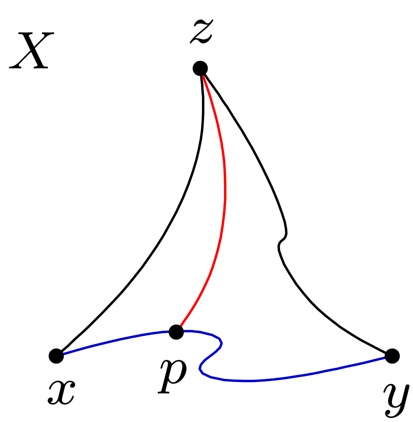
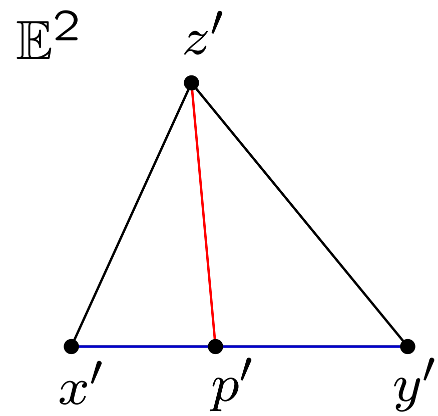
Constructing non-positively curved spaces and groups

Day 1: The basics





Jon McCammond
U.C. Santa Barbara

Outline

I. CAT(κ) and δ-hyperbolic

II. Curvature conjecture

III. Decidability issues

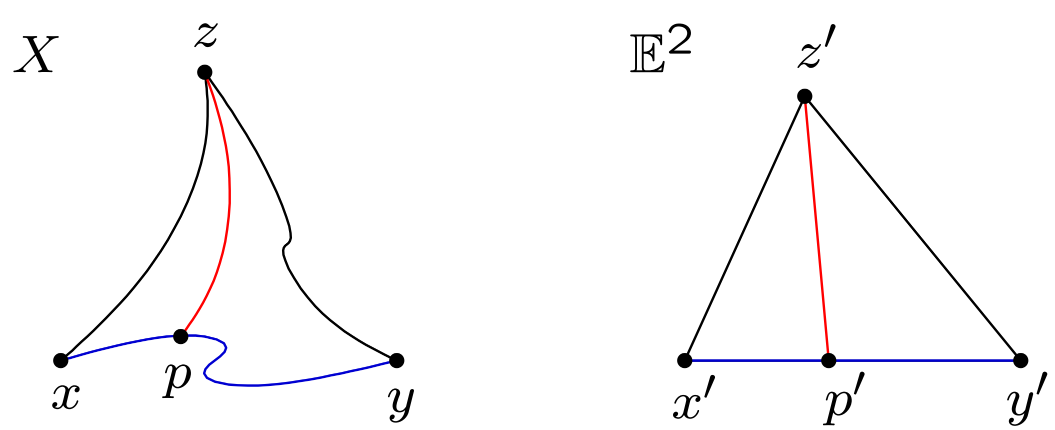
IV. Length spectrum

I. **CAT**(0) **spaces**

Def: A geodesic metric space X is called (globally) CAT(0) if
$\forall$ points $x, y, z \in X$
$\forall$ geodesics connecting x, y, and z
$\forall$ points p in the geodesic connecting x to y

$$d(p, z) \leq d(p', z')$$

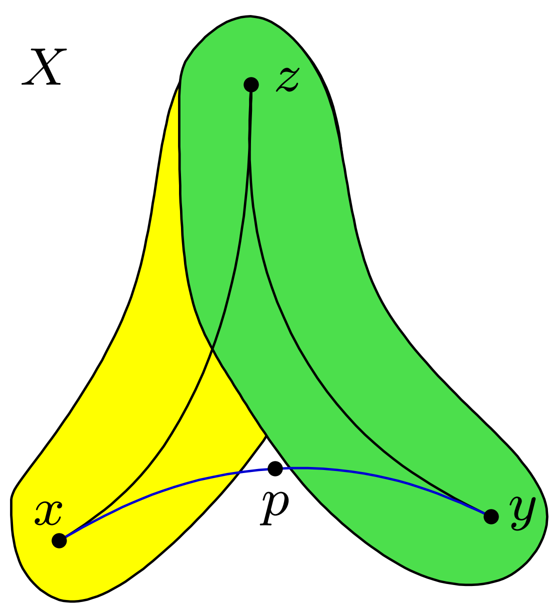
in the corresponding configuration in $\mathbb{E}^2$.



Rem: CAT(1) and CAT(−1) are defined similarly using $\mathbb{S}^2$ and $\mathbb{H}^2$ respectively - with restrictions on x, y, and z in the spherical case, since not all spherical comparison triangles are constructible.

δ-hyperbolic spaces

Def: A geodesic metric space X is *δ-hyperbolic* if
$\forall$ points $x, y, z \in X$
$\forall$ geodesics connecting x,y, and z
$\forall$ points p in the geodesic connecting x to y
the distance from p to the union of the other two geodesics is at most δ.



Rem: Hyperbolic n-space, $\mathbb{H}^n$ is both δ-hyperbolic and CAT(-1).

Local curvature

δ-hyperbolic only implies the large scale curvature is negative. We get no information about local structure.

CAT(0) and CAT(-1) imply good local curvature conditions.

Lem: X is CAT(0) [CAT(-1)] $\Leftrightarrow$
X is locally CAT(0) [CAT(-1)] and $\pi_1 X = 1$ (needs completeness)

Def: A locally CAT(0) [CAT(-1)] space is called *non-positively [negatively] curved*.

CAT(-1) **vs. CAT**(0) **vs.** δ**-hyperbolic**

Thm: CAT(κ) $\Rightarrow$ CAT(κ') when $\kappa \leq \kappa'$.
In particular, CAT(-1) $\Rightarrow$ CAT(0).

Def: A *flat* is an isometric embedding of a Euclidean space $\mathbb{E}^n$, $n > 1$.

Thm: CAT(-1) $\Rightarrow$ CAT(0) + no flats

Thm: CAT(-1) $\Rightarrow$ δ-hyperbolic

In fact, when X is CAT(0) and has a proper, cocompact group action by isometries,
X is δ-hyperbolic $\Leftrightarrow$ X has no flats.
(**Flat Plane Thm**)

CAT(0) groups and hyperbolic groups

Def: A group G is *hyperbolic* if for some δ it acts properly and cocompactly by isometries on some δ-hyperbolic space.

Lem: G is is hyperbolic if for some finite generating set A and for some δ, its Cayley graph w.r.t. A is δ-hyperbolic.

Def: A group G is CAT(0) if it acts properly and cocompactly by isometries on some CAT(0) space.

Rem: Unlike hyperbolicity, showing a group is CAT(0) requires the construction of a CAT(0) space.

CAT(-1) **vs. CAT**(0) **vs. word-hyperbolic**

$$\begin{array}{ccc} \text{CAT}(-1) & & \\ \Downarrow & & \\ \text{CAT}(0)+\text{ no flats} & \Rightarrow & \text{word-hyperbolic} \\ \Downarrow & & \\ \text{CAT}(0)+\text{ no } \mathbb{Z}\times\mathbb{Z} & & \end{array}$$

Flat Torus Thm: $\mathbb{Z}\times\mathbb{Z}$ in $G \Rightarrow \exists$ a flat in X.

Problem: Flat in $X \Leftrightarrow \mathbb{Z}\times\mathbb{Z}$ in G?

Thm(Wise) $\exists$ aperiodic flats in CAT(0) spaces which are not limits of periodic flats.

Rem: This is not even known for VH CAT(0) squared complexes.

Constant curvature complexes

Constant curvature models: $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$.

Def: A *piecewise spherical / euclidean / hyperbolic complex* X is a polyhedral complex in which each polytope is given a metric with constant curvature 1 / 0 / -1 and the induced metrics agree on overlaps. In the spherical case, the cells must be convex polyhedral cells in $\mathbb{S}^n$. The generic term is M_κ-complex, where κ is the curvature.

Thm(Bridson) Compact M_κ complexes are geodesic metric spaces.

Exercise: What restrictions on edge lengths are necessary in order for a PS/PE/PH n-simplex to be buildable?

II. Curvature conjecture

$$
\begin{array}{ccccc}
\text{PH CAT}(-1) & \Rightarrow & \text{CAT}(-1) & & \\
(?) & & \Downarrow & & \\
\begin{array}{c}\text{PE CAT}(0)\\ \text{no flats}\end{array} & \Rightarrow & \begin{array}{c}\text{CAT}(0)\\ \text{no flats}\end{array} & \Rightarrow & \begin{array}{c}\text{word-}\\ \text{hyperbolic}\end{array} \\
\Downarrow & & \Downarrow & & \\
\begin{array}{c}\text{PE CAT}(0)\\ \text{no } \mathbb{Z}\times\mathbb{Z}\end{array} & \Rightarrow & \begin{array}{c}\text{CAT}(0)\\ \text{no } \mathbb{Z}\times\mathbb{Z}\end{array} & &
\end{array}
$$

Conj: These seven classes of groups are equal.

Rem 1: Analogue of Thurston's hyperbolization conjecture.

Rem 2: If Geometrization (Perleman) holds then this is true for 3-manifold groups.

PH CAT(-1) vs. PE CAT(0)

Thm(Charney-Davis-Moussong) If M is a compact hyperbolic n-manifold, then M also carries a PE CAT(0) structure.

Rem: This is open even for compact (variably) negatively-curved n-manifolds.

Thm(N.Brady-Crisp) There is a group which acts nicely on a 3-dim PH CAT(-1) structure, and on a 2-dim PE CAT(0) structure, but not on any 2-dim PH CAT(-1) structure.

Moral: Higher dimensions are sometimes necessary to flatten things out.

Rips Complex

If our goal is to create complexes with good local curvature for an arbitrary word-hyperbolic group, the obvious candidate is the Rips complex (or some variant).

Def: Let $P_d(G, A)$ be the flag complex on the graph whose vertices are labeled by G and which has an edge connecting g and h iff gh^{-1} is represented by a word of length at most d over the alphabet A.

Thm: If G is word-hyperbolic and d is large relative to δ, the complex $P_d(G, A)$ is contractible (and finite dimensional).

Adding a metric to the Rips complex

Let G be a word-hyperbolic group.

Q: Suppose we carefully pick a generating set A and pick a d very large and declare each simplex in $P_d(G, A)$ to be a regular Euclidean simplex with edge length 1. Is the result a CAT(0) space?

Exercise: Is this true when G is free and A is a basis?

Adding a metric to the Rips complex

Let G be a word-hyperbolic group.

Q: Suppose we carefully pick a generating set A and pick a d very large and declare each simplex in $P_d(G, A)$ to be a regular Euclidean simplex with edge length 1. Is the result a CAT(0) space?

Exercise: Is this true when G is free and A is a basis?

A: No one knows!

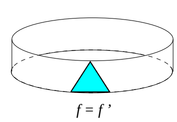
Moral: Our ability to test whether compact constant curvature metric space is CAT(0) or CAT(-1) is *very* primitive.

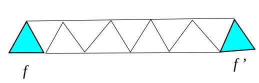
III. Decidability

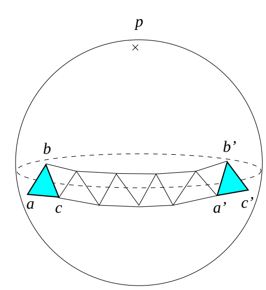
Thm(Elder-M) Given a compact M_κ-complex, there is an "algorithm" which decides whether it is locally CAT(κ).

Proof sketch:

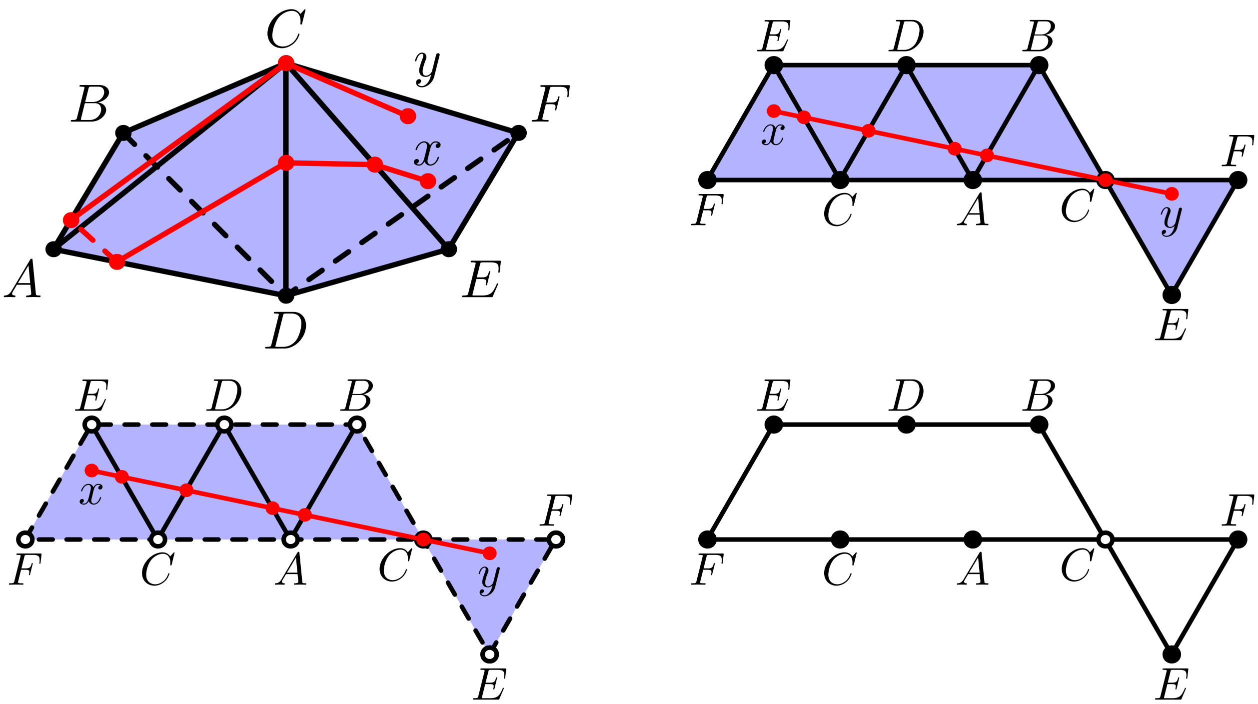
- reduce to galleries in PS complexes
- convert to real semi-algebraic sets
- apply Tarski's "algorithm"







Galleries



A 2-complex, a linear gallery, its interior and its boundary.

Reduction to geodesics in PS complexes

Rem: The link of a point in an M_κ-complex is an PS complex.

Thm: An M_κ-complex is locally CAT(κ)
$\Leftrightarrow$ the link of each vertex is globally CAT(1)
$\Leftrightarrow$ the link of each cell is an PS complex which contains no closed geodesic loop of length less than 2π.

Moral: Showing that PE complexes are non-positively curved or PH complexes are negatively curved hinges on showing that PS complexes have no short geodesic loops.

Geodesics

Def: A *local geodesic* in a M_κ-complex is a concatenation of paths such that
1) each path is a geodesic in a simplex, and
2) at the transitions, the "angles are large" meaning that the distance between the "in" direction and the "out" direction is at least π in the link.

Rem: Notice that there is an induction involved in the check for short geodesics. To test whether a particular curve is a short geodesic, you need to check whether it is short and whether it is a geodesic, but the latter involves checking geodesic distances in a lower dimensional PS complex, but this involves checking geodesic distances in a lower dimensional PS complex...

Unshrinkable geodesics

In practice, we will often restrict our search to unshrinkable geodesics.

Def: A geodesic is *unshrinkable* if there does not exist a non-increasing homotopy through rectifiable curves to a curve of strictly shorter length.

Thm(Bowditch) It is sufficient to search for unshrinkable geodesics.

Cor: In a PS complex it is sufficient to search to for a geodesic which can neither be shrunk nor homotoped til it meets the boundary of its gallery without increasing length.

Converting to Polynomial Equations, I

Spaces and maps:

$$\begin{array}{ccccccc} & & \{x_i\} & \rightarrow & \mathbb{S}^1 & \subset & \mathbb{R}^2 \\ & \swarrow & \downarrow & & & & \\ K & \leftarrow & \mathcal{G} & \rightarrow & \mathbb{S}^n & \subset & \mathbb{R}^{n+1} \end{array}$$

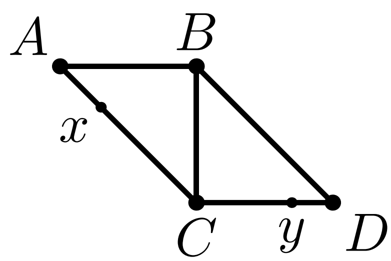
For each 0-cell v in $\mathcal{G}$
- create a vector $\vec{u}_v$ in $\mathbb{R}^{n+1}$

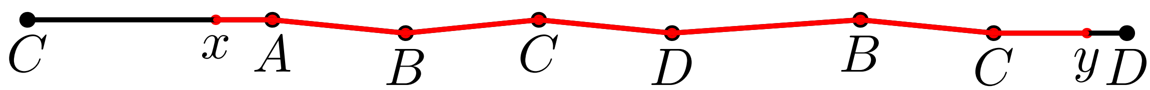
For each x_i
- create a vector $\vec{y}_i$ in $\mathbb{R}^{n+1}$
- a vector $\vec{z}_i$ in $\mathbb{R}^2$.

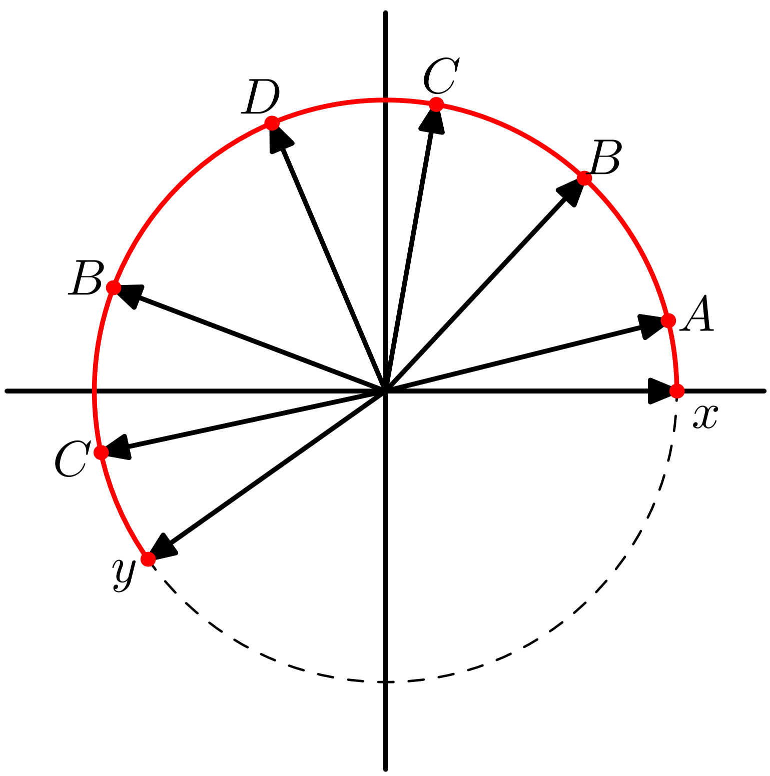
Add equations which stipulate
- they are unit vectors,
- the edge lengths are right,
- $\vec{y}_i$ is a positive linear comb. of certain $\vec{u}_v$,
- the $\vec{z}_i$ march counterclockwise around $\mathbb{S}^1$ starting at $(1, 0)$.

Converting to Polynomial Equations, II

A 1-complex, a gallery and its model space.







Real semi-algebraic sets

Def: A *real semi-algebraic set* is a boolean combination ($\cup$, $\cap$ and complement) of real algebraic varieties.

Inducting through dimensions, it is possible to show that there is a real semi-algebraic set in which the points are in one-to-one correspondence with the closed geodesics in the circular gallery $\mathcal{G}$.

Punchline: Tarski's theorem about the decidability of the reals implies that there is an algorithm which decides whether a real semi-algebraic set is empty or not.

Rem: It is still not known whether there is an algorithm to decide whether a particular complex supports a CAT(0) metric.

Why is this so hard?

Problems with high codimension (≥ 2) can often be quite hard.

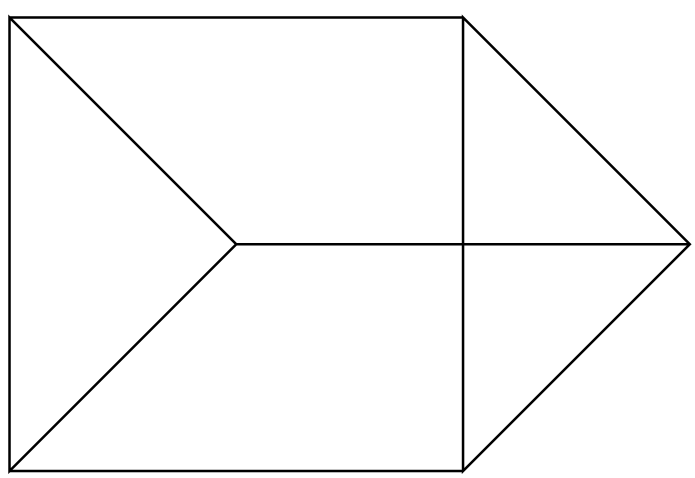
Q: What is the unit volume 3-polytope with the smallest 1-skeleton (measured by adding up the edge lengths)?

Why is this so hard?

Problems with high codimension (≥ 2) can often be quite hard.

Q: What is the unit volume 3-polytope with the smallest 1-skeleton (measured by adding up the edge lengths)?

A: No one knows, but the best guess is a triangular prism.



IV. Length spectra

Def: The lengths of open geodesics from x to y is the *length spectrum from x to y*.

Thm(Bridson-Haefliger) The length spectrum from x to y in a compact M_κ-complex is discrete.

Def: The lengths of closed geodesics in a space is simply called its *length spectrum*.

Thm(N.Brady-M) The length spectrum of a compact M_κ-complex is discrete.

Proof sketch:

- Suppose not and reduce to a single gallery.
- Closed geodesics are critical points of d.
- d is real analytic on a compact set containing the tail of the sequence
- d extends to real analytic function on a larger open set.
- $\therefore$ only finitely many critical values.

Totally geodesic surfaces

Def: A surface $f : D \to X$ is *totally geodesic* if $\forall d \in D$, Lk(d) is sent to a local geodesic in Lk($f(d)$).

Cor: If D is a totally geodesic surface in a NPC PE complex then the points in the interior of D with negative curvature have curvatures bounded away from 0.

Rem: In a 2-dimensional NPC PE complex, every null-homotopic curve bounds a totally geodesic surface. This fails in dimension 3 and higher, and is one of the key reasons why theorems in dimension 2 fail to generalize easily to higher dimensions.