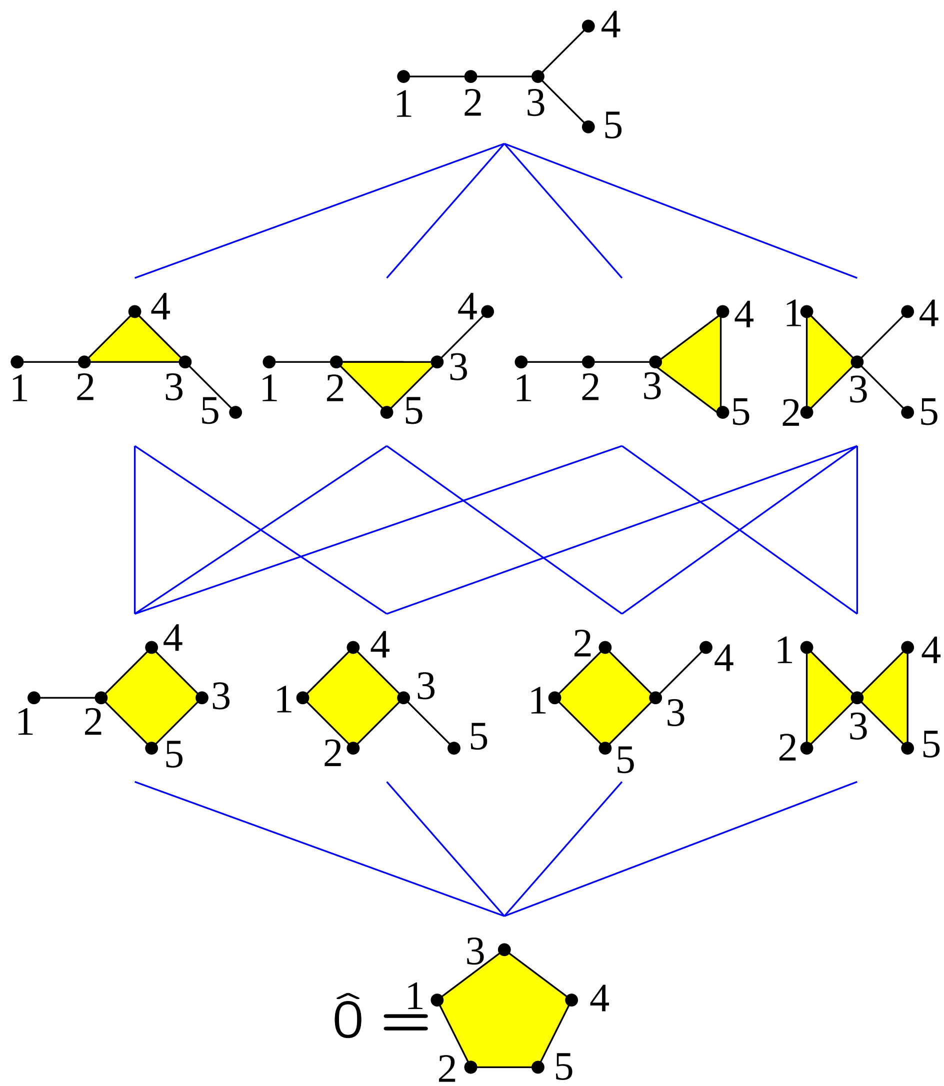
Hypertree Poset

The hypertrees on $[n]$ form a very nice poset, that is surprisingly unstudied in combinatorics.

The elements of HT_n are n-vertex hypertrees with the vertices labelled by $[n] = \{1, \dots, n\}$. The order relation is given by:
$\tau < \tau' \Leftrightarrow$ each hyperedge of τ' is contained in a hyperedge of τ.

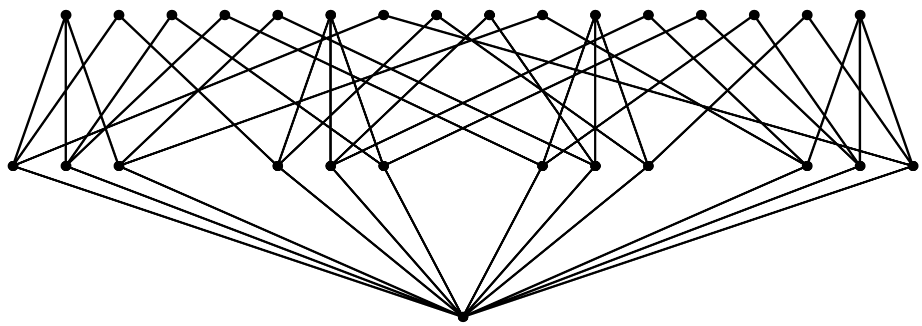
The hypertree with only one edge is $\hat{0}$, also called the *nuclear* element. If one adds a formal $\hat{1}$ such that $\tau < \hat{1}$ for all $\tau \in \mathsf{HT}_n$, the resulting poset is $\widehat{\mathsf{HT}}_n$.

An interval in HT_5



Properties of $\mathbf{HT}_n$

The Hasse diagram of HT_4 is



Thm: $\widehat{\mathrm{HT}}_n$ is a finite lattice that is graded, bounded, and Cohen-Macaulay.

- Finite and Bounded are easy.

- Lattice is easy based on the similarities between HT_n and the partition lattice. (Lattice is the key element in the McCullough-Miller proof that MM_n is contractible.)

Properties of MM_n

The McCullough-Miller space, MM_n, is the geometric realization of a poset of *marked* hypertrees. The marking is similar (and related) to the marked graph construction for outer space.

Some Useful Facts:

• MM_n admits $P\Sigma_n$ and $OP\Sigma_n$ actions.

• The fundamental domain for either action is the same, it's finite and isomorphic to the order complex of HT_n (also known as the *Whitehead* poset).

• The isotropy groups for the $OP\Sigma_n$ action are free abelian; the isotropy groups are free-by-(free abelian) for the action of $P\Sigma_n$.

ℓ^2-Betti Numbers

We compute the ℓ^2-Betti numbers of $OP\Sigma_{n+1}$ via its action on MM_{n+1}. In order to do this we have to switch to an algebraic standpoint, using group cohomology with coefficients in the group von Neumann algebra $\mathcal{N}(G)$.

We also are really computing the equivariant ℓ^2-Betti numbers of the action of $OP\Sigma_{n+1}$ on MM_{n+1}. We can get away with this because

Lemma. *The ℓ^2-cohomology of $\mathbb{Z}^n$ is trivial.*

Lemma. *Let X be a contractible G-complex. Suppose that each isotropy group G_σ is finite or satisfies $b_p^{(2)}(G_\sigma) = 0$ for $p \geq 0$. Then $b_p^{(2)}(X, \mathcal{N}(G)) = b_p^{(2)}(G)$ for $p \geq 0$.*

(cf. Lück's *L^2-Invariants: Theory and Applications ...*)

Reduction to Euler characteristics

In looking at the resulting equivariant spectral sequence we find we are really looking at the homology of

$$\mathsf{HT}^{\circ}_{n+1} = \mathsf{HT}_{n+1} - \{\text{the nuclear vertex}\}$$

(this is the singular set for the $OP\Sigma_{n+1}$ action.)

Since this poset is Cohen-Macaulay, all we really care about is

$$\operatorname{rank}\Big(H_{n-2}(\mathsf{HT}^{\circ}_{n+1})\Big) = |\widetilde{\chi}(\mathsf{HT}^{\circ}_{n+1})|$$

and so computing the ℓ^2-Betti numbers of the group $OP\Sigma_{n+1}$ has boiled down to computing the Euler characteristic of the poset $\mathsf{HT}^{\circ}_{n+1}$.

Reduction to Möbius functions

Realizing we need to compute $\widetilde{\chi}(\mathsf{HT}^{\circ}_{n+1})$ we start filling up chalk boards with Hasse diagrams and compute ...

$$\chi(\mathsf{HT}^{\circ}_{4}) = 28 - 36 = -8$$
$$\chi(\mathsf{HT}^{\circ}_{5}) = 310 - 855 + 610 = 65$$

etc.

Luckily, Euler characteristics are well studied in enumerative combinatorics. In particular we can get to the Euler characteristic of $\mathsf{HT}^{\circ}_{n+1}$ by studying the Möbius function μ of $\widehat{\mathsf{HT}}_{n+1}$.

Fact: If μ is the Möbius function of $\widehat{\mathsf{HT}}_{n+1}$ then $\mu(\hat{0}, \hat{1}) = \widetilde{\chi}(\mathsf{HT}^{\circ}_{n+1})$

$$\widetilde{\chi}(\mathsf{HT}^{\circ}_{4}) = -9$$
$$\widetilde{\chi}(\mathsf{HT}^{\circ}_{5}) = 64$$

The Calculation and Its Corollaries

Using various recursion formulas for Möbius functions, and a functional equation for the number of hypertrees, it only takes 3 or 4 pages of work to show:

Thm: $\widetilde{\chi}(\mathsf{HT}^{\circ}_{n+1}) = (-1)^n n^{n-1}$.

Cor 1: *The ℓ^2-Betti numbers of $OP\Sigma_{n+1}$ are trivial, except $b^{(2)}_{n-1} = n^{n-1}$. It follows that*

$$b^{(2)}_{n-1}(O\Sigma_{n+1}) = \frac{n^{n-1}}{(n+1)!} \ .$$

Cor 2: *The ℓ^2-Betti numbers of $P\Sigma_{n+1}$ are trivial, except $b^{(2)}_{n} = n^{n}$. It follows that*

$$b^{(2)}_{n}(\Sigma_{n+1}) = \frac{n^{n}}{(n+1)!} \ .$$

More recent computations

Theorem C. If $G = G_1 * \cdots * G_n$ then
$\chi(OWh(G)) = \chi(G)^{n-2}$ and
$\chi(Wh(G)) = \chi(G)^{n-1}$.

Theorem D. If all the G_i are finite then
$\chi(FR(G)) = \chi(G)^{n-1} \prod |Inn(G_i)|$
$\chi(Aut(G)) = \chi(G)^{n-1} |\Omega|^{-1} \prod |Out(G_i)|$
$\chi(Out(G)) = \chi(G)^{n-2} |\Omega|^{-1} \prod |Out(G_i)|$

(Jensen-M-Meier, almost a preprint '04)

In general, Euler characteristics are not this nice:

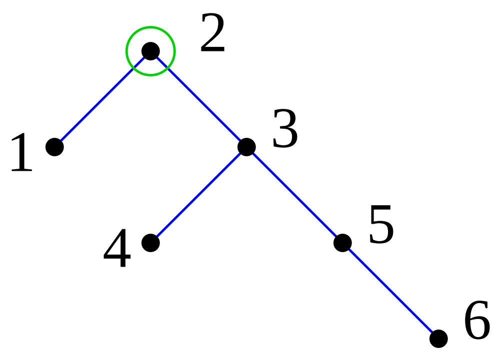
$$\chi(Out(F_{12})) = -\frac{37539377353473689 9347}{2191186722816000}$$

(Smillie-Vogtmann, '87)

A hint at the underlying combinatorics

$$m : \text{Rooted trees} \to \text{Monomials}$$
$$T \mapsto \prod_i x_i^{\deg i} \text{ (rooted degree)}$$

Example:



$$\mapsto \quad x_1^0 x_2^2 x_3^2 x_4^0 x_5^1 x_6^0 = x_2^2 x_3^2 x_5^1$$

Thm: $\sum_T m(T) = (x_1 + x_2 + \cdots x_n)^{n-1}$
where the sum is over all rooted trees on $[n]$

Thm: $\sum_T m(T) = \binom{n-1}{k-1}(x_1 + x_2 + \cdots x_n)^{n-k}$
where the sum is over all planted forests on $[n]$ with k components.