Roots, Ratios and Ramanujan

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots}}}},$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}} \text{ and}$$

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Outline

I. Iteration
II. Fractions
III. Radicals
IV. Fractals
V. Conclusion

I. Iteration

Bill Thurston

[Story about Thurston at the 1999 Cornell Topology Festival]

Calculator Iteration

Let $\sin^{[n]}(x)$ denote $\sin(\sin(\sin(\cdots \sin(x))\cdots))$

Q: What is $\lim_{n\to\infty} \sin^{[n]}(x)$?

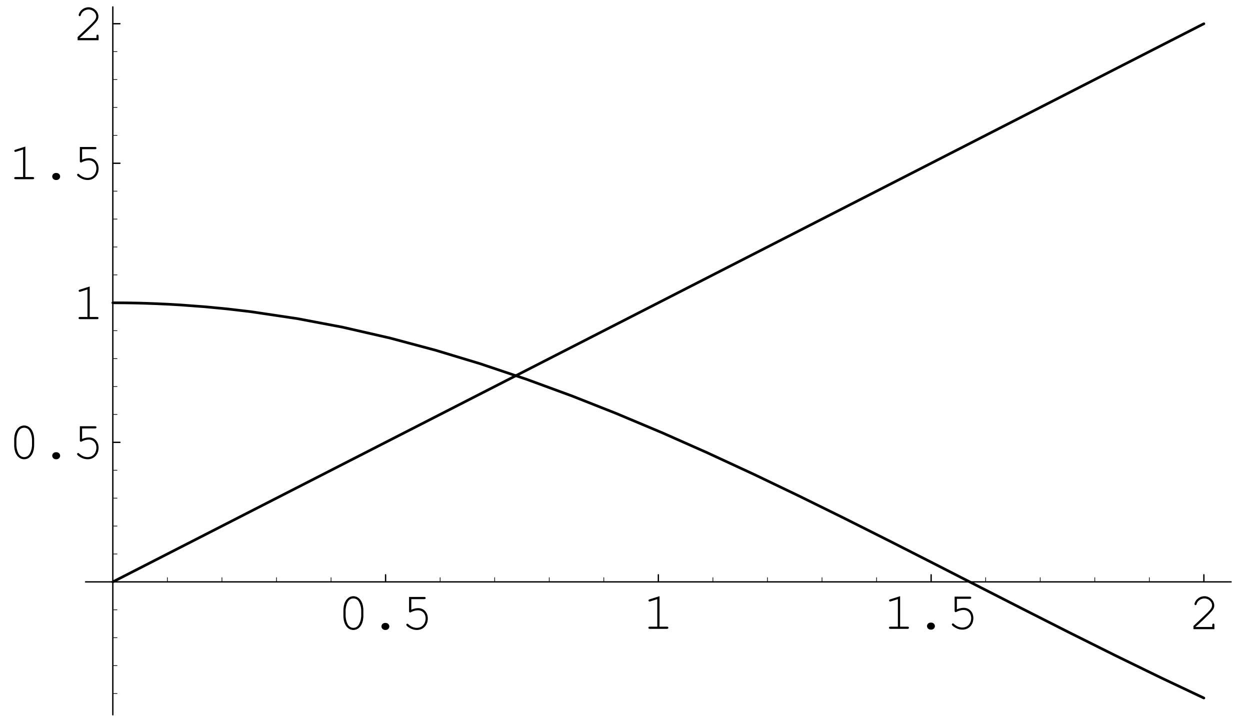
A: 0.

Q: What about $\lim_{n\to\infty} \cos^{[n]}(x)$?

A: 0.739085133215160641655312087 6... (independent of x!!!)

Why! [Actually it matters whether your calculator is in radian or degree mode]

A Pictorial Explanation



Towers

Q: If $x^{x^{x^{x^{x^{\text{etc}}}}}} = 2$ what is x?

A: First the only possible answers are the solutions to $x^2 = 2$. Thus we focus on $x = \pm\sqrt{2}$. Moreover, $x = -\sqrt{2}$ is questionable since $(-\sqrt{2})^{-\sqrt{2}}$ is usual left undefined! Even negative bases with rational exponents are dodgy:

$$-1 = \sqrt[3]{-1} = (-1)^{\frac{1}{3}} = (-1)^{\frac{2}{6}} = ((-1)^2)^{\frac{1}{6}} = \sqrt[6]{(-1)^2} = \sqrt[6]{1} = 1$$

Of course, we would need to decide what $x^{x^{x^{x^{x^{\text{etc}}}}}}$ means for $x = \sqrt{2}$ before we declare it a solution. Definitions matter.

II. Fractions

Freshman addition is defined as $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. This definition is surprisingly useful.

Start with $\frac{0}{1}$ and $\frac{1}{0}$ and repeatedly insert new fractions in the list.

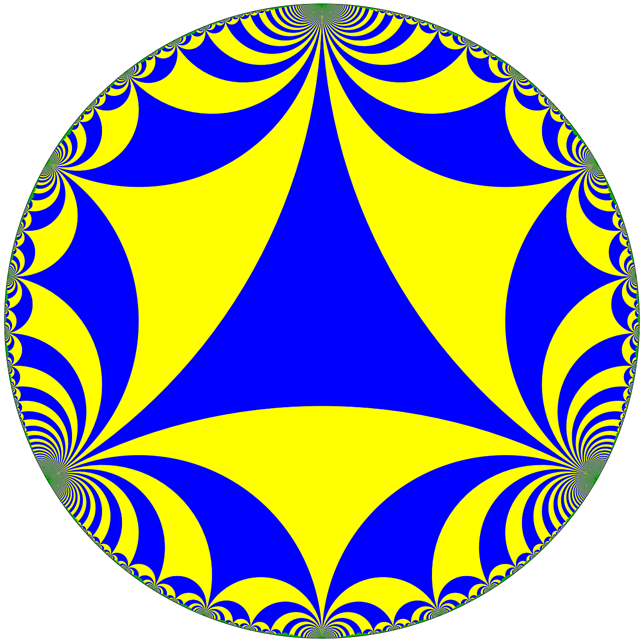
- They stay in increasing order
- They are always in least terms
- Every positive fraction eventually shows up

For example, after three iterations we have $\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}$

Farey Tiling

This pattern can be labeled so that the corners of the triangles are rationals in reduced form and the third corner is found by doing freshman addition or subtraction.

This is secretly a tiling of the hyperbolic plane.



Continued Fractions

Restricted continued fractions are those with positive integer denominators and unit numerators. The set of finite restricted continued fractions = the set of positive rational numbers (a fact that is intricately related to the Euclidean algorithm).

$$\frac{25}{7} = 3 + \frac{4}{7} = 3 + \frac{1}{7/4} = 3 + \frac{1}{1+\frac{3}{4}} = 3 + \frac{1}{1+\frac{1}{4/3}} = 3 + \frac{1}{1+\frac{1}{1+\frac{1}{3}}}$$

More generally every non-rational positive real number can be uniquely represented as

$$\pi = 3 + \frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{292+\frac{1}{1+\cdots}}}}} \qquad e = 2 + \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{4+\cdots}}}}}$$

Continued Fractions: General Definition

In general we let each a_i and b_i represent a complex number and consider

$$\frac{a_1}{b_1+\frac{a_2}{b_2+\frac{a_3}{b_3+\frac{a_4}{b_4+\cdots}}}}$$

Convergence is determined by looking at the limit of the sequence of finite answers:

$$\{\frac{a_1}{b_1}, \frac{a_1}{b_1+\frac{a_2}{b_2}}, \frac{a_1}{b_1+\frac{a_2}{b_2+\frac{a_3}{b_3}}}, \ldots\}$$

Repeating patterns converge to numbers that solve quadratic equations.

Continued Fractions: Examples

Special Numbers and Entire Functions:

$$\pi = \cfrac{4}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cdots}}}}} \qquad e = 2 + \cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\cfrac{6}{6+\cdots}}}}}$$

$$\tan^{-1}(x) = \cfrac{x}{1+\cfrac{x^2}{3+\cfrac{(2x)^2}{5+\cfrac{(3x)^2}{7+\cfrac{(4x)^2}{9+\cdots}}}}} \qquad \tan(x) = \cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{x^2}{7-\cfrac{x^2}{9-\cdots}}}}}$$

In general, a nice power series implies a nice continued fraction. (These last two are convergent and true for every complex x where the function is defined.)

Ramanujan's Continued Fraction

If $u := \cfrac{x}{1+\cfrac{x^5}{1+\cfrac{x^{10}}{1+\cfrac{x^{15}}{1+\cfrac{x^{20}}{1+\cdots}}}}}$ and $v := \cfrac{\sqrt[5]{x}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\cfrac{x^4}{1+\cdots}}}}}$ then

$$v^5 = u \cdot \frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}$$

About this and two other continued fraction evaluations Hardy wrote "A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because if they were not true, no one would have had the imagination to invent them."

III. Continued Radicals

Q: Let $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}}$.
What is x?
(Convergence is based on the sequence $\sqrt{a_0}, \sqrt{a_0 + \sqrt{a_1}}, \ldots.$)

A: As before the only possibility is where $x = \sqrt{2 + x}$, so $x^2 = 2 + x$ and $x^2 - x - 2 = (x - 2)(x + 1) = 0$. Thus $x = 2$ or $x = -1$. Since $x = -1$ doesn't make sense the only reasonable answer is $x = 2$.

Now the hard work begins. The approximations are strictly increasing, they stay below 2, but they get arbitrarily close. Done.

Continued Radicals

For $a > 0$ let $r(a) = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}}}$.

The only possibility for $r(a)$ is the solution of $x = \sqrt{a + x}$ which implies that $x^2 - x - a = 0$. Thus $x = \frac{1 \pm \sqrt{1+4a}}{2}$ and, in fact, only the plus sign is possible.

Ex: Golden ratio $= \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}}$

Ex: $2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}}}$.

Iterated Square Roots with Signs

What about $x = \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \cdots}}}}}}$

where the signs are chosen once and for all.

If the signs repeat $+--$ forever, then $x = 2\cos(\pi/7)$! In other words

$$2\cos(\pi/7) = \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \cdots}}}}}}}$$

If the signs repeat $+-$ forever, then $x = 2\cos(\pi/5)$.
What's going on here?

Main Theorem

Thm: Let $\epsilon : \mathbb{N} \to \{\pm 1\}$ be a fixed sequence of signs and abbreviate $\epsilon(i)$ as ϵ_i.

- For every sequence $\epsilon_0\sqrt{2+\epsilon_1\sqrt{2+\epsilon_2\sqrt{2+\epsilon_3\sqrt{\cdots}}}}$ converges.

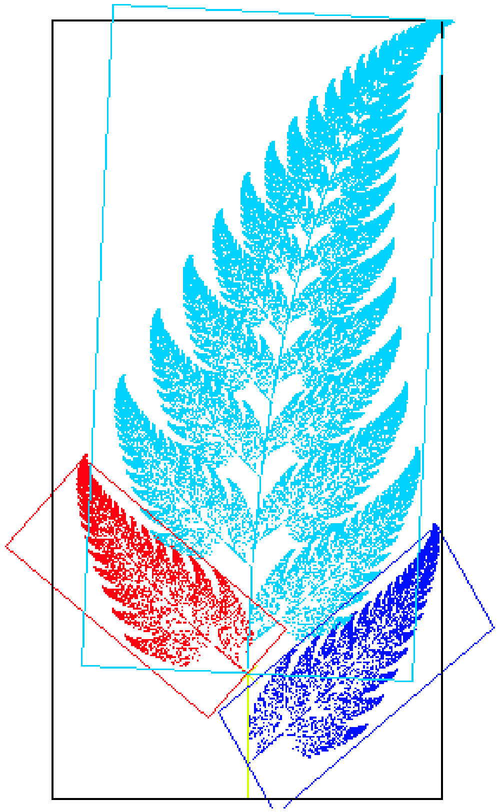
- It converges to a real number in $[-2,2]$.
- Every $x \in [-2,2]$ is the answer for some set of signs.
- Every $x \in [-2,2]$ is the answer for at most 2 sets of signs.
- The signs eventually repeat iff $x = 2\cos(a\pi)$ for some $a \in \mathbb{Q}$.

- Finally, and most surprisingly, these claims heavily depend on the fact that we are using 2's under the square root.

IV. Fractals

A fractal is an object that has some type of self-similarity. Consider a fern.

This picture was produced with iterated linear transformations. It's not a real fern.



Iterating Functions

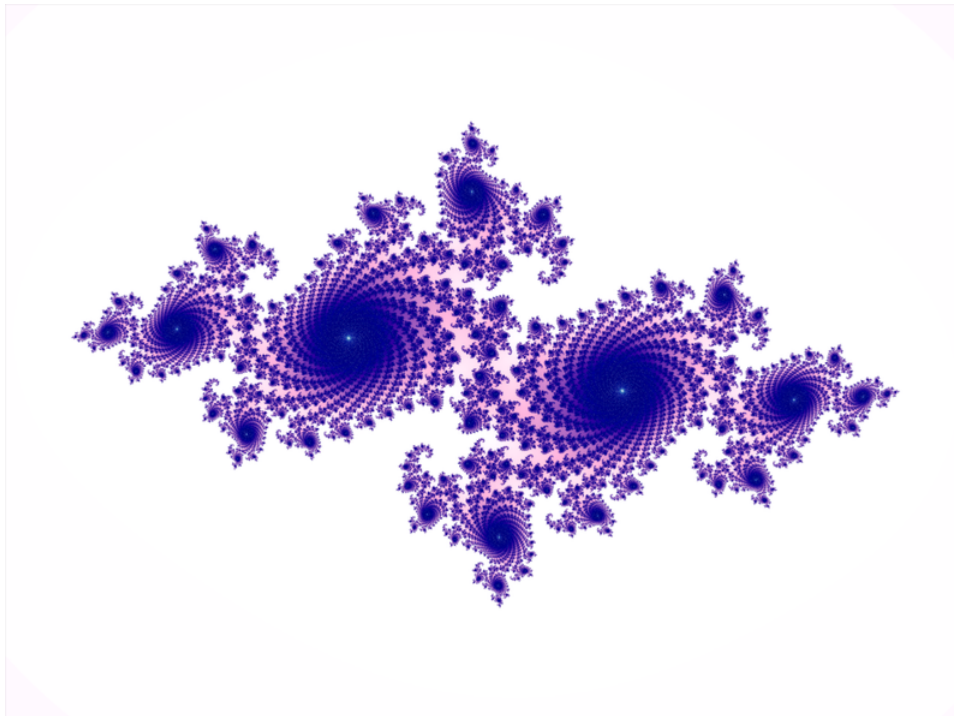
What happens when you iterate the function $f_c(z) = z^2 + c$ where c is a fixed complex number.

The **Mandelbrot set** $\mathbb{M}$ is the set of $c \in \mathbb{C}$ where the forward iterates of 0, $\{0, f_c(0), f_c(f_c(0)), \ldots\}$, stay bounded.

The **Julia set for** c in $\mathbb{M}$ is the set of starting points whose forward iterates stay bounded.

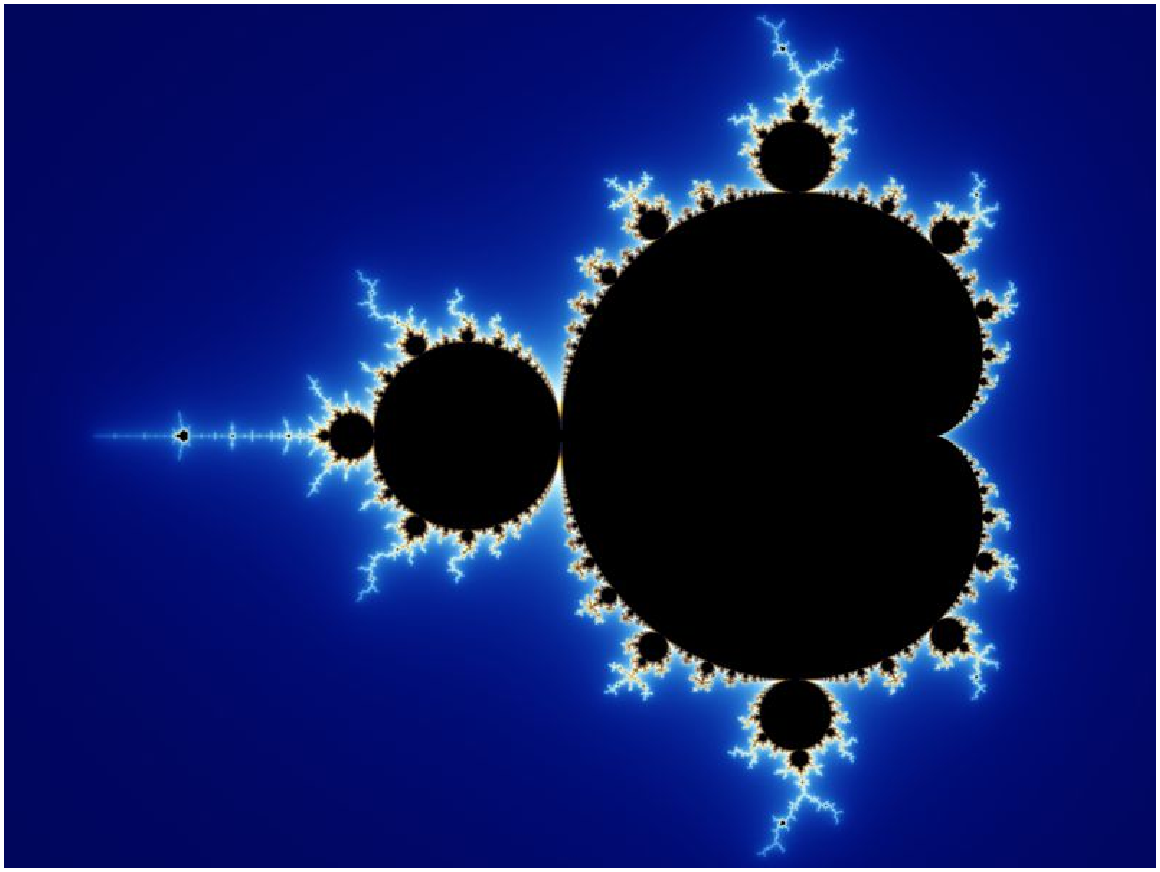
This makes the Mandelbrot set something like an index for the Julia sets. [The next several images are from Wikipedia]

Julia Sets

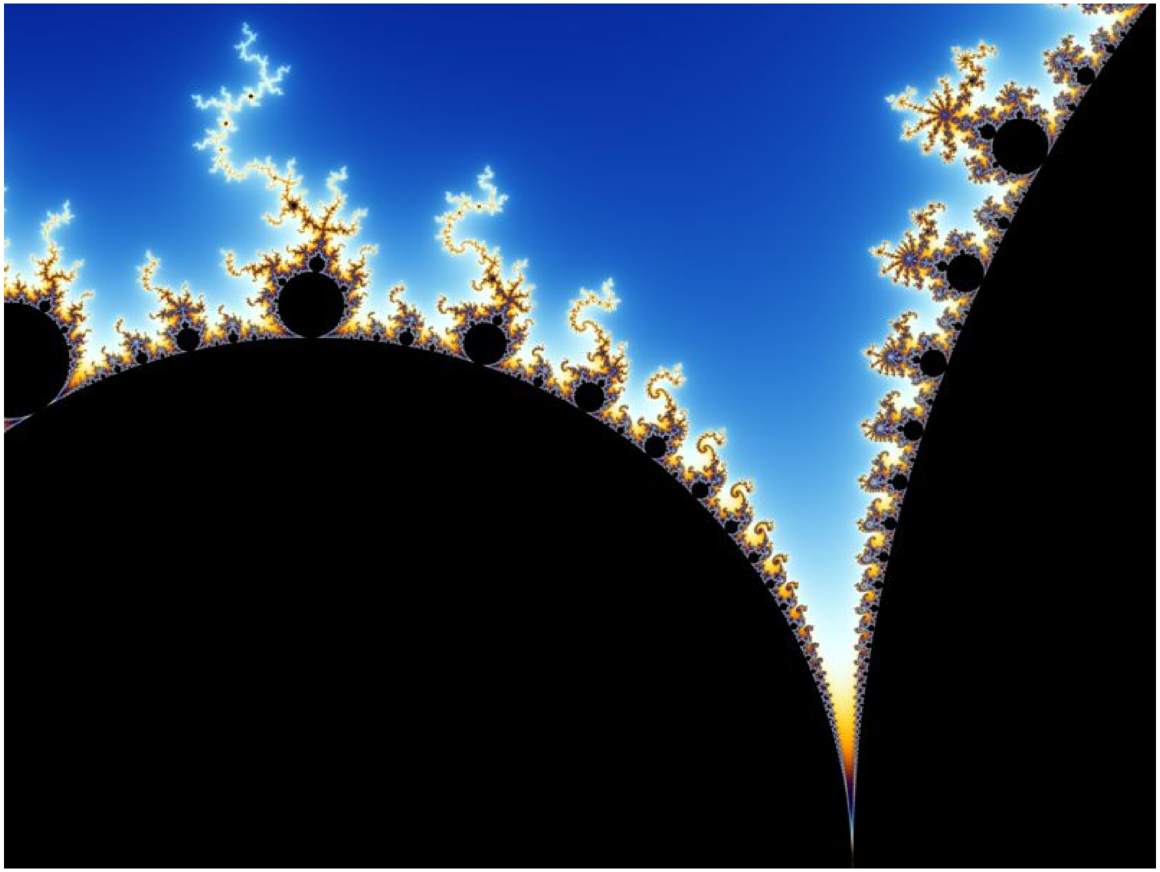


Julia set for $c = -(0.7268953477\ldots) + (0.188887129\ldots)i$.

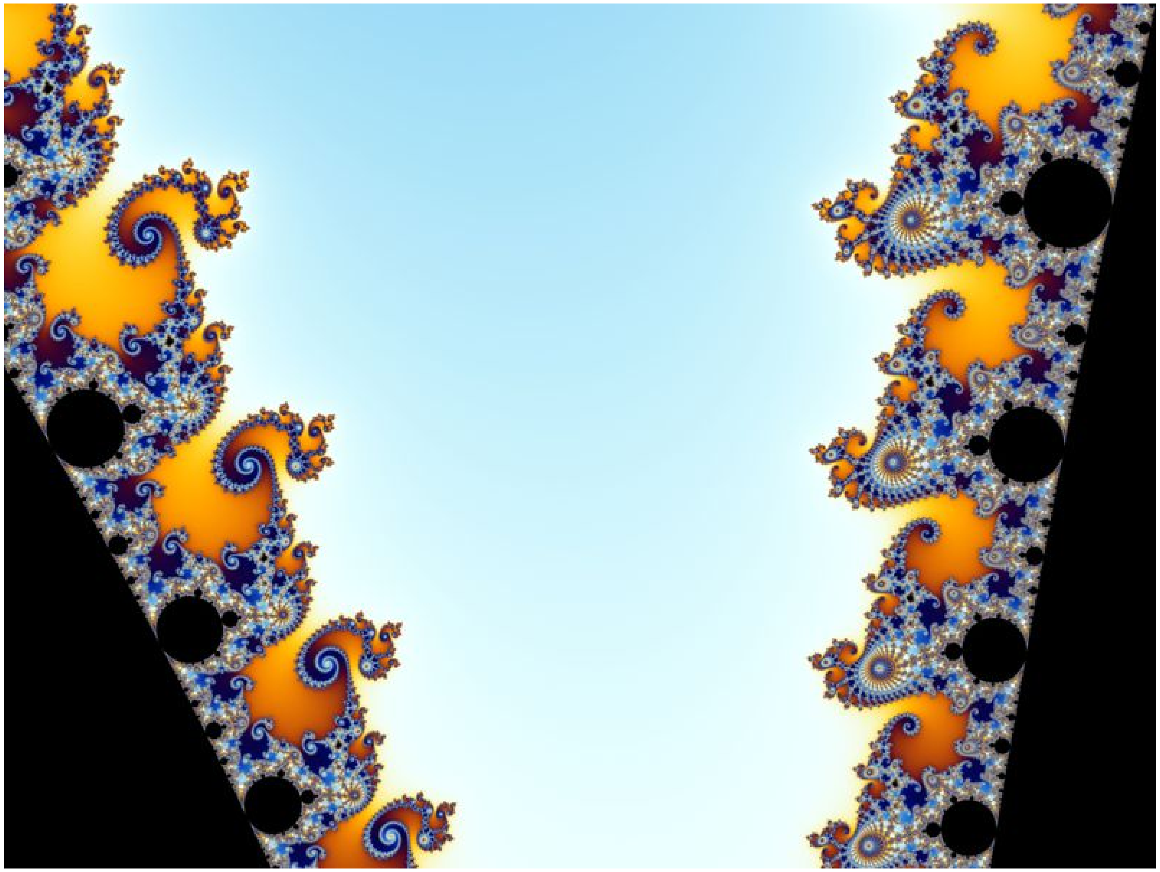
Mandelbrot Set



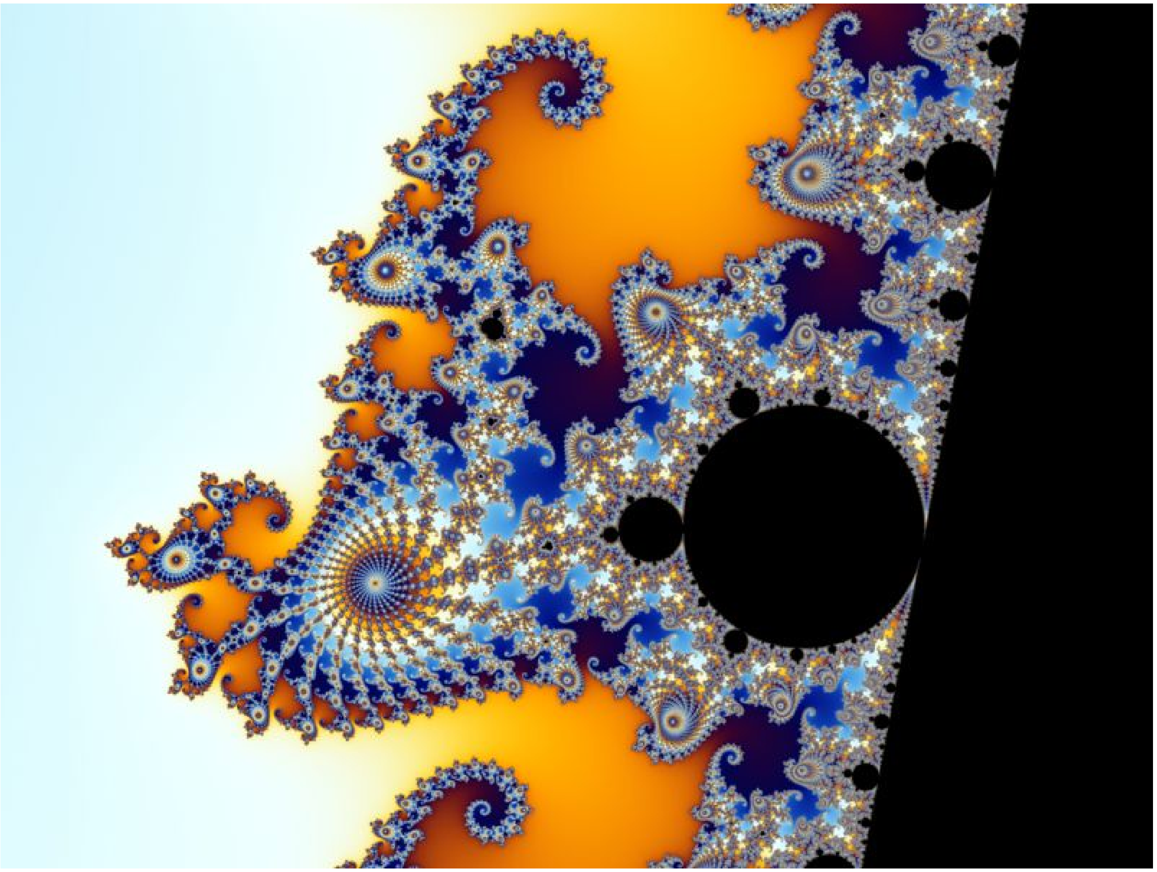
Mandelbrot Set: Zoom 1



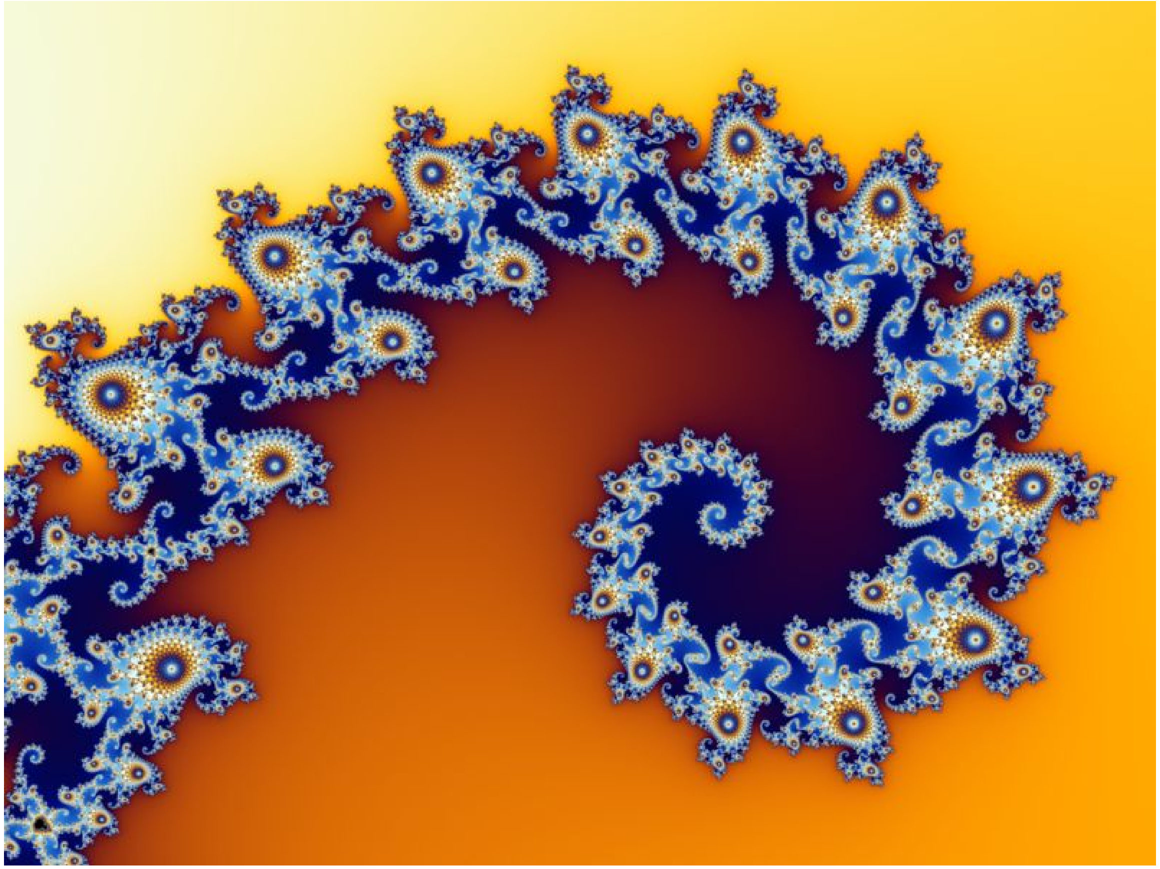
Mandelbrot Set: Zoom 2



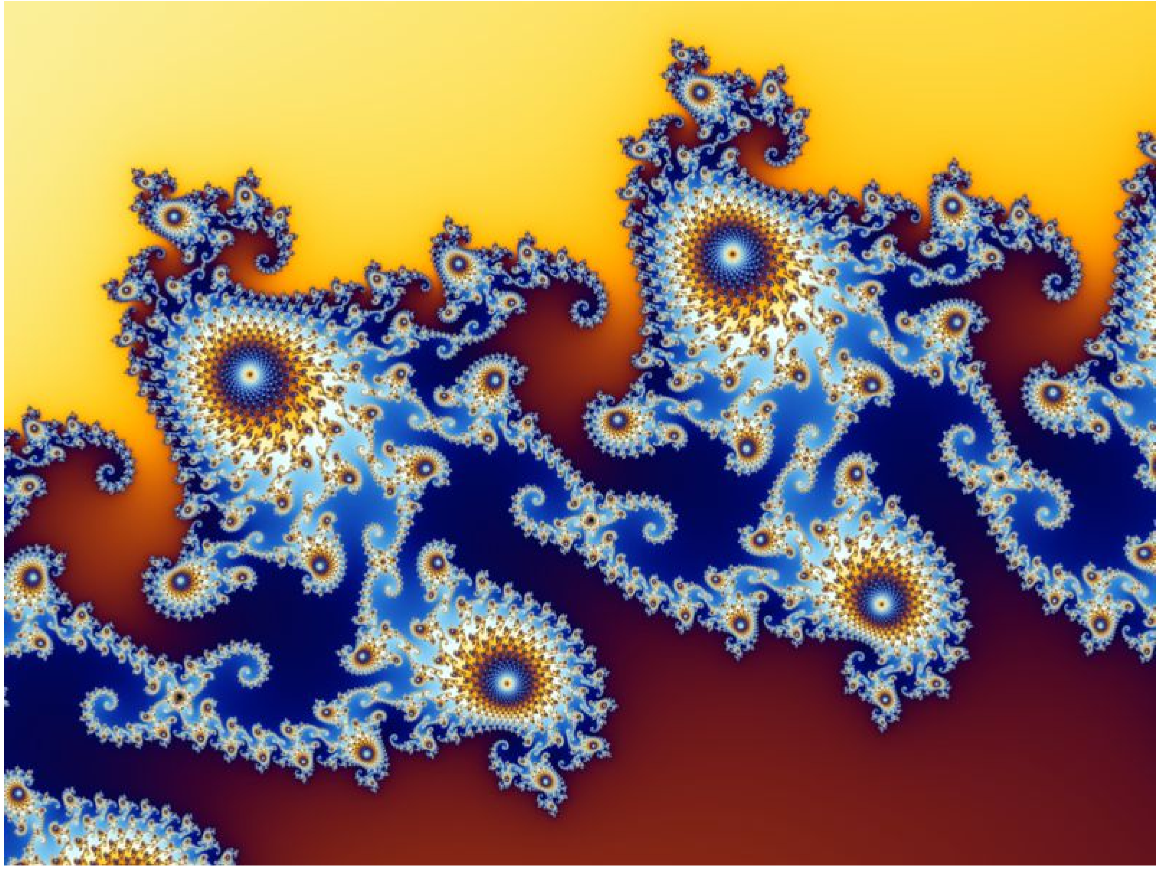
Mandelbrot Set: Zoom 3



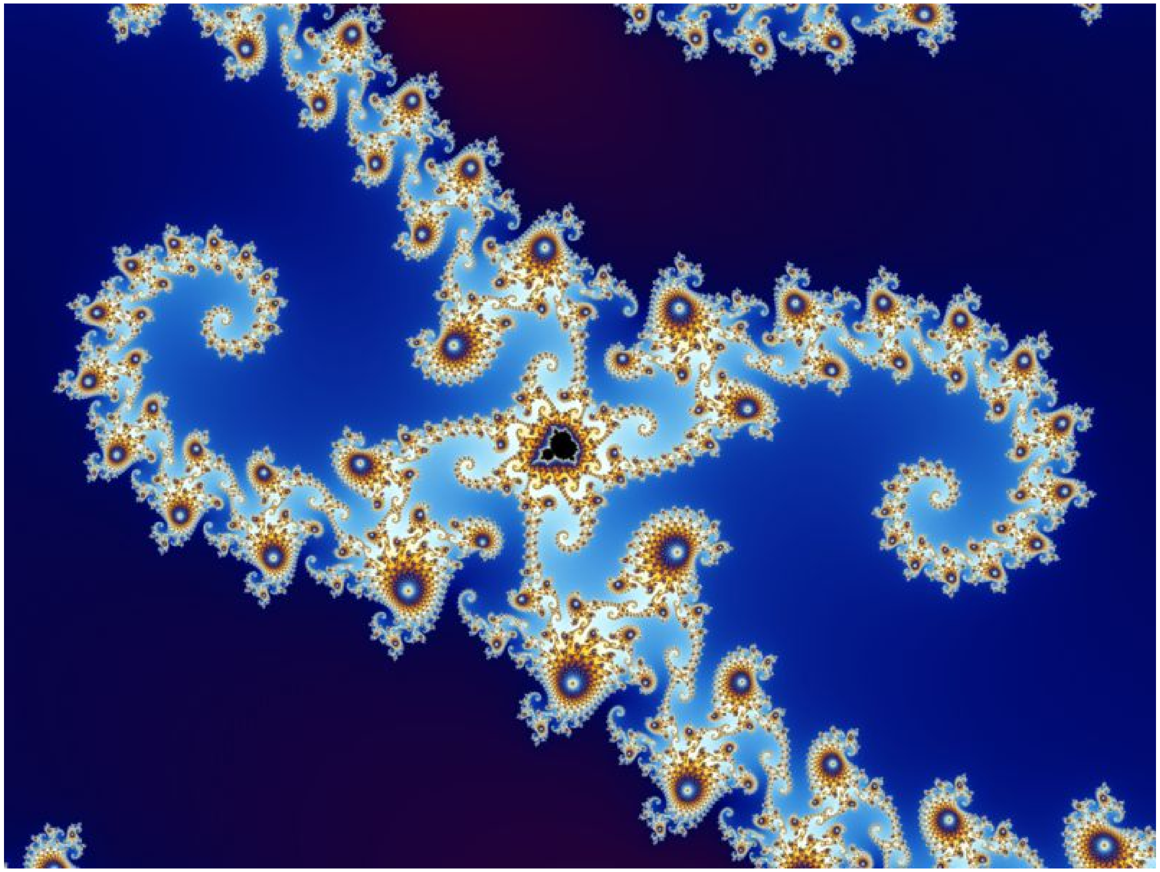
Mandelbrot Set: Zoom 4



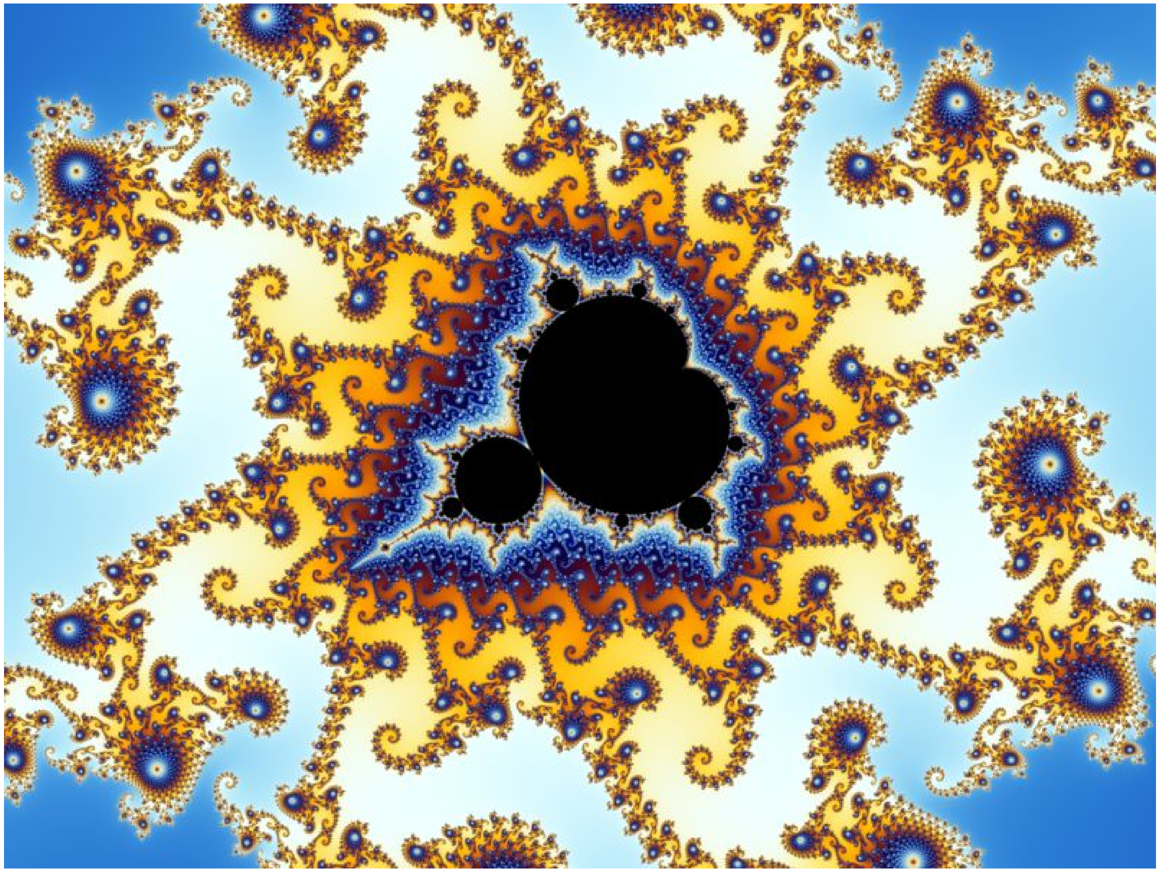
Mandelbrot Set: Zoom 5



Mandelbrot Set: Zoom 6



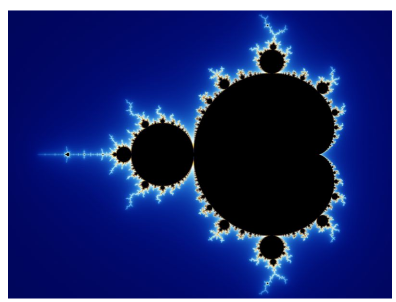
Mandelbrot Set: Zoom 7



IV. Conclusion: Mandelbrot, Radicals and Ramanujan

A double angle formula for cosine is $\cos 2\theta = 2\cos^2\theta - 1$. This can be rewritten as $2 + (2\cos 2\theta) = (2\cos\theta)^2$. For a fixed θ, let $x_0 = 2\cos\theta$, $x_1 = 2\cos 2\theta$, $x_2 = 2\cos 4\theta$, $x_3 = 2\cos 8\theta$, $x_i = 2\cos(2^i\theta)$.

By the double angle formula, $2 + x_{i+1} = (x_i)^2$. In other words, $x_{i+1} = (x_i)^2 - 2 = f_{-2}(x_i)$, and x_i's are the sequence of forward iterates of the function $f_{-2}(x) = x^2 - 2$. This is known as the tip of the Mandelbrot set and f_{-2} is called the doubling map.



Connections

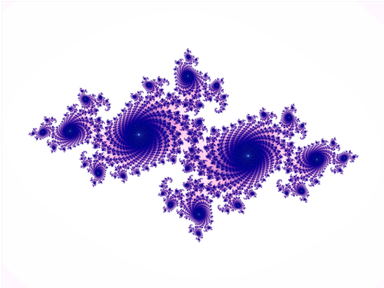
There are strong connections between:

- the double angle formula for cosine,
- the doubling map $x \mapsto x^2 - 2$,
- its (multivalued) inverse $x_i = \pm\sqrt{2 + x_{i+1}}$, and
- binary decimal expansions for real numbers

An Explanation

The explanation of the Main Theorem is that iterated square roots with 2's under the radical are closely connected to iterates of the doubling map, and the Julia set for the doubling map, the map indexed by the tip of the Mandelbrot set, is the interval $[-2, 2]$.

If we replace 2 by another number c in the Mandelbrot set, we get convergence to essentially arbitrary point in ITS Julia set.



Ramanujan and Continued Radicals

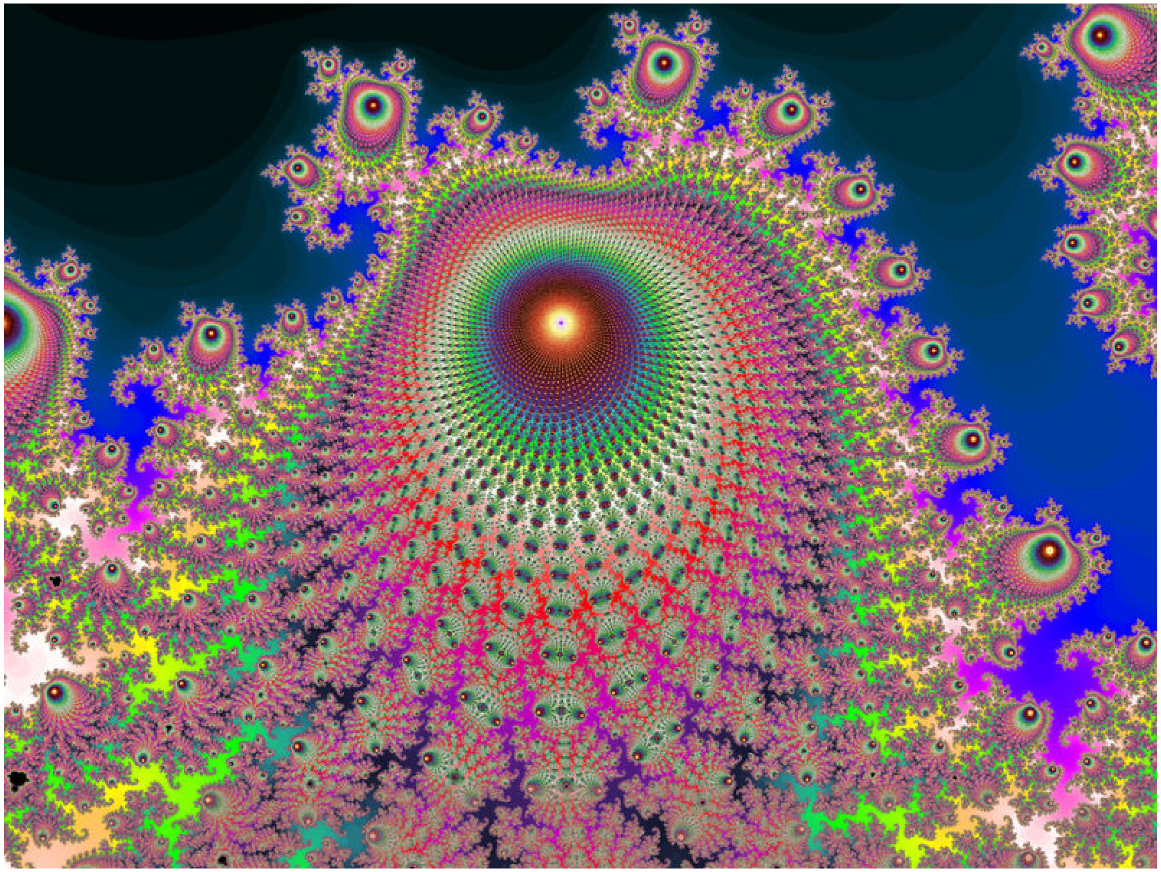
In Chapter 12 in his notebooks, Ramanujan claims that

$$2\cos\theta = (2+2\cos 2\theta)^{1/2} = (2+(2+2\cos 4\theta)^{1/2})^{1/2}$$
$$= (2+(2+(2+2\cos 8\theta)^{1/2})^{1/2})^{1/2} = \cdots$$

Bruce Berndt comments "We state Entry 5(i) as Ramanujan records it. But, as we shall see, Entry 5(i) is valid only for $\theta = 0$. [...] We suggest to the readers that they attempt to develop more thoroughly the theory of infinite radicals"

If Ramanujan's $+$'s are replaced by appropriate $+$ or $-$ signs, what he wrote is correct for every θ and connected to some very interesting mathematics.

Another Corner of the Mandelbrot Set



(Thank you for your attention)