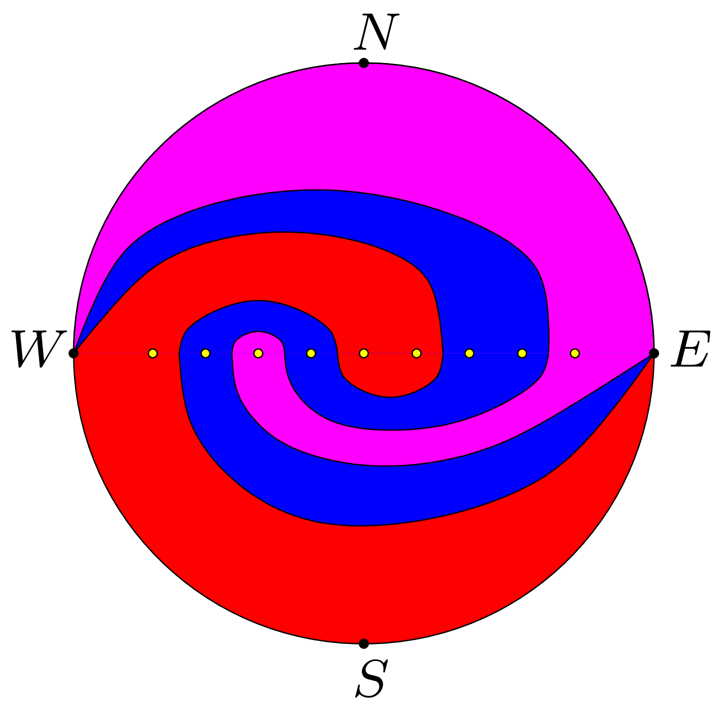
Garside structures for free groups
(and other Artin groups)



Jon McCammond
U.C. Santa Barbara

(joint work with Noel Brady,
John Crisp, and Anton Kaul)

Overview

I. Coxeter groups and Artin groups

II. Garside structures

III. Garside structures for free groups

IV. Garside-like structures for Artin groups

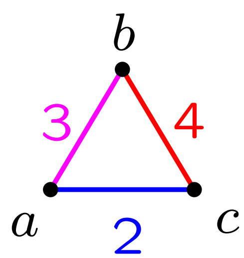
I. Coxeter groups and Artin groups

Let Γ be a finite graph with edges labeled by integers greater than 1, and let $(a,b)^n$ be the length n prefix of $(ab)^n$.

Def: The *Artin group* A_Γ is generated by its vertices with a relation $(a,b)^n = (b,a)^n$ whenever a and b are joined by an edge labeled n.

Def: The *Coxeter group* W_Γ is the Artin group A_Γ modulo the relations $a^2 = 1$ $\forall a \in \mathbf{Vert}(\Gamma)$.

Graph



Artin presentation

$$\langle a,b,c \mid aba = bab, ac = ca, bcbc = cbcb \rangle$$

Coxeter presentation

$$\left\langle a,b,c \mid \begin{array}{c} aba = bab, ac = ca, bcbc = cbcb \\ a^2 = b^2 = c^2 = 1 \end{array} \right\rangle$$

Natural yet mysterious

Artin groups are "natural" in the sense that they are closely tied to the complexified version of the hyperplane arrangements for finite reflection groups.

But they are "mysterious" in the sense that it is unknown if

1. They have a decidable word problem
2. They are torsion-free
3. They have finite (dimensional) $K(\pi, 1)$s
4. They are linear
5. The positive monoid injects into the group

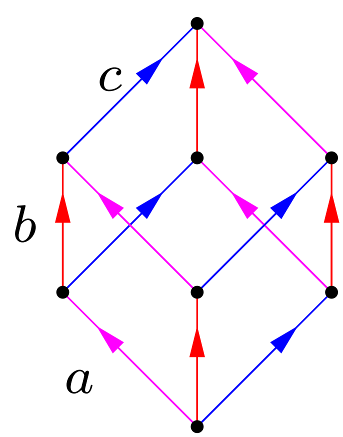
Actually 5 was recently shown to be true by Luis Paris, but the proof is still mysterious.

II. Garside structures

Let G be a group, let A be a generating set, and let Δ be an element of G.

This data defines a *Garside structure* for G if the edge-labeled poset of directed paths in the Cayley graph of G with respect to A is a balanced lattice (lattice in the combinatorial sense) and includes edges labeled by each of the elements of A.

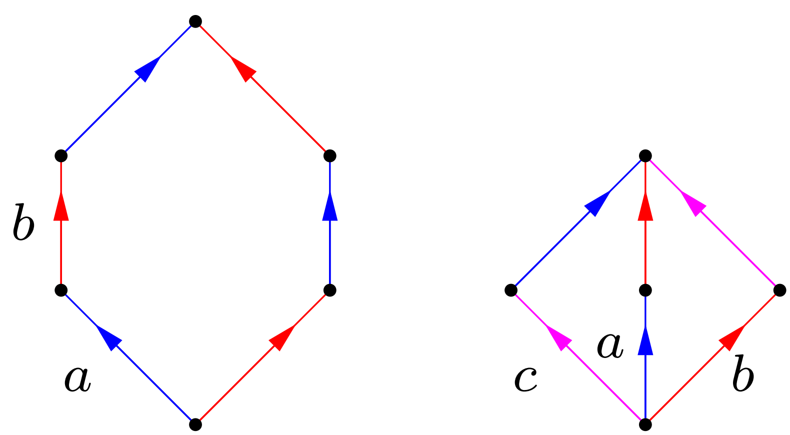
Balanced means that the words readable starting at the bottom are the words readable ending at the top.



A Garside structure for $\mathbb{Z}^3$ is shown.

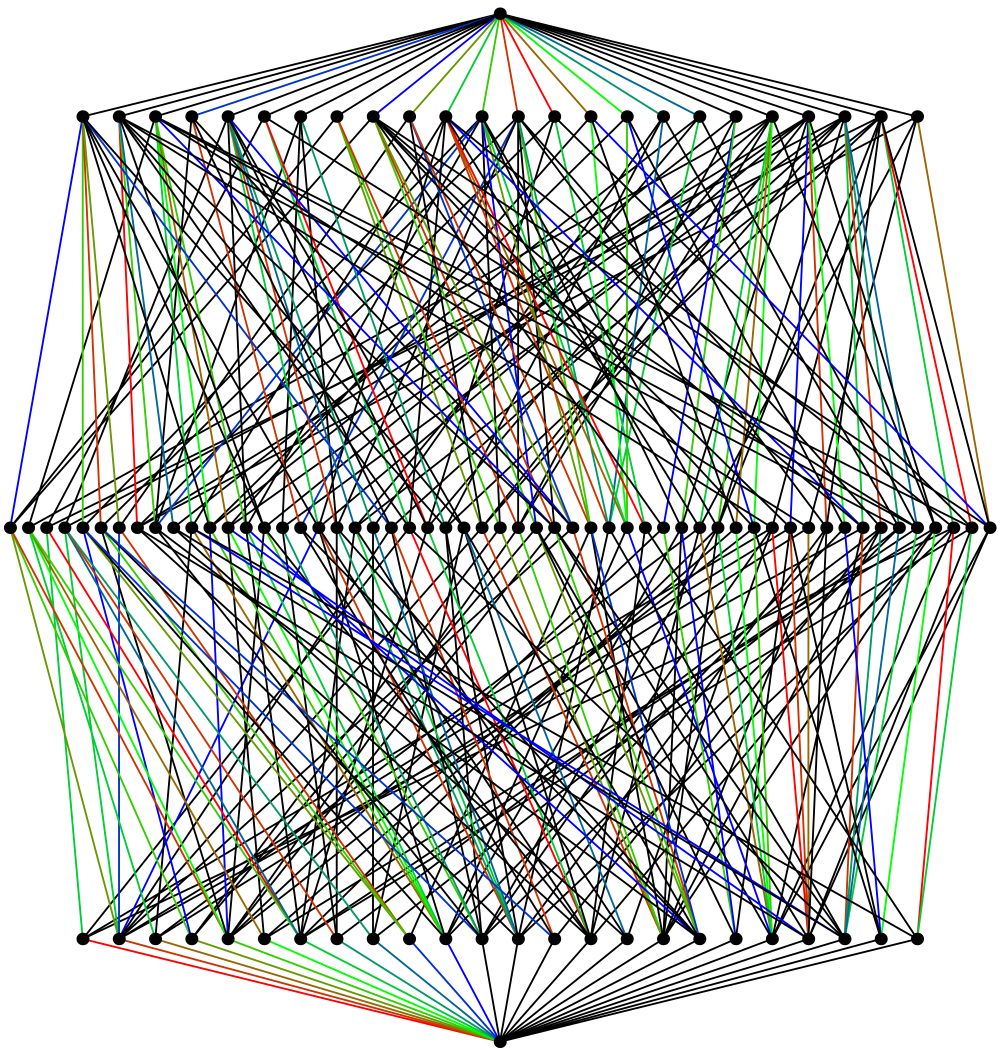
Examples of Garside structures

Braid groups and other finite-type Artin groups each have two Garside structures. For the 3-string braid group the two posets are shown. The second one is the *dual* of the first.



$$\langle a, b | aba = bab \rangle = \langle a, b, c | ab = bc = ca \rangle$$

The dual F_4 Poset



Why "dual"?

$\mathbf{S}$ = standard generators

$\mathbf{T}$ = set of all "reflections"

$\mathbf{c}$ = a Coxeter element

$\mathbf{w}_0$ = the lift of the longest element in W

n = the rank (dimension) of W

N = # reflections = # of positive roots

h = Coxeter number = order of $\mathbf{c}$

	Classical monoid	Dual monoid
Set of atoms	$\mathbf{S}$	$\mathbf{T}$
Product of atoms	$\mathbf{c}$	$\mathbf{w}_0$
Number of atoms	n	N
Regular degree	h	2
$\boldsymbol{\Delta}$	$\mathbf{w}_0$	$\mathbf{c}$
Length of $\boldsymbol{\Delta}$	N	n
Order of $p(\boldsymbol{\Delta})$	2	h

What Garside structures are good for

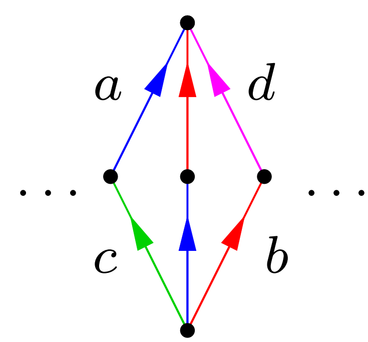
If G is a group with a Garside structure, then it

1. has a presentation derived from the poset
2. is the group of fractions of this presentation
3. has a decidable word problem
4. has a finite (dimensional) $K(\pi, 1)$
5. is torsion-free.

Thus finding Garside structures for Artin groups would be a very good thing. The hardest part is almost always showing that the poset is a lattice.

III. Garside structures for free groups

Let F_n be a free group with basis $x_1, x_2, \ldots, x_n$ and let $\Delta = x_1 x_2 \cdots x_n$. We can start building a Garside structure by continuing to add paths (and generators) to create a balanced graded poset.

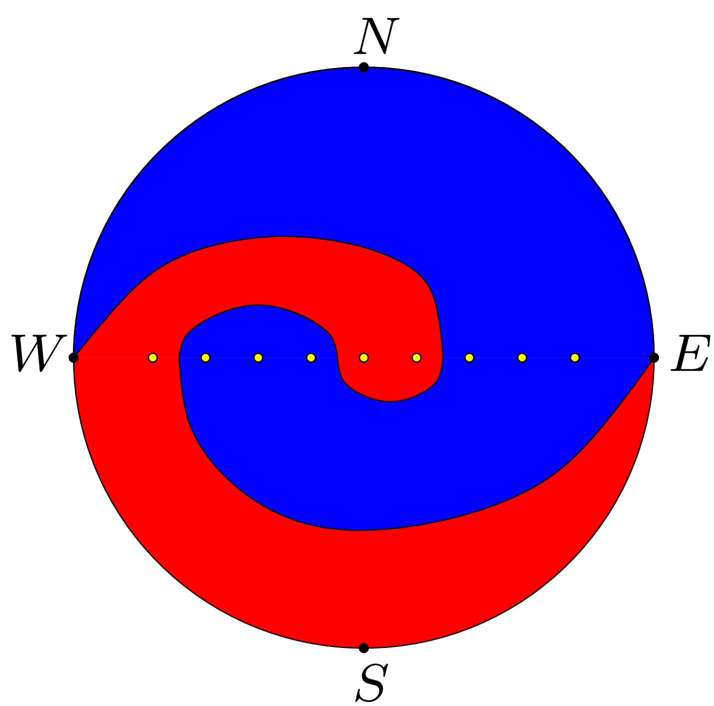


The construction in this case leads to a universal cover which is an infinitely branching tree cross the reals with a free $\mathbf{F}_2$ action.

A more topological definition

Let $\mathbf{D}^*$ denote the unit disc with n puntures and 4 distinguished boundary points, N, S, E and W.

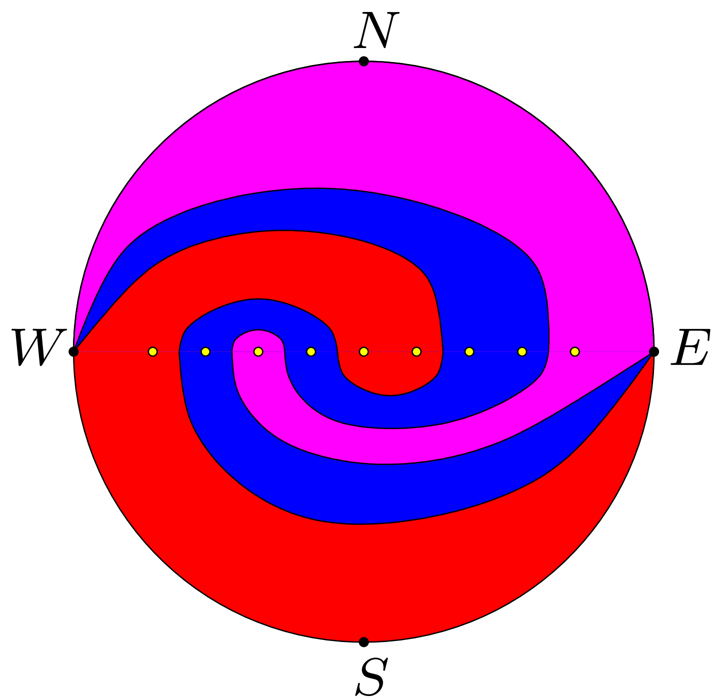
Def: A *cut-curve* is an isotopy class (in $\mathbf{D}^*$) of a path from E to W (rel endpoints, of course).



Notice that cut-curves divide $\mathbf{D}^*$ into two pieces, one containing S and the other containing N. Its *height* is the number of puncture in the lower piece.

Poset of cut-curves

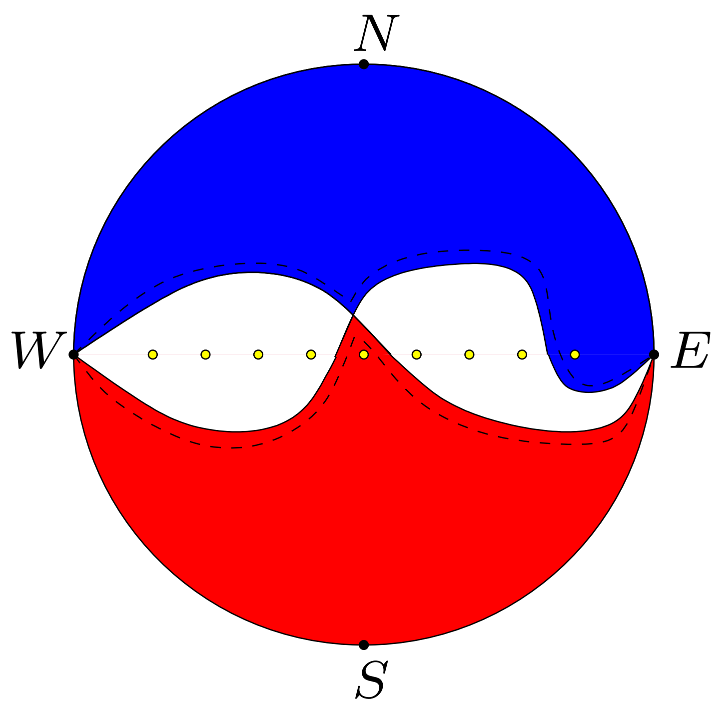
Let $[c]$ and $[c']$ be cut-curves. We write $[c] < [c']$ if there are representatives c and c' which are disjoint (except at their endpoints) and c is "below" c'.



Notice that if representative c is given, then we can tell whether $[c] < [c']$ by keeping c fixed and isotoping c' into a "minimal position" with respect to c (i.e. no football shaped regions with no punctures).

Proving the lattice property

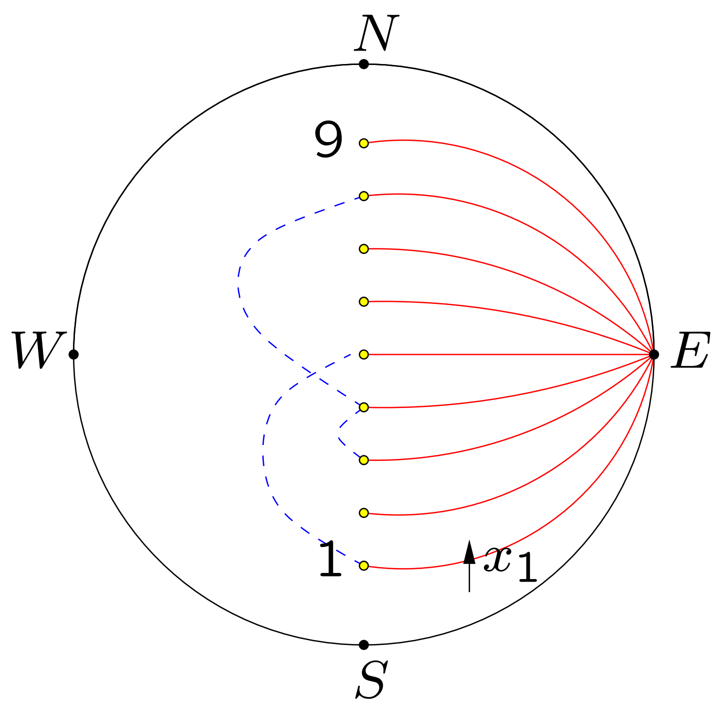
Lemma The poset of cut-curves is a lattice.



Proof: Suppose $[c]$ is above $[c_1]$ and $[c_2]$. Place representatives c_1 and c_2 in minimal position with respect to each other (i.e. no football regions) and then isotope c so that it is disjoint from both. This c is above the dotted line. Thus the dotted line represents a least upper bound for $[c_1]$ and $[c_2]$.

IV. Garside structures for Artin groups

For a general Artin group, we start with a specific marking of $\mathbf{D}^*$ (in the form of cuts) and draw arcs connecting the punctures which avoid the cuts.



From the graph Γ we define a subgroup H of the braid group which is generated by powers of half-twists along the arcs with the powers determined by the labels on the edges.

Garside structure for A_Γ

From H we can define a poset either algebraically (using double cosets inside the braid group) or topologically using the action on $\mathbf{D}^*$.

Algebraic: Consider the double cosets $B_k \times B_{n-k}\beta H$ where β is a braid in B_n. The poset order is intersection.

We can show, in full generality that the balanced poset we get yields a presentation of the correct Artin group.

The only missing piece (but it's a major piece) is showing that the poset is a lattice. [But we can show this in some special cases.]