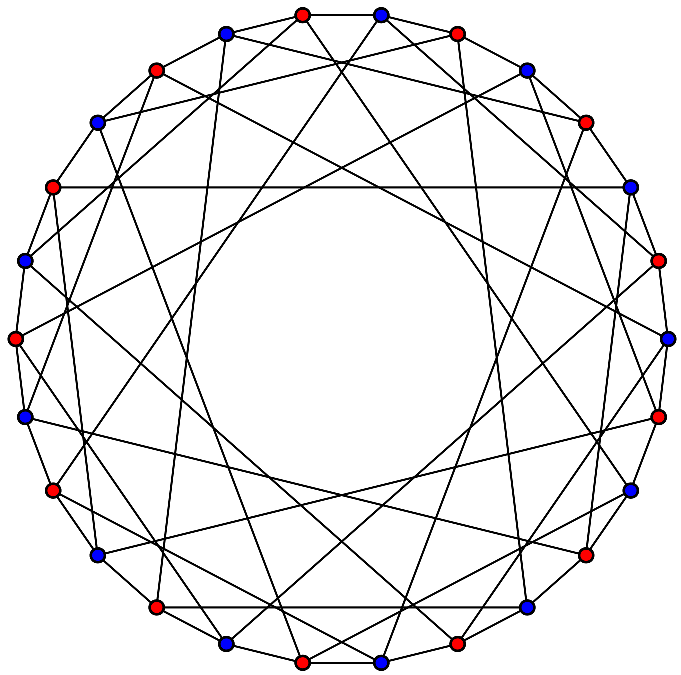
Coxeter groups and Artin groups

Day 2: Symmetric Spaces and Simple Groups



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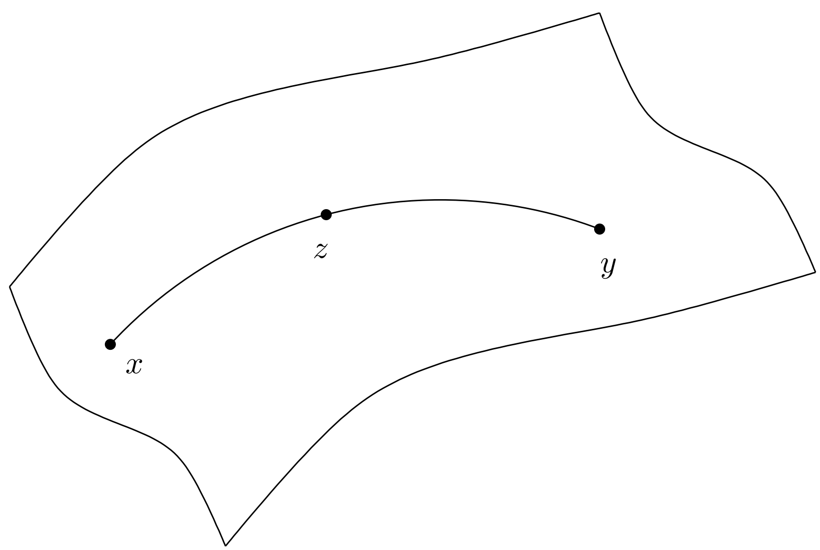
Symmetric Spaces

Def: A Riemannian manifold is called a **symmetric space** if at each point p there exists an isometry fixing p and reversing the direction of every geodesic through p. It is **homogeneous** if its isometry group acts transitively on points.

Prop: Every symmetric space is homogeneous.

Proof:



Symmetric Spaces: First Examples

Ex: $\mathbb{S}^n$, $\mathbb{R}^n$, and $\mathbb{H}^n$ are symmetric spaces.

Def: The projective space $\mathbb{K}P^n$ is defined as $(\mathbb{K}^{n+1} - \{\vec{0}\})/\mathbb{K}^*$.

Ex: n-dimensional real, complex, quarternionic, and, for $n \leq 2$, octonionic projective spaces are symmetric.

Rem: $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are straightforward since $\mathbb{R}$ and $\mathbb{C}$ are fields. The quarternionic version needs more care (due to non-commutativity) and the octonionic version only works in low dimensions (due to non-associativity).

Rem: The isometries of a symmetric space form a Lie group.

Lie Groups

Def: A **Lie group** is a (smooth) manifold with a compatible group structure.

Ex: Both $\mathbb{S}^1 \cong$ the unit complex numbers and $\mathbb{S}^3 \cong$ the unit quaternions are (compact) Lie groups.

It is easy to see that $\mathbb{S}^1$ is a group under rotation. The multiplication on $\mathbb{S}^3$ is also easy to define: Once we pick an orientation for $\mathbb{S}^3$ there is a unique isometry of $\mathbb{S}^3$ sending x to y ($x \neq -y$) that rotates the great circle C containing x and y and rotates the circle orthogonal to C by the same amount in the direction determined by the orientation.

Lie Groups and Coxeter Groups: a quick rough sketch

Continuity forces the product of points near the identity in a Lie group to be sent to points near the identity, which in the limit gives a Lie algebra structure on the tangent space at 1.

An analysis of the resulting linear algebra shows that there is an associated discrete affine reflection group and these affine reflection groups have finite reflection groups inside them.

The classifcation of finite reflection groups can then be used to classify affine reflection groups, Lie algebras and Lie groups.

Lie Groups and Symmetric Spaces

Rem: The isometry group of a symmetric space X is Lie group G and the stabilizer of a point is a compact subgroup K. Moreover, the points in X can be identified with the cosets of K in G.

Thm: Every symmetric space arises in this way and the classification of Lie groups quickly leads to a classification of symmetric spaces.

Ex: Isom($\mathbb{O}P^2$) = Lie group $F_4 \rightsquigarrow$
Lie algebra $F_4 \rightsquigarrow$ affine reflection group $F_4 \rightsquigarrow$
finite reflection group F_4 = Isom(24-cell).

These are usually distinguished via fonts and other markings.

Finite Projective Planes

The construction of $\mathbb{K}P^2$ still works perfectly well when $\mathbb{K}$ is a *finite* field instead of $\mathbb{R}$ or $\mathbb{C}$. Lines through the origin in the vector space $\mathbb{K}^3$ become (projective) points in $\mathbb{K}P^2$, planes through the origin in $\mathbb{K}^3$ become (projective) lines in $\mathbb{K}P^2$.

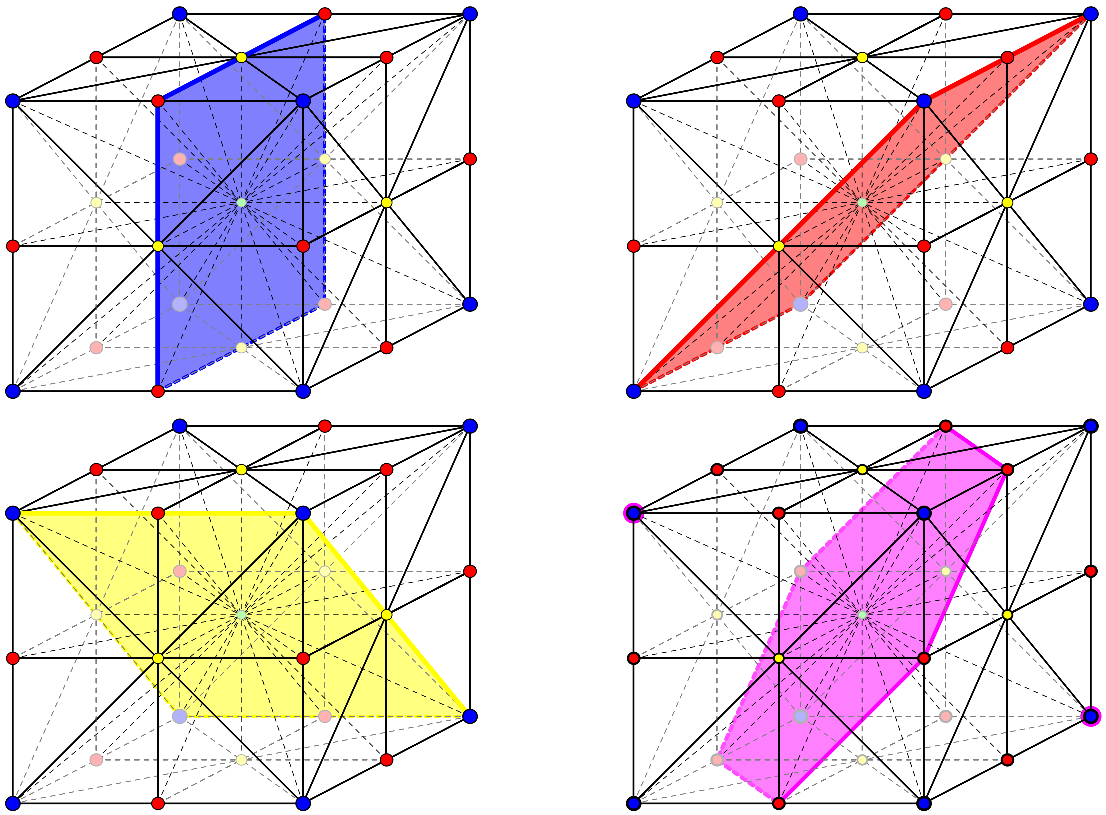
If the field $\mathbb{K}$ has q elements, then $\mathbb{K}P^2$ has
- $q^2 + q + 1$ points and
- $q^2 + q + 1$ lines.

Moreover, there is the usual duality between points and lines.

Rem: Finite projective spaces can be viewed as discrete analogs of symmetric spaces and their automorphism groups as discrete analogs of Lie groups.

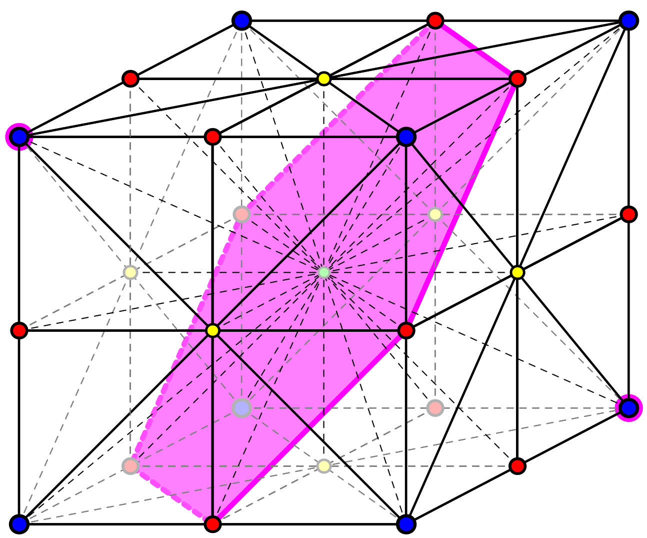
A Sample Finite Projective Plane

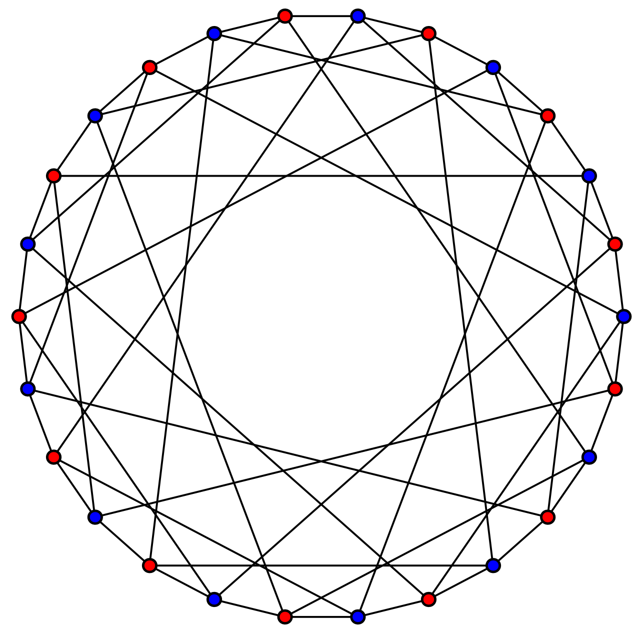
The finite projective plane over $\mathbb{F}_3$ with its $9+3+1=13$ points and 13 lines can be visualized using a cube.



Incidence Graphs

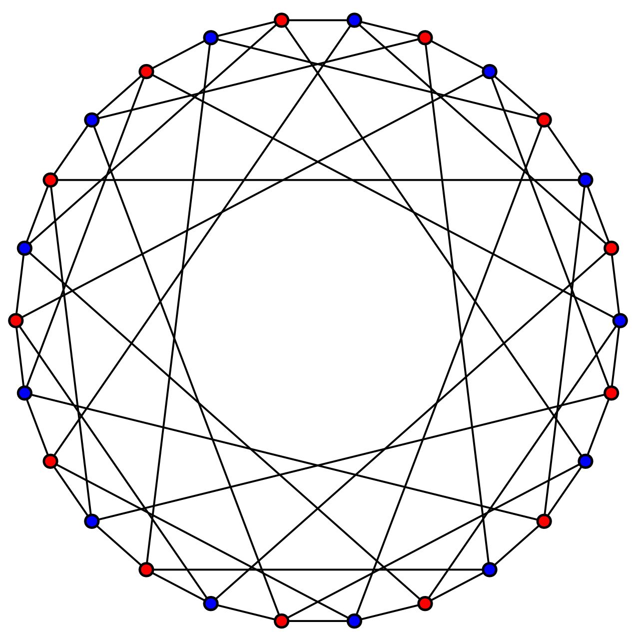
Let $\text{Inc}(\mathbb{K}P^2)$ denote the incidence graph of $\mathbb{K}P^2$: draw a red dot for every projective point in $\mathbb{K}P^2$, a blue dot for every projective line and connect a red dot to a blue one iff the point lies on the line. $\text{Inc}(\mathbb{F}_3P^2)$ is shown.





Buildings

The incidence graphs of finite projective planes are highly symmetric and examples of **buildings**. They have diameter 3, girth 6, distance transitive, and every pair of points lies on an embedded hexagon. Their automorphism groups are very large.



Coxeter presentations

Def: A **Coxeter presentation** is a finite presentation $\langle S \mid R \rangle$ with only two types of relations:

- a relation s^2 for each $s \in S$, and
- at most one relation $(st)^m$ for each pair of distinct $s, t \in S$. A group defined by such a presentation is called a **Coxeter group**.

Ex: $\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^3 \rangle$

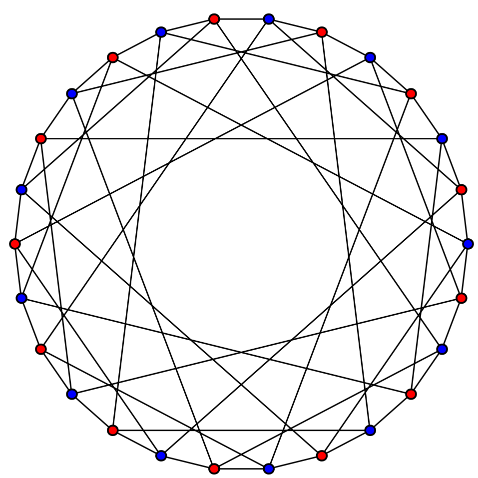
Thm: Every finite reflection group has a Coxeter presentation that can be read off of its Dynkin diagram.

Proof: If W is the isometry group of a regular polytope then the fact that this is a presentation of W following from the fact that the 2-skeleton of the dual of the subdivided polytope is simply-connected. The general proof is similar.

Small Type Coxeter Groups

Def: A Coxeter group $W = \langle S \mid R \rangle$ is **small type** if for every pair of distinct $s, t \in S$, either $(st)^2$ or $(st)^3$ is a relation in R.

Rem: Dynkin diagrams of small type Coxeter groups correspond to arbitrary simplicial graphs, so graphs such as Inc($\mathbb{F}_3 P^2$) can be used to define (very large) Coxeter groups.



The Bilinear Form

The normal vectors $\{\vec{n}_i\}$ arising from the basic reflections of a finite reflection group determine a matrix $M = [\vec{n}_i \cdot \vec{n}_j]_{(i,j)}$ where the dot products are 1 along the diagonal and $\cos(\pi - \pi/n)$ otherwise (where n is label on the edge connecting v_i and v_j or $n = 3$ when there is no label, or $n = 2$ when there is no edge).

The formula can be followed blindly for any Dynkin diagram. The resulting real symmetric matrix M is called the **Coxeter matrix** for the corresponding Coxeter group.

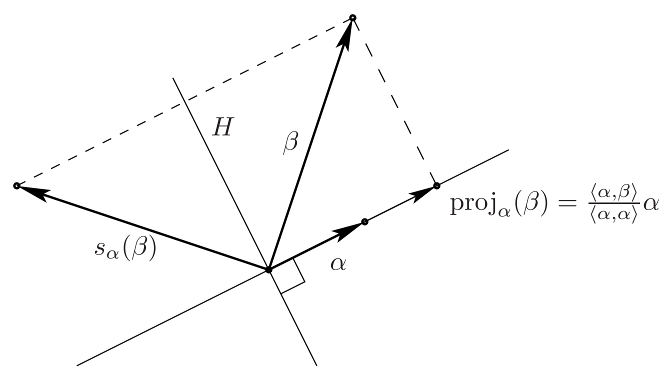
Define a bilinear form on $V = \mathbb{R}^n$ by setting $B(\vec{x}, \vec{y}) = \vec{x}\ M\ \vec{y}^{\,T}$.

Linear Representations

Let $V = \mathbb{R}^n$, let Γ be a Dynkin diagram and let W be the Coxeter group it defines. Using the bilinear form B we can define a (linear) representation of W. For each generator s_i define a reflection $\rho_i : V \to V$ by setting

$$\rho_i(\vec{v}) = \vec{v} - 2\frac{B(\vec{e_i},\vec{v})}{B(\vec{e_i},\vec{e_i})}\vec{e_i}$$

This mimics the usual formula for a reflection.



Generalized Orthogonal Groups

For any bilinear form B, let $O(V,B)$ denote the set of invertible linear transformations T of the n-dimensional vector space V that preserve this bilinear form: $B(T\vec{x},T\vec{y}) = B(\vec{x},\vec{y})$.

Rem: $O(V,B)$ is a subset of $GL(V)$ and it inherits a Lie group structure.

Thm: The homomorphism $W \to O(V,B)$ is an embedding.

Rem: The orbit of a (non-isotropic) vector in V under the action of $O(V,B)$ sweeps out a symmetric space.

Ex: Hyperboloid model.

Types of Coxeter Groups

Let W be a Coxeter group and let B be its matrix. If B has

- no non-positive eigenvalues, then W is **spherical**.
- one non-positive eigenvalue and it is $= 0$, then W is **affine**.
- one non-positive eigenvalue and it is < 0, then W is **hyperbolic**.
- more than one non-positive eigenvalue then W is **higher rank**.

Ex: The Coxeter group defined by:

- a hexagon is affine,
  Spectrum $= [4^1\ 3^2\ 1^2\ 0^1]$
- $\mathrm{Inc}(\mathbb{F}_3P^2)$ is hyperbolic,
  Spectrum $= [6^1\ (2+\sqrt{3})^{12}\ (2-\sqrt{3})^{12}\ (-2)^1]$
- the 1-skeleton of the 4-cube is higher rank.
  Spectrum $= [6^1\ 4^4\ 2^4\ 0^4\ (-2)^1]$

Coxeter Groups and Symmetric Spaces

Thm: Every Coxeter group acts faithfully on some symmetric space with the generators acting by reflection. This action is proper and discontinuous but only rarely cocompact.

Rem: The type of the Coxeter group indicates the type of symmetric space it acts on. Spherical ones act on $\mathbb{S}^n$, affine ones on $\mathbb{R}^n$, hyperbolic ones on $\mathbb{H}^n$ and the ones of higher rank on one of the more unusal symmetric spaces.

Ex: The small type Coxeter group W defined by the graph Inc($\mathbb{F}_3P^2$) acts on 25-dimensional hyperbolic space $\mathbb{H}^{25}$.

Coxeter Elements

Def: A **Coxeter element** in a Coxeter group is the product or its standard generating set in some order.

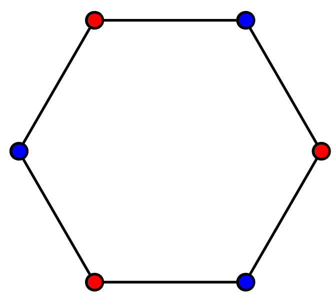
Thm: If the Dynkin diagram has no loops then all of its Coxeter elements are conjugate. In particular, Coxeter elements in finite Coxeter groups are well-defined up to conjugacy.

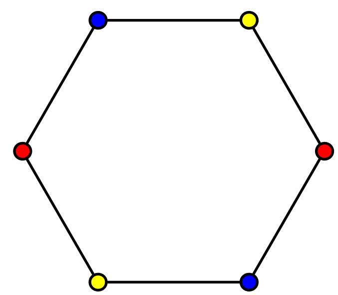
Rem: Coxeter elements are in bijection with acyclic orientations of the Dynkin diagram, and conjugacy classes of Coxeter elements are in bijection with equivalence classes of such orientations where the equivalence relation is generated by "reflection functors". (Closely related to quivers in representation theory)

Distinct Coxeter Elements

Ex: Consider the small type Coxeter group defined by a hexagon.

- there are $6! = 720$ orderings of the generators,
- but only $2^6 - 2 = 62$ different group elements,
- that fall into 5 distinct conjugacy clases.





Representatives are the 2 cyclically ordered Coxeter elements, the bipartite Coxeter element, and the 2 antipodal Coxeter elements.

Classification of Finite Simple Groups

Recall the classification theorem for finite simple groups.

Thm: Every finite simple groups is either

1. Cyclic ($\mathbb{Z}_p$, p prime),
2. Alternating (Alt_n, $n \geq 5$),
3. A finite group of Lie type,
4. One of 26 sporadic exceptions.

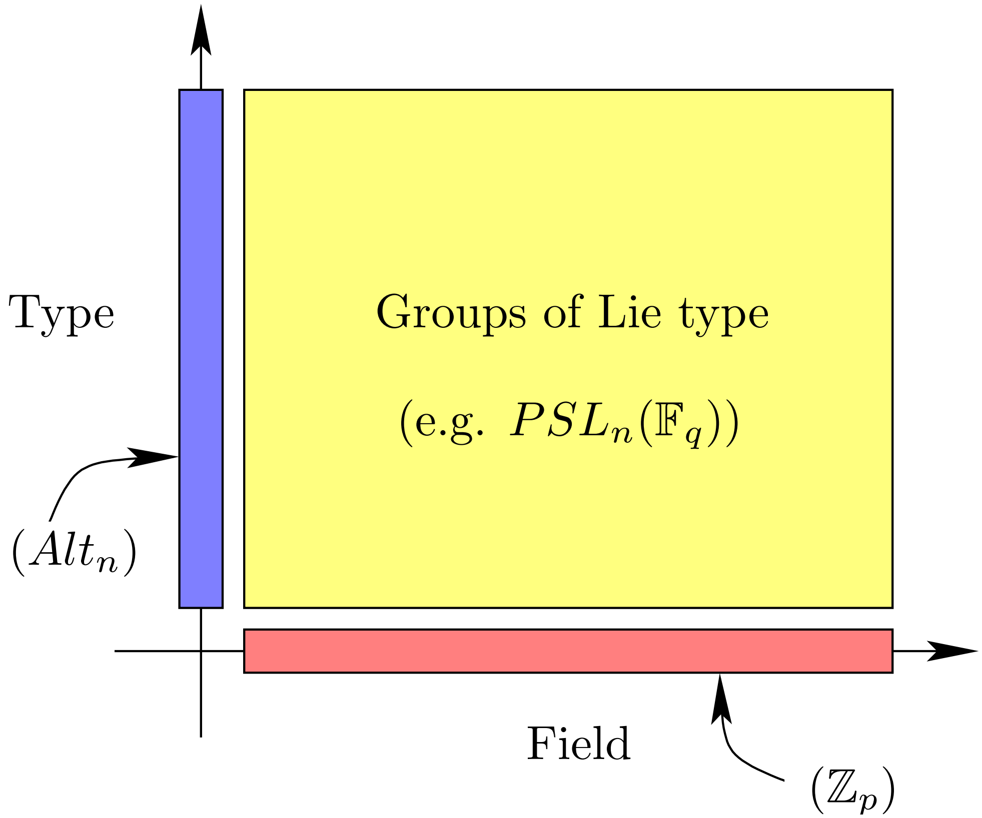
Finite Groups of Lie Type

The finite groups of Lie type are 16 infinite families of finite groups all of the form $X_n(q)$ where X_n is a Cartan-Killing type and q is a power of a prime. The X_n indicates the bilinear form and dictates the construction, and q is order of the finite field over which the construction is carried out.

$A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, $E_8(q)$, $E_7(q)$, $E_6(q)$, $F_4(q)$, $G_2(q)$.

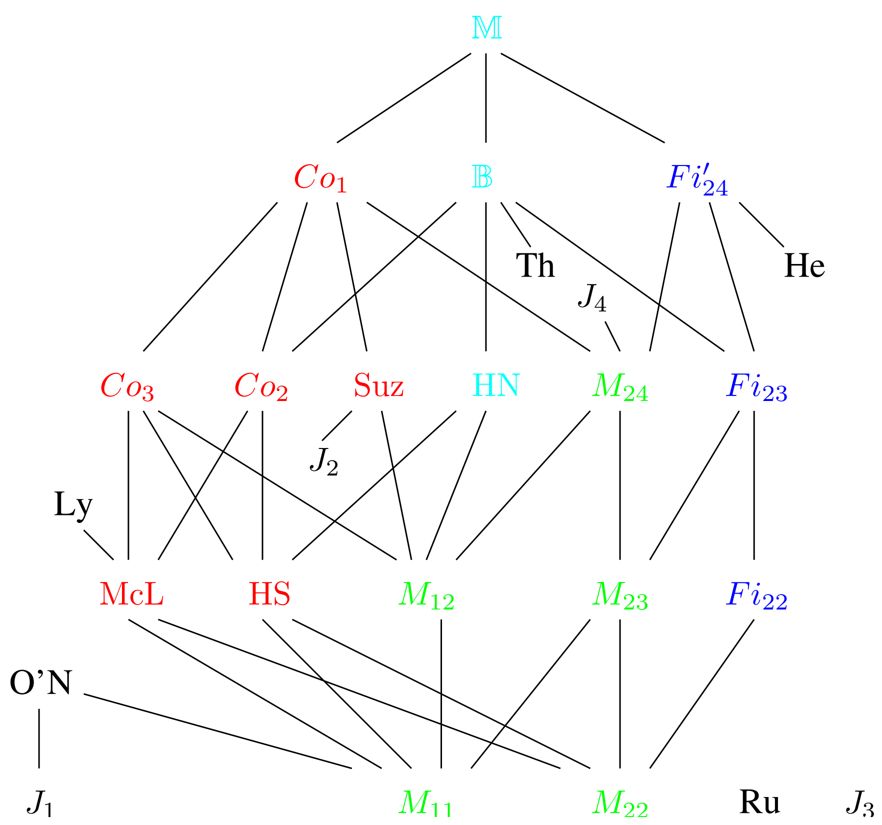
Type $A_n(q)$ comes from the automorphism groups of finite projective spaces over $\mathbb{F}_q$. In addition to these 9, diagram symmetries lead to twisted versions ($^2A_n(q)$, $^2D_n(q)$, $^3D_4(q)$, $^2E_6(q)$) including some ($^2B_2(q)$, $^2G_2(q)$, $^2F_4(q)$) whose construction is characteristic dependent.

A Mnemonic for the Classification



(plus 26 sporadic exceptions)

The Sporadic Finite Simple Groups



A line means that one is an image of a subgroup in the other.

The Monster Finite Simple Group

The **Monster** finite simple group $\mathbb{M}$ is the largest of the sporadic finite simple groups with order

$$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71$$

(which is 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000, or $\sim 10^{54}$)

The **Bimonster**, $\mathbb{M} \wr \mathbb{Z}_2$, is a related group of size $\sim 10^{108}$.

A One-relator Coxeter Presentation of the Bimonster

Thm: If W is the small type Coxeter group defined by $\mathrm{Inc}(\mathbb{F}_3 P^2)$, and $u \in W$ is the fourth power of the antipodal Coxeter element of a hexagonal subgraph in $\mathrm{Inc}(\mathbb{F}_3 P^2)$, then $W/\langle u \rangle \cong \mathbb{M} \wr \mathbb{Z}_2$.