Part V
Consequences

Part V is devoted to proving various consequences of the axioms for general small cancellation presentations. Section 12 shows that the Cayley category of a general small cancellation presentation is contractible. Section 13 represents a detour into an area which is useful in the construction of general relators and general small cancellation presentations. In this section it is shown that a given $\mathcal{R}$-category which is $(1-3\alpha-2\epsilon)$-closed can be embedded in another $\mathcal{R}$-category which is α-closed in a unique and minimal way. Finally, in Section 14 the finite subgroups of a general small cancellation presentation are shown to be contained within the automorphism groups of the general relators.

12 Cayley Categories Revisited

In this section the Cayley category of a general small cancellation is shown to be proper and contractible. Then using these results the torsion elements in the group are shown to occur only as automorphisms of general relators. At the end of the section the conditions under which the Poincaré construction of a general small cancellation presentation is an Eilenberg-MacLane space are described.

12.1 Cayley Categories are Proper

The next several lemmas show that Cayley categories are proper, and they provide conditions under which $\mathcal{R}$-categories can be embedded into Cayley categories.

Lemma 12.1 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{8}$, and let C be its Cayley category. If B is an $\mathcal{R}$-category which is connected, collapsed and $(1-3\alpha)$-closed with respect to $\mathcal{R}$, and $f : B \to C$ is an $\mathcal{R}$-functor, then f is an embedding of B in C and B is simply connected.*

Proof: Let W be a path in B between vertices u and v. The restriction on α guarantees that $1-3\alpha \geq \frac{1}{2}+\alpha$. Since B is $(1-3\alpha)$-closed, by Lemma 10.4 W can be assumed to be $(1-3\alpha)$-free. If u and v are sent to the same vertex in C, then by Lemma 8.11 W is equivalent to 1 in G. Since W is also $(1-3\alpha)$-free, by Lemma 10.18 W is the empty word and u and v are identical in B. Thus f is injective on vertices, and by Lemma 6.5, it is an embedding.

Since by Lemma 4.7 every topological loop based at a vertex is homotopic to a loop in the 1-skeleton, to show B is simply connected it is sufficient to show that each loop in the 1-skeleton of B is contractible to a point. Let W be a loop in B based at v. Since B is embedded in C, W is a loop in C, and by Lemma 8.11 it is equivalent to 1. Thus by Lemma 10.4 and Lemma 10.18, the loop W is homotopic in B to the empty loop based at v. This shows that B is simply connected. □

Lemma 12.2 *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation, and let C be its Cayley category. If B is a connected, collapsed $\mathcal{R}$-category which is $(1-3\alpha)$-closed with respect to $\mathcal{R}$, then the universal cover of B can be embedded in C. As a special case, note that when W is a word which is $(1-3\alpha)$-free with respect to $\mathcal{R}$, then it is read in C as a simple path.*

Proof: Let B' be the universal cover of B. Since B' is an $\mathcal{R}$-category, by Lemma 6.8 there is a functor from B' to the Poincaré construction of the presentation, and since by definition it is simply connected, this functor lifts to a functor $f : B' \to C$ by Lemma 1.13. Since B' is connected, simply connected, collapsed, and $(1-3\alpha)$-closed, by Lemma 12.1 the functor f is an embedding. The final statement is immediate. $\Box$

Lemma 12.3 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. The Cayley category of the presentation is proper, meaning that the underlying circular category is also a circular complex.*

Proof: To prove that the Cayley category is proper, it is sufficient to show that every characteristic functor is injective on vertices by Lemma 3.1. Let C/c be a particular closed cone of C which is isomorphic with a general relator $R \in \mathcal{R}$, and let $\phi_c : C/c \to C$ be its characteristic functor. Notice that C/c is collapsed, connected and α-closed by Lemma 9.8, by the definition of circular cones, and by Axiom 1, respectively. Since $\alpha < 1 - 3\alpha$ for all general small cancellation presentations, Lemma 12.1 is applicable, and it shows that ϕ_c is an embedding and is consequently injective on vertices. $\Box$

12.2 Torsion Elements

The next several lemmas show that torsion elements correspond to automorphisms of general relators.

Lemma 12.4 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation and a general relator $R \in \mathcal{R}$ has a non-trivial automorphism, then G contains a torsion element.*

Proof: Suppose R is a general relator which has a non-trivial automorphism $f : R \to R$. If u is a vertex in R and $f(u) = v$, then u and v must be distinct by Lemma 6.3. Let W be a path in R from u to v. Since f is an automorphism, W^i is readable in R starting at u for all natural numbers i. Since R is finite, there is an n such that W^n is a loop in R. Using an attaching functor for R, W^n is also a loop in the Cayley category of the presentation, and thus by Lemma 8.10 and Lemma 4.5 it must be equivalent to 1 in G. However, since the Cayley category is proper by Lemma 12.3, W itself is not a loop, and by Lemma 8.10 and Lemma 4.5 again, W is not equivalent to 1. Thus W is a torsion element of G. $\Box$

Lemma 12.5 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. If X^n is readable as a loop with non-trivial winding number in a general relator*

$R \in \mathcal{R}$ *for some* $n > 1$, *but* X *itself is not readable as a loop in* R, *then the path* X *induces a non-trivial automorphism of* R.

Proof: Since X^n is a loop with non-trivial winding number, by one of the defining properties of a relator metric, $d_R(X^n) \geq 1 \geq \alpha$. Thus by Axiom 1 and Lemma 2.1 there is an automorphism of R which sends the initial vertex of the path X to its terminal vertex. $\square$

If X is a path in R such as is described in the statement of the lemma, then X is called a rotation of R, and the automorphism represented by X is said to rotate R.

Lemma 12.6 *Let* $G = \langle A|\mathcal{R}\rangle$ *be a general small cancellation presentation. If* Y *is a word of order* n, $1 < n < \infty$ *in* G, *which is cyclically reduced in the free group and whose cycle is* $(1-4\alpha)$*-free with respect to* $\mathcal{R}$, *then there exists a general relator* $R \in \mathcal{R}$ *which is rotated by* Y.

Proof: Since $Y^n = 1$ in G and cyclically reduced, by Lemma 10.15 there exists a connected and simply connected $\mathcal{R}$-diagram Δ with boundary cycle Y^n over a $C(6)$-map. Let Δ be chosen so that it is a diagram of minimal rank having these properties. By Lemma 10.17 Δ contains an exposed 2-cell D representing a general relator R with internal degree at most 3. Let U be the path which reads the unique boundary arc of D. Since each of the internal arcs has length less than α, $d_R(U) > 1-3\alpha$ by Lemma 5.18. Because the cycle of Y is $(1-4\alpha)$-free by assumption, $|U| > |Y|$. Let $U = U_1U_2$ where U_1 is the initial segment of U of length $|Y|$. Since $d_R(U) \leq d_R(U_1) + d_R(U_2)$ by the definition of the relator metric d_R, and since $d_R(U_1) < 1-4\alpha$ by assumption, it must be the case that $d_R(U_2) > \alpha$.

If U is viewed as a path in the loop Y^n, then U_2 can be viewed as the overlap between this path and another path reading U obtained by shifting the start vertex of the path $|Y|$ edges. Since each path individually is readable in R and the overlap U_2 has $d_R(U_2) > \alpha$, by Axiom 1 and Lemma 2.1 the union of the two paths is readable in R. When the path is shifted $|Y|$ more edges, the overlap is again long relative to R, and the reading of the word Y^n can be continued. In this way an arbitrarily long word periodic in Y can be read in the relator R. Since R is finite, Y^i must be a loop for some integer i, and in particular, since R embeds into the Cayley category of the presentation by Lemma 12.3, Y^n must be a loop in R. If the winding number of Y^n is 0 then by Lemma 7.15 there is a connected and simply connected $\mathcal{R}$-diagram with boundary Y^n of rank less than the rank of R, contradicting the choice of Δ. Thus the winding number of Y^n is non-zero. $\square$

Lemma 12.7 *Let* $G = \langle A|\mathcal{R}\rangle$ *be a general small cancellation presentation with* $\alpha \leq \frac{1}{10}$. *If* X *is a word of order* n, $1 < n < \infty$, *then* X *is conjugate in* G *to a word* Y *such that a power of* Y, *namely* Y^n, *is readable as a loop with non-zero winding number in one of the general relators* $R \in \mathcal{R}$, *and such that* Y *is cyclically reduced in the free group and the cycle of* Y *is* $(\frac{1}{2}+\alpha)$*-free with respect* *to $\mathcal{R}$. Moreover, the general relator R is uniquely determined by the word X, and the word Y represents a non-trivial automorphism of R.*

Proof: By Lemma 10.5 the cycle X is conjugate in G to a cycle Y which is cyclically reduced in the free group and $(\frac{1}{2}+\alpha)$-free with respect to $\mathcal{R}$. Because X and Y are conjugate in G, they have the same order. Also, since $\alpha \leq \frac{1}{10}$, the cycle Y is also $(1-4\alpha)$-free. Lemma 12.6 can now be applied to show that Y^n is readable as a loop with non-zero winding number in a general relator R. Next, by Lemma 9.11, the general relator R for which this is true is unique. And finally, by Lemma 12.5, Y represents a non-trivial automorphism of R. □

Since Lemma 12.7 guarantees that the general relator rotated by an element of G is unique, any rank function which turns $G = \langle A|\mathcal{R}\rangle$ into a graded presentation also provides a rank function which assigns a finite number to every automorphism of finite order based on the rank of the general relator which it rotates.

Lemma 12.8 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation with $\alpha \leq \frac{1}{10}$, then G is torsion-free iff all of the general relators in $\mathcal{R}$ have no non-trivial automorphisms.*

Proof: The result is a combination of Lemma 12.4 and Lemma 12.7. □

12.3 Cayley Categories are Contractible

The lemmas below show that the Cayley category of a general small cancellation presentation is always contractible as a topological space. The heart of the proof is contained in the rather long proof of Lemma 12.11. A quick sketch of the argument goes as follows. Given an arbitrary continuous function from a 2-sphere to the geometric realization of an $\mathcal{R}$-category C, the Simplicial Approximation Theorem can be applied to a subdivision of C to yield a simplicial map. After a sequence of technical intermediary steps, the focus devolves upon $\mathcal{R}$-functors from $\mathcal{R}$-spheres to C, at which point it is possible to use general small cancellation theory to show that this continuous map is contractible to a point, the original map is contractible to a point, and thus the second homotopy group of C is trivial. Lemma 12.10 establishes this important special case. More generally, the same results hold when the Cayley category is replaced by an arbitrary connected and simply connected subcategory of the Cayley category.

Lemma 12.9 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let Δ be an $\mathcal{R}$-sphere. If the boundary cycle of each 2-cell in Δ is cyclically reduced, then Δ must contain an arc which is long with respect to one of the 2-cells which it borders.*

Proof: Let D be one of the 2-cells in Δ. When the interior of D is removed the result is a planar, connected and simply connected $\mathcal{R}$-diagram. If this planar diagram is a $C(7)$-map then by Lemma 10.17 it contains an exposed cell D'

whose unique boundary arc measures more than $1-3\alpha$ in its relator metric. In the latter case, the unique boundary arc of D' forms a long arc between D and D' in the original $\mathcal{R}$-sphere. If, on the other hand, it is not a $C(7)$-map, then since the boundary cycles of the 2-cells are cyclically reduced, by Lemma 7.4 there must be an internal arc which is long relative to one of the 2-cells which it borders. □

Lemma 12.10 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, let C be the Cayley category, let Δ be an $\mathcal{R}$-sphere, and let f be a continuous function from Δ to C. If f is a label-preserving map when restricted to the 1-skeleton of the $\mathcal{R}$-sphere, and f sends every 2-cell into an instance of the general relator which the cell represents, then f is homotopic to a constant map.*

Proof: Suppose that a subword of the form aa^{-1} can be read in the boundary of some 2-cell D in Δ. If the vertex in the middle of the subword has degree 2 in Δ then the entire subword can be identified to a point in Δ. If the vertex in the middle has a degree greater than 2 and the start and end vertices of the subword are distinct in Δ then the two edges can be identified so that only the boundary cycle of D is altered. If the vertex in the middle of the subword has a degree greater than 2 and the start and end vertices of the subword are not distinct then the identification of the two edges creates two spheres which are identified along an edge. Up to homotopy type this can be modified so that the spheres are identified at a point. In all three cases the result is a modification of the original $\mathcal{R}$-sphere and of the original map f. In addition, it is clear that the new map is homotopic to a constant map iff the old map is homotopic to a constant map. Since the new configuration also has a lower type, these operations can be repeated until no such subwords occur. Thus the boundary cycles of the 2-cells in Δ can be assumed to be cyclically reduced without loss of generality. Lemma 12.9 can now be applied to show that there exists a 2-cell D in Δ which must contain a long arc.

The remainder of the proof is by induction on the type of Δ. To begin, consider a sphere with at most two 2-cells. Since the cases of spheres with only one 2-cell, or two 2-cells with different ranks, are easily seen to be impossible, assume that Δ contains two 2-cells representing relators of the same rank. Since the ranks are the same, they form a cancellable pair and they represent the same general relator R. In this case, the assumptions about f guarantee that the image of Δ is contained in an instance of R in C. The general relator R is embedded in C by Lemma 12.3. Since R is contractible, the result follows.

Next suppose that the lemma has been shown for all $\mathcal{R}$-spheres of strictly lower type, and that Δ contains at least three 2-cells. In this case, there must be a 2-cell in Δ which does not border the long arc. If the interior of this 2-cell is removed then the remainder of the sphere can be viewed as a connected and simply connected $\mathcal{R}$-diagram in the plane. The modifications used in Section 10 to remove long interior arcs can then be mimicked in the $\mathcal{R}$-sphere Δ. When the arc in question is negative relative to the boundary cycle of D, for example, then the process described for removing it in Lemma 10.11 was divided into three separate cases. The first two cases create new $\mathcal{R}$-sphere structures of strictly lower type. In the third case a one-point product of two lower-type $\mathcal{R}$-spheres is created. In each case it is easy to see how to create modified functions to go with the modified configurations, so that the new functions are homotopic to a constant map iff the old function is homotopic to a constant map as well.

If the long arc is between two 2-cells of different ranks, then the arc can simply be erased. This creates a new $\mathcal{R}$-sphere structure of strictly lower type, but without changing the function f. If the long arc is between two 2-cells of the same rank then these form a cancellable pair in Δ. The removal of the cancellable pair creates a one-point product of $\mathcal{R}$-spheres of lower type. Once again it is easy to see how to create modified functions to go with the modified configurations, so that the new functions are homotopic to a constant map iff the old function is homotopic to a constant map as well.

Finally, if the long arc borders D in two distinct ways, then because both instances are sent by f to the same path in the general relator which D represents, the orientations of the two instances must be opposite with respect to the boundary of D. In particular, this shows that no long arcs bordering the same cell in two distinct ways with both orientations positive can occur in this context. Thus, a long arc which borders the same cell in two distinct ways can always be dealt with by focusing on the long negative arc. Since all four types of long arcs lead to modifications which result in configurations of strictly lower type, the inductive step completes the proof. □

Lemma 12.11 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation and C is its Cayley category, then $\pi_2(C) = 0$.*

Proof: To show that $\pi_2(C) = 0$ it is enough to show that all maps from the unit 2-sphere are homotopic to constant maps. Due to its length the proof will be divided into steps.

Step 1: Let $f_1 : S^2 \to C$ be an arbitrary continuous function from the unit 2-sphere to the Cayley category C. Since by Lemma 9.8 the general relators in $\mathcal{R}$ are thin, there is a set $\mathcal{R}'$ of standard representatives for $\mathcal{R}$ by Lemma 5.19. Then by Lemma 6.14 there is a deformation retraction of C onto a 2-dimensional $\mathcal{R}'$-category C' such that the 1-skeleton remains fixed throughout. The deformation retraction creates a homotopy of f_1 with a map f_2 from the unit 2-sphere whose image is in C'.

Step 2: Since C is a circular complex, by Lemma 12.3, and the 1-skeletons of C and C' are the same, C' has a simplicial graph as a 1-skeleton. Moreover, because of the procedure used in Lemma 6.14 there is a 1 to 1 correspondence between the 2-cells in C' and the circular cones of height at least 2 in C. In particular, if D_1 and D_2 are 2-cells in C' with the same boundary, then the corresponding closed circular cones C/c_1 and C/c_2 in C share a representative. By Lemma 9.11, Axiom 1, and the fact that C is collapsed, $c_1 = c_2$ and $D_1 = D_2$. Combined with the above, this shows that C' is a circular complex. Thus by Lemma 2.9 $\mathrm{Chain}(C')$ is a simplicial complex.

Step 3: Using the Simplicial Approximation Theorem, there is a simplicial subdivision Δ_1 of S^2 and a map $g_1 : \Delta_1 \to \mathrm{Chain}(C')$, such that f_2 and g_1 are homotopic maps, and g_1 is simplicial in the broad sense. The map g_1 cannot yet be assumed to be simplicial in the strict sense adopted in this article since g_1 may be degenerate on simplices; g_1 may send triangles to edges or vertices, or send edges to vertices.

Step 4: Consider the largest connected subcomplex of Δ which is sent to the same point in $\mathrm{Chain}(C')$. If this subcomplex is all of Δ then g_1 factors through a complex in which the entire sphere is identified to a point. If not, then each of the components of the complement of this subcomplex are simply connected, and g_1 factors through a complex in which the boundary of the component has been identified to a point. Lemma 7.5 can be used on the component, and on Δ minus the component, to show that the result is homeomorphic to the one-point product of spheres. This procedure need only be repeated a finite number of times since the total number of simplices is strictly decreasing. When the process stops a new structure Δ_2 and a map $g_2 : \Delta_2 \to \mathrm{Chain}(C')$ have been created, such that g_2 is a simplicial map which is a factor of g_1 and Δ_2 is a one-point product of spheres. In particular g_2 is a factor of a map homotopic with f_1. As a result of the identifications, however, the structure Δ_2 is not a simplicial complex, because multiple edges can have the same endpoints and distinct triangles can have the same boundaries. These multiplicities will be considered one sphere at a time.

Step 5: Two distinct triangles with the same boundaries form a sphere which is sent to a triangle in $\mathrm{Chain}(C')$. Thus the sphere in Δ_2 can be collapsed to a triangle and the map g_2 factors through this new structure. Similarly, if there are two edges with the same vertices, they are necessarily part of the same sphere. If this sphere contains nothing else, then the entire sphere can be collapsed to an edge, and the map g_2 factors through this new structure. If one of the disks bounded by the two edges contains no other vertices or edges, but the other disk does, then this one side can be collapsed to an edge, leaving a sphere formed by the other side. Finally, if both disks bounded by the edges contain other vertices or edges then the identification of the edges creates two spheres joined along an edge. Up to homotopy, these spheres can be moved so that they are joined only at one of the endpoints of the edge. If these reductions are carried out until no such situations exist, the result is a simplicial complex Δ_3 and a simplicial map g_3 such that Δ_3 is the one point product of edges, triangles, and simplicial spheres, and g_3 is a factor of g_2. In particular g_3 is a factor of a map homotopic to f_1.

Step 6: Every vertex in Δ_3 is sent by g_3 to a vertex in $\mathrm{Chain}(C')$ which corresponds to an object in C'. If every vertex in Δ_3 is labeled by the height of the corresponding object in C', then all triangles in Δ_3 contain vertices of heights 0, 1, and 2 since they must be distinct, and these are the only three heights of objects in C'. By selecting a finite tree embedded in a sphere and then peeling it off one edge at a time, it is possible to assume, up to homotopy, that every sphere is attached to the other elements in the one-point product at only one vertex, and that it is a vertex of height 0. Call this new one-point product Δ_4 and the corresponding map g_4. By construction g_4 is a factor of a map homotopic to f_1.

Step 7: Let v be a vertex of height 1 in one of the simplicial spheres in Δ_4, and let E be the set of open edges connecting v to vertices of height 0 in the same simplicial sphere. Since $g_4(v)$ is an element in $\mathrm{Chain}(C')$ of height 1, it corresponds to an edge e in C'. If the ends opposite v of the selected edges in E all correspond to the same endpoint of the edge e in C', or if E contains more than two edges, then cut the sphere from v along each of the edges in E, and fill in the hole with a 2-cell whose boundary cycle has length at least 4. This can be done up to homotopy by extending the map to the 2-cell by sending it all to the simplicial subdivision of the edge e. Because an edge in a simplicial complex is contractible, there is such an extension. Then repeat this process for every vertex of height 1 in the simplicial sphere in Δ_4 which satisfies the conditions. The result is a sphere which is simplicial except for a number of 2-cells which have been added. Moreover, if every edge in this sphere is considered labeled by the edge in $\mathrm{Chain}(C')$ (as opposed to the edge in C') to which it is mapped, then the boundary cycles of the 2-cells are Dyck words over this alphabet.

Step 8: Pick one of the new 2-cells, say D, to remove first. Since the order in which the reducible subwords are removed plays no role in the proof of Lemma 7.6, first identify those pairs of adjacent edges in ∂D where the vertex between the edges has height 1. If the only region on the other side of the edges is one of the new 2-cells then the identification creates a vertex of degree 1 which can be removed along with the edge to which it is attached. If one of the edges borders a simplicial triangle then simply identify the edges. If both of the edges border simplicial triangles then these triangles already shared the edge formed by the vertices of height 1 and 2 by the construction so far. When the edges formed by the vertices of height 0 and 1 are identified, the two triangles share two edges and all three vertices. Up to homotopy, both triangles and their common edges can be removed and the remaining edges identified.

Notice that once all of the identifications of this type have occurred the subcomplex determined by the vertices of height 0 and 1 forms a simplicial subdivision of a graph. As the added 2-cells are removed using the procedure in Lemma 7.6, some care will be taken to preserve this property. In particular, at this point the remaining boundary of the selected 2-cell can be viewed as a simplicial subdivision of a cycle sent to the edge e in C'. Thus the remaining boundary is not only a Dyck word when edges are labeled by the edges in $\mathrm{Chain}(C')$ to which they are sent, but it is also the simplicial subdivision of a Dyck word labeled by edges of C'. The rest of the identifications are done in pairs according to the identifications which would take place when the boundary is viewed as a Dyck word in this second sense. Because the identified boundary is contractible, the 2-cell can be viewed as collapsing into the identified boundary. When this procedure has been used to remove all of the new 2-cells, the result is a simplicial complex Δ_5 which is the one-point product of edges, triangles, and simplicial spheres. In addition, each of the simplicial spheres is the simplicial subdivision of a 2-dimensional cell category. There is also a simplicial map g_5 which is a factor of a map homotopic to f_1. Finally, by perhaps adding a finite number of edges it is possible to assume that all of the one-point products in Δ_5 involving spheres occur at vertices of height 0. In this case, the spheres can now be replaced with the 2-dimensional cell categories of which they are subdivisions without affecting the fact that they are in a one-point product attached at vertices.

Step 9: Consider one of the spheres in the domain of g_5, viewed as a 2-dimensional cell category. Let D be a cell in one of these spheres, and let v be the unique vertex contained in the interior of D when the sphere is viewed as a simplicial subdivision of this sphere. If R' is the particular 2-cell in C' determined by $g_5(v)$ then the boundary of D is sent to a cycle in the boundary of R'. Since there are no constraints on the winding numbers of the cycles, the 2-dimensional cell category is not necessarily an $\mathcal{R}'$-sphere, but the boundary cycle of D is at the least equivalent in the free group to an integral power of the word read by the boundary of R' . Alterations will now be made to Δ_5 so that the spheres remaining in the one-point product are all $\mathcal{R}'$-spheres.

Let D and R' be as above, and assume that the winding number n of the image of ∂D in $\partial R'$ is not 1. Pick a representative U of R' which is based at a vertex in the image of ∂D. If the interior of D is removed then the remainder of the sphere can be embedded in the plane so that the boundary of the connected and simply connected map which results from this action is the same as the original boundary of D. By attaching n copies of 2-cells labeled by U (or U^{-1}) to the appropriate vertex, the resulting planar diagram has a boundary cycle U' which has winding number 0 when it is pushed into R'. In particular, the resulting boundary is a Dyck word. By Lemma 7.6 the boundary edges can be identified so that the result is a one-point product of spheres and edges. Because R' is contractible, it is clear that there is a continuous map from Δ_5 to the resulting space (although the map is not cellular), and that there is a cellular map from this space to C' which is a factor of a map homotopic to f_1. Since the net result of the process described above is to remove a 2-cell whose boundary cycle has a winding number other than 1 without introducing any other such cells, the process can be repeated a finite number of times to produce a one-point product of edges, triangles, and $\mathcal{R}'$-spheres called Δ_6 for which there is a map g_6 such that g_6 is a factor of a map homotopic to f_1, and such that g_6 restricted to one of the $\mathcal{R}'$-spheres is a cellular map.

Step 10: Finally, since the cycles in $\mathcal{R}'$ are cyclically reduced by the definition of standard representatives, it follows that each of the $\mathcal{R}'$-spheres mapped into C' corresponds to an $\mathcal{R}$-sphere mapped into C satisfying the hypothesis of Lemma 12.10. Thus by Lemma 12.10 the $\mathcal{R}'$-spheres mapped into C' are homotopic to constant maps. And more broadly, since each of the elements in the one-point product is homotopic to a constant map, so is g_6, and as a result, the same is true of f_1. $\square$

Lemma 12.12 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation, and C is the Cayley category of the presentation, then C is contractible.*

Proof: From the definition of a Cayley category of a general presentation it is immediate that C is connected and simply connected. Thus $\pi_1(C) = 0$, and by Lemma 12.11, $\pi_2(C) = 0$ as well. Using Hurewicz's Theorem, which says that the first non-trivial homotopy group is isomorphic to the first non-trivial homology group, these facts imply that $H_i(C) = 0$ for $i = 1, 2$. Since by Lemma 6.15 $H_i(C) = 0$ for all $i > 2$, all of the homology groups are trivial. Using Hurewicz's Theorem in the other direction, all of the homotopy groups must also be trivial, making C a weakly contractible space. Finally, by Lemma 1.7, C is also contractible. □

Lemma 12.13 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let C be the Cayley category of the presentation. If B is an arbitrary subcategory of C then $\pi_2(B) = 0$. If B is an arbitrary connected and simply connected subcategory of C then B is contractible.*

Proof: To prove the first statement it only needs to be noted that the proofs of Lemma 12.10 and Lemma 12.11 do not make use of the fact that C is the full Cayley category of the general small cancellation group G. The second statement follows from a similar observation about Lemma 12.12; the only properties which are needed in its proof are that the subcategory under consideration be connected and simply connected. □

12.4 Eilenberg-MacLane Spaces

A topological space is called an Eilenberg-MacLane space iff it has exactly one non-trivial homotopy group, say $\pi_n(C, v) = G \neq 1$, in which case the space is called a $\mathrm{K}(G, n)$-space. It is known that with the restriction that G be abelian if n is greater than 1, the space $\mathrm{K}(G, n)$ always exists and is unique up to homotopy equivalence. It is also well-known that a polyhedron is a $\mathrm{K}(G, 1)$-space iff its fundamental group is G and its universal cover is contractible. The next lemma will extend the results of Lemma 12.8 to include Eilenberg-MacLane spaces. Notice that the group G is assumed to be finitely presented.

Lemma 12.14 *If $G = \langle A|\mathcal{R}\rangle$ is a finitely presented general small cancellation presentation with $\alpha \leq \frac{1}{10}$, then the following five conditions are equivalent.*
(1) the group G is torsion-free
(2) all of the general relators in $\mathcal{R}$ have no non-trivial automorphisms
(3) the universal cover of the Poincaré construction is collapsed
(4) the universal cover of the Poincaré construction is contractible
(5) the Poincaré construction is a $K(G, 1)$-space.

Proof: ($1 \Leftrightarrow 2 \Leftrightarrow 3$) The first three conditions are equivalent by Lemma 12.8 and Corollary 8.6. ($3 \Rightarrow 4$) The third condition implies that the universal cover of the Poincaré construction is the same as the Cayley category of G. Thus by Lemma 12.12 the fourth condition is also satisfied. ($4 \Leftrightarrow 5$) The last two are equivalent, since the fundamental group of the Poincaré construction is the group G by definition. ($5 \Rightarrow 1$) Finally, notice that the Poincaré construction of the presentation can be subdivided to yield a finite-dimensional simplicial complex. Since it is known that a finite-dimensional $\mathrm{K}(G, 1)$ space implies that the cohomological dimension of G is finite, which in turn implies that G is torsion-free (see [3]), the proof is complete. □

Lemma 12.14 is reminiscent of Lyndon's Theorem which states that the Poincaré construction of a 1-relator group is a $K(G,1)$-space iff the relator is simple. The above lemma verifies several cases of Lyndon's Theorem and extends the result to the realm of general small cancellation theory. The fact that finitely presented, automorphism-free general small cancellation presentations have finite and immediately constructible Eilenberg-MacLane spaces is also worth emphasizing. The usual argument for the existence of Eilenberg-MacLane spaces involves iteratively killing off the generators of higher and higher homotopy groups and it has a distinctly non-constructive flavor. One consequence of the existence of these finite Eilenberg-MacLane spaces is that the groups constructed in this way have finite cohomological dimension, thus providing a rich source of examples of such groups. In addition, these groups also provide an illustration of a theorem of J. Cannon (see [4]). Cannon proved that a finitely generated group G acts geometrically on some contractible geometry iff G has a finite Eilenberg-MacLane space $K(G,1)$. In this instance, the contractible geometry alluded to is the Cayley category of the presentation under the natural metric on its simplicial subdivision.

13 Closures of $\mathcal{R}$-Categories

In the first half of this section one generic and two specific constructions are introduced. The generic construction is the μ-closure of an $\mathcal{R}$-category with respect to a given set of general relators. The specific constructions are the straightline construction and the circular construction. These constructions are shown, under certain conditions, to exist, to be collapsed, and, most importantly, to be α-closed. The first half concludes with a few lemmas relating the paths and cycles read in the specific constructions to the constructions themselves. The process described above uses an infinite series of local closures. In the second half of the section the process is refined so that only a finite number of local closures are needed in most cases. The main construction is contained in Lemma 13.25.

13.1 Existence of Minimal Closures

Recall that an $\mathcal{R}$-category C is called μ-closed if whenever a word U is read in C by a functor g, and it is also read in a general relator $R \in \mathcal{R}$ by a functor f with $d_R(U) \geq \mu$, then there exists a functor $h : R \to C$ with $hf = g$. The lemmas below show that every $\mathcal{R}$-category B which is embedded in a μ-closed $\mathcal{R}$-category C is contained in a unique, minimal, μ-closed $\mathcal{R}$-category in C. This $\mathcal{R}$-category will be called the μ-closure of B in C.

Lemma 13.1 *Let $\mathcal{R}$ be a set of general relators, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If $\mathcal{F}$ is a non-empty collection of $\mathcal{R}$-categories embedded in a fixed, collapsed $\mathcal{R}$-category C, and all of the categories in $\mathcal{F}$ are μ-closed with respect to $\mathcal{R}'$ for some μ, then the intersection of the $\mathcal{R}$-categories in $\mathcal{F}$ is also μ-closed with respect to $\mathcal{R}'$.*

Proof: Let R be a general relator in $\mathcal{R}'$ and let U be a path in R with $d_R(U) > \mu$. Suppose that U is also readable in an $\mathcal{R}$-category B in $\mathcal{F}$. Since B is μ-closed with respect to $\mathcal{R}'$, there is a characteristic functor from R to B such that the reading of U in R is sent to the reading of U in B. In other words, this characteristic functor on R is a closed cone of B. Since B is embedded in C, there is also a characteristic functor from R to C, and since C is collapsed, by Lemma 6.3 this is the only $\mathcal{R}$-functor from R to C which sends the reading of U in R to the reading of U in C. If U is read in the intersection of all of the $\mathcal{R}$-categories in $\mathcal{F}$ then this particular characteristic functor on R is a closed cone in each of the $\mathcal{R}$-categories, and thus it is a closed cone in the intersection. This shows that the intersection is μ-closed with respect to $\mathcal{R}'$. $\square$

Lemma 13.2 *Let $\mathcal{R}$ be a set of general relators, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If B is an $\mathcal{R}$-category which is embedded in an $\mathcal{R}$-category C and C is μ-closed with respect to $\mathcal{R}'$ for some constant μ, then there is a unique minimum $\mathcal{R}$-category in C which contains B and is μ-closed with respect to $\mathcal{R}'$.*

Proof: Let $\mathcal{F}$ be the set of all $\mathcal{R}$-categories in C which contain B and are μ-closed with respect to $\mathcal{R}$. Since C itself satisfies these conditions, $\mathcal{F}$ is non-empty . Let B' be the intersection of all of the categories in $\mathcal{F}$. The intersection is non-empty since it at least contains B, and by Lemma 13.1, the intersection is also μ-closed with respect to $\mathcal{R}$. Thus B' is itself an element in the set, and it is contained in every other $\mathcal{R}$-category satisfying these conditions. $\square$

The $\mathcal{R}$-category described in the lemma is called the μ-closure of B in C with respect to $\mathcal{R}'$. If C/c is a closed cone in C but not in B which is isomorphic with $R \in \mathcal{R}'$ and U is a path in the intersection of C/c and B with $d_R(U) \geq \mu$, then C/c is certainly a cone which must be included in any μ-closure of B with respect to $\mathcal{R}'$. Such a closed cone is called a cone which must be added to B immediately. The union of B and all of the closed cones in C which must be added to B immediately is called the local μ-closure of B in C with respect to $\mathcal{R}'$. The local μ-closure of B in C respect to $\mathcal{R}'$ will be denoted $\mathrm{L}(B)$ with μ, C, and $\mathcal{R}'$ understood from context. This operation can be iterated. In particular, let $B = \mathrm{L}^0(B)$, let $\mathrm{L}(B) = \mathrm{L}^1(B)$, and let $\mathrm{L}^{k+1}(B) = \mathrm{L}(\mathrm{L}^k(B))$ for all k. Notice that clearly, $\mathrm{L}^i(B) \subset \mathrm{L}^j(B) \subset C$ for all integers $i \leq j$. Finally, let $\mathrm{L}^\infty(B) = \cup_{i=0}^\infty \mathrm{L}^i(B)$. Again it is clear that $\mathrm{L}^\infty(B) \subset C$. The next lemma shows that L^∞ is identical with the μ-closure of B in C with respect to $\mathcal{R}'$.

Lemma 13.3 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, let B and C be $\mathcal{R}$-categories, with B embedded in C, let $\mathcal{R}'$ be a subset of $\mathcal{R}$, and let μ be a constant. If C is itself μ-closed with respect to $\mathcal{R}'$, then $L^\infty(B)$ is the μ-closure of B in C with respect to $\mathcal{R}'$.*

Proof: First note that $\mathrm{L}^\infty(B)$ is itself μ-closed with respect to $\mathcal{R}'$. To see this let U be a path in $\mathrm{L}^\infty(B)$ which is also readable in a general relator $R \in \mathcal{R}'$ with $d_R(U) > \mu$. Since C is μ-closed there is an $\mathcal{R}$-functor $f : R \to C$ which sends the reading of U in R to the path U in $\mathrm{L}^\infty(B)$. Since every edge in the finite path U in $\mathrm{L}^{\infty}(B)$ is contained in one of the $\mathrm{L}^{k}(B)$, the $\mathrm{L}^{k}(B)$ with the largest superscript contains the entire path U, and the closed cone which is the image of R under f is included in $\mathrm{L}^{k+1}(B)$. Thus $\mathrm{L}^{\infty}(B)$ is μ-closed. If B' denotes the μ-closure of B in C with respect to $\mathcal{R}'$, then by Lemma 13.1 the fact that $\mathrm{L}^{\infty}(B)$ is μ-closed with respect to $\mathcal{R}'$ implies that $B' \subset \mathrm{L}^{\infty}(B)$. On the other, by construction it is clear that $\mathrm{L}^{k}(B)$ must be contained in B' for all k, so that their union, $\mathrm{L}^{\infty}(B)$, is also contained in B'. Finally, since $B' \subset \mathrm{L}^{\infty}(B)$ and $\mathrm{L}^{\infty}(B) \subset B'$, they must be identical. □

Lemma 13.4 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, let $\mathcal{R}'$ be a subset of $\mathcal{R}$. For all $\mathcal{R}$-categories B embedded in a Cayley category and for all μ, there exists a μ-closure of B in C with respect to $\mathcal{R}'$. Moreover, the automorphism group of the μ-closure of B contains the automorphism group of B itself.*

Proof: Since B is embedded in the Cayley category, and the Cayley category is 0-closed, and thus μ-closed, the result follows by Lemma 13.2. The second statement follows from Lemma 13.3 and the obeservation that paths which show that a closed cone must be added at one location are sent by automorphisms to paths which show that corresponding closed cones must be added at all of the corresponding locations. □

Lemma 13.5 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, let $\mathcal{R}'$ be a subset of $\mathcal{R}$, and let B be a connected, collapsed $\mathcal{R}$-category with fundamental group H. If B is $(1-3\alpha)$-closed with respect to $\mathcal{R}'$, then there is exists an $\mathcal{R}$-category C in which B is embedded, which is μ-closed with respect to $\mathcal{R}'$, and which does not properly contain any other $\mathcal{R}$-category which satisfies these conditions. In addition, the inclusion of B in C induces an isomorphism on their fundamental groups, and both B and C are collapsed.*

Proof: Let B' be the universal cover of B. By Lemma 12.2 B' embeds in the Cayley category of the presentation, by Lemma 1.15 the group of automorphisms of B' contains a subgroup isomorphic to H, and by Lemma 13.4 it has a μ-closure in the Cayley category. Next, by Lemma 13.3 the automorphism group of the μ-closure contains the automorphism group of B' and thus contains the subgroup isomorphic with H. By Lemma 12.1 the μ-closure of B' is simply connected. If the μ-closure of B' is viewed as a separate $\mathcal{R}$-category not embedded in the Cayley category, and then quotiented by the action of the subgroup of the automorphism group isomorphic with H, then the result is an $\mathcal{R}$-category which satisfies the statements of the lemma. Finally, the fact that B and C are collapsed follows from the fact that their universal covers embed in the Cayley category which is collapsed by definition. □

In the case where either B or its universal cover embeds in the Cayley category of a general small cancellation presentation, it possible to speak unambiguously of the μ-closure of B with respect to $\mathcal{R}'$. Although the above lemma proves the existence of a canonical $\mathcal{R}$-category containing B which is μ-closed with respect to $\mathcal{R}'$ it does not guarantee that the resulting $\mathcal{R}$-category will be finite. The proof that the μ-closure is finite will be postponed until after a more detailed investigation of the structure of μ-closures.

13.2 Closures of Words and Cycles

The next few lemmas describe some of the functorial properties of α-closures, and these are then used to create a well-defined α-closed $\mathcal{R}$ category associated with every word and cycle.

Lemma 13.6 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If $f : B \to C$ is an $\mathcal{R}$-functor between connected, collapsed $\mathcal{R}$-categories, B is $(1-3\alpha)$-closed with respect to $\mathcal{R}'$, and C is μ-closed with respect to $\mathcal{R}$, then a μ-closure of B with respect to $\mathcal{R}'$ exists, and there is a unique $\mathcal{R}$-functor from the μ-closure of B to C which, when restricted to B, is equal to f.*

Proof: Let B' be the μ-closure of B which exists by Lemma 13.5, and let B'/b be a closed cone in B' which is isomorphic to a general relator $R \in \mathcal{R}'$ but which is not a closed cone in B. If U is a path in the intersection of B'/b and B with $d_R(U) > \mu$, then since C is μ-closed there is a unique $\mathcal{R}$-functor from B'/b to C which sends the path U in B'/b to the image of the path U in B under the functor f. Next, by Lemma 1.2 the obvious $\mathcal{R}$-functor from the disjoint union of B'/b and B to C factors through the quotient of the disjoint union obtained by identifying the two paths reading U, which in turn factors through the collapse of this $\mathcal{R}$-category by Lemma 6.1. Repeating this procedure for all of the closed cones which must be added immediately to B shows that there exists a functor from $\mathrm{L}^1(B)$ to C which extends the functor f. More repetitions of the procedure show that there exists a functor from $\mathrm{L}^k(B)$ to C which extends all of the previous functors. Finally, this shows that there is a functor from L^∞ to C which extends f. By Lemma 6.3 such a functor is unique. □

Lemma 13.7 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If B and B' are $(1-3\alpha)$-closed $\mathcal{R}$-categories with B embedded in B', μ and μ' are constants with $1-3\alpha \geq \mu \geq \mu'$, and $\mathcal{R}'$ and $\mathcal{R}''$ are subsets of $\mathcal{R}$ with $\mathcal{R}' \subset \mathcal{R}''$, then the μ-closure of B with respect to $\mathcal{R}'$ is contained in the μ'-closure of B' with respect to $\mathcal{R}''$.*

Proof: Let $f : B \to B'$ be the $\mathcal{R}$-functor which embeds B in B', and apply Lemma 13.6. The conclusion is immediate. □

Lemma 13.8 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If X and Y are words which are $(1-3\alpha)$-free with respect to $\mathcal{R}'$ and equivalent to each other in the group $G' = \langle A|\mathcal{R}'\rangle$, then there is an isomorphism from the α-closure of the abstract path X with respect to $\mathcal{R}'$ to the α-closure of the abstract path Y with respect to $\mathcal{R}'$ which sends the endpoints of X to the endpoints of Y.*

Proof: Let B and C be the α-closures of X and Y respectively, and notice that both B and C are connected and collapsed. By Lemma 11.3, the path Y is readable in B between the same vertices as X. By Lemma 13.6 this reading of Y extends to a functor $f : C \to B$. Similarly, there is a functor $g : B \to C$ extending the reading of X in C between the same vertices as Y. Since fg is a functor B to itself which fixes a vertex, by Lemma 6.3 it is the identity functor, and similarly, gf is the identity functor on C. Thus, f is the desired isomorphism. □

Lemma 13.9 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If X and Y are cycles which are $(1-3\alpha)$-free with respect to $\mathcal{R}'$ and conjugate to each other in the group $G' = \langle A|\mathcal{R}'\rangle$, then there is an isomorphism from the α-closure of the abstract loop X with respect to $\mathcal{R}'$ to the α-closure of the abstract loop Y with respect to $\mathcal{R}'$.*

Proof: Let B and C be the α-closures of X and Y respectively, and let Δ be an annular $\mathcal{R}'$-diagram with no long internal arcs, which shows that X and Y are conjugate in G'. By Lemma 11.3, the loop Y is readable in B, and by Lemma 13.6 this reading of Y extends to a functor $f : C \to B$. Similarly, there is a functor $g : B \to C$ extending the reading of X in C. If the same diagram Δ is used in both applications of Lemma 11.3 to create the readings of the loop X in C and Y in B, then the functor fg will fix the loop X, and the functor gf will fix the loop Y. Then by Lemma 6.3 fg is the identity functor on B, and gf is the identity functor on C. Thus, f is the desired isomorphism. □

Lemma 13.10 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{8}$. If $\mathcal{R}'$ is a subset of $\mathcal{R}$ and W is an arbitrary word or cycle, then the α-closure of a Dehn reduction of W is a well-defined $\mathcal{R}'$-category.*

Proof: This follows immediately from Lemma 13.8 and Lemma 13.9. □

The α-closure of a Dehn reduction of a word W with respect to $\mathcal{R}'$ is called the straightline construction on W with respect to $\mathcal{R}'$ and it will be denoted $\mathrm{str}(W, \mathcal{R}')$. Similarly, the α-closure of a Dehn reduction of a cycle W with respect to $\mathcal{R}'$ is called the circular construction on W with respect to $\mathcal{R}'$ and it will be denoted $\mathrm{cir}(W, \mathcal{R}')$. The 1-skeletons of $\mathrm{str}(W, \mathcal{R}')$ and $\mathrm{cir}(W, \mathcal{R}')$ can be viewed as automata by specifying the initial and terminal vertices of the original path W as the start and stop states of the automata. In addition an orientation of $\mathrm{cir}(W, \mathcal{R})$ is induced by the orientation of the loop W.

The most important special cases are those in which $\mathcal{R}'$ is compatible with the grading of the general relators, and these cases merit an abbreviated notation. In particular, the constructions $\mathrm{str}(W, \mathcal{R}(k))$ and $\mathrm{cir}(W, \mathcal{R}(k))$ will be denoted as $\mathrm{str}_k(W)$ and $\mathrm{cir}_k(W)$ respectively, and $\mathrm{str}(W, \mathcal{R})$ and $\mathrm{cir}(W, \mathcal{R})$ as $\mathrm{str}(W)$ and $\mathrm{cir}(W)$. Notice that since $\mathcal{R}_1 = \emptyset$, the definitions of $\mathrm{str}_1(W)$ and $\mathrm{cir}_1(W)$ given here coincide with those given in Section 5.

Lemma 13.11 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{8}$, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If X is an arbitrary word and Y is an arbitrary cycle which is not equivalent to 1 in $G' = \langle A|\mathcal{R}'\rangle$, then $str(X, \mathcal{R}')$ is simply connected while the fundamental group of $cir(Y, \mathcal{R}')$ is $\mathbf{Z}$.*

Proof: Without loss of generality assume that X and Y are Dehn-reduced with respect to $\mathcal{R}'$. Then the abstract path X is simply connected, the abstract loop Y has a fundamental group of $\mathbf{Z}$, and by Lemma 13.5 the functor embedding them into their α-closures with respect to $\mathcal{R}'$ induces an isomorphism on the fundamental groups. □

Lemma 13.12 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{8}$, let C be its Cayley category, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If X is an arbitrary path in C, then there is a unique embedding of $str(X, \mathcal{R}')$ into C which sends the start and stop states of $str(X, \mathcal{R}')$ to the start and end vertices of the path X.*

Proof: By Lemma 10.2 there exists a Dehn reduction of X with respect to $\mathcal{R}'$. Call this not necessarily unique reduction X'. Since C is μ-closed for all μ, by Lemma 10.4 X' is readable in C between the same vertices as X. By construction $\mathrm{str}(X, \mathcal{R}) = \mathrm{str}(X', \mathcal{R})$, and it is a connected, collapsed, and $(1 - 3\alpha)$-closed $\mathcal{R}'$-category. The latter is true since $1 - 3\alpha \geq \alpha$. Thus by Lemma 12.2 and Lemma 13.11, $\mathrm{str}(X, \mathcal{R}')$ embeds in C. Since by definition the automorphisms of a Cayley category act transitively on its vertices there is an embedding of $\mathrm{str}(X, \mathcal{R}')$ which sends its start state to the start vertex of the path X, and by Lemma 6.3 this embedding is unique. Finally, since $\mathrm{str}(X, \mathcal{R}')$ can be constructed from X', X' is certainly accepted by $\mathrm{str}(X, \mathcal{R}')$ which shows that the stop state of $\mathrm{str}(X, \mathcal{R}')$ has been sent to the end vertex of the path X', which is also the end vertex of the path X. □

Lemma 13.13 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{8}$, let C be its Cayley category, and let B be an arbitrary subcategory of C which is α-closed with respect to $\mathcal{R}$. If X is a path in B then the embedding of $str(X)$ into C is contained in B. As a consequence, if X is a path in $str(Y)$, then there is an embedding of $str(X)$ into $str(Y)$ which sends the start and stop states of $str(X)$ to the start and end vertices of the path X.*

Proof: Let X' be a Dehn reduction of X. By Lemma 10.4, X' is readable in B between the same endpoints. By Lemma 13.12, $\mathrm{str}(X) = \mathrm{str}(X')$ can be uniquely embedded in C so that it starts and ends at the same vertices as X and X'. If $\mathrm{str}(X)$ were not contained in B, the intersection of the two would be an α-closed $\mathcal{R}$-category containing X', which contradicts the definition of $\mathrm{str}(X)$ as the minimum such $\mathcal{R}$-category. Thus $\mathrm{str}(X)$ is contained in B. The second assertion follows immediately from the first once $\mathrm{str}(Y)$ is embedded in C by Lemma 13.12. □

13.3 Decidability

A word is said to be accepted by $\mathrm{str}(W, \mathcal{R}')$ if it readable as a path in the construction between the same vertices as the Dehn reduction of W with respect to $\mathcal{R}'$. By analogy, a cycle is said to be accepted by $\mathrm{cir}(W, \mathcal{R}')$ if it is readable as a loop in the construction with a winding number of 1 and with the same orientation as W.

Lemma 13.14 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, let $\mathcal{R}'$ be a subset of $\mathcal{R}$, and let X and Y be words which are $(1-3\alpha)$-free with respect to $\mathcal{R}'$. The following statements are equivalent.*
(1) X and Y are equivalent in $G' = \langle A|\mathcal{R}'\rangle$
(2) $str(X, \mathcal{R}')$ and $str(Y, \mathcal{R}')$ are isomorphic as automata
(3) Y is a word accepted by $str(X, \mathcal{R}')$
(4) X is a word accepted by $str(Y, \mathcal{R}')$.

Proof: $(1 \Rightarrow 2 \Rightarrow 3)$ By Lemma 13.8 the first condition implies the second, and the second condition clearly implies the third. $(3 \Rightarrow 1)$ Suppose that Y is accepted by $\mathrm{str}(X, \mathcal{R}')$. By Lemma 13.12 $\mathrm{str}(X, \mathcal{R}')$ embeds in the Cayley category of $G' = \langle A|\mathcal{R}'\rangle$. This shows that X and Y are readable in the Cayley category between the same endpoints, and thus by Lemma 8.11 they are equivalent. $(1 \Leftrightarrow 4)$ Since the first three conditions have been shown to be equivalent, and the first condition is symmetric with respect to X and Y, it follows that the fourth condition is equivalent as well. □

Lemma 13.15 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{8}$, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. Words X and Y are equivalent in $G' = \langle A|\mathcal{R}'\rangle$ iff there is a start- and endpoint-preserving isomorphism between $str(X, \mathcal{R}')$ and $str(Y, \mathcal{R}')$.*

Proof: By Lemma 10.2 there exist Dehn reductions of X and Y with respect to $\mathcal{R}'$. Call these not necessarily unique reductions X' and Y', respectively. By Lemma 10.3 they are $(\frac{1}{2} + \alpha)$-free and thus $(1-3\alpha)$-free with respect to $\mathcal{R}'$ since $\alpha \leq \frac{1}{8}$. Thus Lemma 13.14 shows that X' and Y' are equivalent in G' iff there is a start- and endpoint-preserving isomorphism between $\mathrm{str}(X', \mathcal{R}')$ and $\mathrm{str}(Y', \mathcal{R}')$. Since X and X' are equivalent in G', Y and Y' are equivalent in G', and by definition $\mathrm{str}(X, \mathcal{R}') = \mathrm{str}(X', \mathcal{R}')$ and $\mathrm{str}(Y, \mathcal{R}') = \mathrm{str}(Y', \mathcal{R}')$, the proof is complete. □

Lemma 13.16 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, let $\mathcal{R}'$ be a subset of $\mathcal{R}$, and let X and Y be cycles which are $(1-3\alpha)$-free with respect to $\mathcal{R}'$. The following statements are equivalent.*
(1) X and Y are conjugate in $G' = \langle A|\mathcal{R}'\rangle$
(2) $cir(X, \mathcal{R}')$ and $cir(Y, \mathcal{R}')$ are isomorphic as oriented $\mathcal{R}'$-categories
(3) Y is a cycle accepted by $cir(X, \mathcal{R}')$
(4) X is a cycle accepted by $cir(Y, \mathcal{R}')$.

Proof: ($1 \Rightarrow 2 \Rightarrow 3$) By Lemma 13.9 the first condition implies the second, and the second condition clearly implies the third. ($3 \Rightarrow 1$) Suppose that Y is accepted by $\mathrm{cir}(X, \mathcal{R}')$. Since the abstract loop X is $(1-3\alpha)$-free with respect to $\mathcal{R}'$, it is also $(1-3\alpha)$-closed. And by Lemma 13.11 its fundamental group is $\mathbf{Z}$ so that it is possible to speak of winding numbers. Let P be a path in $\mathrm{cir}(X, \mathcal{R}')$ from the basepoint of the loop X to the basepoint of the loop Y. Then $PYP^{-1}X^{-1}$ is a loop of winding number 0, and it lifts to a loop in the universal cover of $\mathrm{cir}(X, \mathcal{R}')$. By Lemma 12.2, the universal cover of $\mathrm{cir}(X, \mathcal{R}')$ embeds in the Cayley category of G' so that $PYP^{-1}X^{-1}$ is readable as a loop in the Cayley category. By Lemma 8.11, this word is equivalent to 1 in G', showing that X and Y are conjugate in G'. Thus the first three conditions are equivalent. ($1 \Leftrightarrow 4$) Since both of the first two are symmetric with respect to X and Y, it follows that the fourth condition is equivalent to the others as well. $\Box$

Lemma 13.17 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{8}$, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. Cycles X and Y are conjugate in $G' = \langle A|\mathcal{R}'\rangle$ iff there is an orientation-preserving isomorphism between $\mathrm{cir}(X, \mathcal{R}')$ and $\mathrm{cir}(Y, \mathcal{R}')$.*

Proof: By Lemma 10.5 there exist Dehn reductions of X and Y with respect to $\mathcal{R}'$. Call these not necessarily unique reductions X' and Y', respectively. Also by Lemma 10.5 they are $(\frac{1}{2}+\alpha)$-free and thus $(1-3\alpha)$-free with respect to $\mathcal{R}'$ since $\alpha \leq \frac{1}{8}$. Thus Lemma 13.16 shows that X' and Y' are conjugate in G' iff there is an orientation-preserving isomorphism between $\mathrm{cir}(X', \mathcal{R}')$ and $\mathrm{cir}(Y', \mathcal{R}')$. Since X and X' are conjugate in G', Y and Y' are conjugate in G', and by definition $\mathrm{cir}(X, \mathcal{R}') = \mathrm{cir}(X', \mathcal{R}')$ and $\mathrm{cir}(Y, \mathcal{R}') = \mathrm{cir}(Y', \mathcal{R}')$, the proof is complete. $\Box$

13.4 Finite Closures

Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let $\mathcal{R}(k)$ be the general relators in $\mathcal{R}$ of rank at most k. The efficient construction of the α-closure of an $\mathcal{R}$-category with respect to $\mathcal{R}(k)$ which is given below is reminiscent of a construction of Zimin. For example, if B is an $\mathcal{R}$-category which is sufficiently closed to begin with and all of the general relators in B are in $\mathcal{R}(3)$, then B can be made α-closed by a local closure in rank 1, then rank 2, then rank 1 again, then rank 3, then rank 1, then rank 2, then rank 1. The sequence of ranks has the characteristic Zimin structure, so that like Zimin's construction, the process can be conceptualized as involving three steps. Of these, the first and the third are identical and amount to inductively applying the procedure in the previous rank. After the first step the construction is α-closed in all ranks strictly less than k, but slightly less closed in rank k itself. The second step, which is the only step involving relators of rank k, adds relators of rank k in order to make the result μ-closed in rank k for some small constant μ. This procedure, however, disrupts the closures in the lower ranks. Thus, the third step is needed in which the procedure used to make the lower ranks α-closed is

Rank	Start	After Step 1	After Step 2	After Step 3
1	$1-3\alpha-2\epsilon$	α	3α	α
2	$1-3\alpha-2\epsilon$	α	3α	α
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$k-1$	$1-3\alpha-2\epsilon$	α	3α	α
k	$1-3\alpha-2\epsilon$	$1-3\alpha$	$\alpha-2\epsilon$	α
$(k+1)$	(δ)			$(\delta+2\epsilon)$

Figure 31: Levels of completeness

applied again. The final result is a structure which is α-closed with respect to all of the general relators in $\mathcal{R}(k)$. Because of the back-and-forth nature of the levels of completeness, Figure 31 has been provided to illustrate the process. The final row of the table shows the result of the study of Zimin-type words in the next rank. Once this has been shown, then the entire three-step process is ready to become the first step, or the third step, in the process in the next larger rank.

Because of their connection with the above procedure, the following words will be called Zimin-type words. A rank k Zimin-type word is a word $W = XYZ$ where X and Z are rank $(k-1)$ Zimin-type words and Y is a word readable in a rank k relator. To start the definition, notice that a rank 1 Zimin-type word must be the empty word, since no rank 1 general relators exist. For simplicity, Zimin-type words will be referred to as Zimin words even though this designation is usually reserved for words of a much more restricted form.

Lemma 13.18 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{12}$. If XYZ is a path in a general relator $R \in \mathcal{R}$ with $d_R(XYZ) \geq 1-3\alpha$, and X, Y and Z are $(\frac{1}{2}+\alpha)$-free words with respect to R then at least two of the three words are long in R.*

Proof: Suppose on the contrary that two of the words, say Y and Z, are short in R. Then $1-3\alpha \leq d_R(XYZ) \leq d_R(X)+d_R(Y)+d_R(Z) < d_R(X)+2\alpha$. Thus $d_R(X) > 1-5\alpha \geq \frac{1}{2}+\alpha$, since $\alpha \leq \frac{1}{12}$, contradiction. □

Lemma 13.19 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{10}$, let $\mathcal{R}'$ be a subset of $\mathcal{R}$, let C be a collapsed $\mathcal{R}$-category and let S and T be connected $\mathcal{R}$-categories embedded in C which are each α-closed with respect to $\mathcal{R}'$. Suppose further that S and T intersect and that u and v are vertices in S and T respectively. If X is the shortest path from u to the intersection of S and T, and Y is the shortest path from the endpoint of X to v, then XY is $(1-3\alpha)$-free with respect to $\mathcal{R}'$.*

Proof: First, note that since S and T are α-closed with respect to $\mathcal{R}'$, by Lemma 11.5 they are geodesic-closed with respect to $\mathcal{R}'$ as well, so that X is contained in S, Y is contained in T, and both paths are geodesics with respect to $\mathcal{R}'$. Because X and Y are geodesics with respect to $\mathcal{R}'$, by Lemma 10.3 they are $(\frac{1}{2}+\alpha)$-closed and thus $(1-3\alpha)$-closed with respect to $\mathcal{R}'$. Thus if XY is not $(1-3\alpha)$-closed with respect to $\mathcal{R}'$ it is because there are non-empty words X' and Y' and a general relator $R \in \mathcal{R}'$ such that X' is a final segment of X, Y' is an initial segment of Y, $X'Y'$ is readable in R and $d_R(X'Y') > 1-3\alpha$. Since none of the edges of X are in T, $d_R(Y') < \alpha$, because otherwise R would be included in T and X' would consist of edges in T, contradiction. By one of the properties of relator metrics, $d_R(X') > 1-4\alpha$. But since $\alpha \leq \frac{1}{10}$, $\frac{1}{2}+\alpha \leq 1-4\alpha$, so that by Lemma 10.3, X' is not a geodesic, contradiction. Therefore the assumption that XY is not $(1-3\alpha)$-free is false. $\square$

If U is a subword of a word X and UV^{-1} is a representative of a general relator R, then replacing U with V in X is called a simple substitution.

Lemma 13.20 *Let X be readable in a general relator R and let Y be a word obtained from X by simple substitutions which shorten the length of the word. If none of the simple substitutions involve a representative of R, then $d_R(X) = d_R(Y)$.*

Proof: Assume that Y is obtained from X by a single simple substitution of V for a subword U in X, and that S is the general relator of which UV^{-1} is a representative. Since the substitution shortens the length, $2|U| > |U|+|V| \geq |S|$ so that $|U| > \frac{1}{2}|S|$ and by Axiom 4 and Axiom 7, $d_S(U) > \frac{1}{2} - \delta > \alpha$. Thus by Axiom 1, there is a functor from S to R which extends the reading of U in R. Since by assumption none of the simple substitutions involve a representative of R, it must be that the image of S is contained in ∂R by Lemma 2.1. This shows that V is readable in R between the same vertices as U, and that the new path Y is homotopic to X in ∂R relative to their endpoints. Thus, by one of the properties of relator metrics $d_R(X) = d_R(Y)$. In the general case, Y is formed by a finite number of iterations of this procedure. $\square$

Lemma 13.21 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{8}$, let C be a collapsed $\mathcal{R}$-category and let $\mathcal{F}$ be a family of connected $\mathcal{R}$-categories embedded in C which are each α-closed with respect to $\mathcal{R}(k)$ and whose union is C. If U is a path in C then there exists a path V in C with $|V| \leq |U|$ which is homotopic to U relative to its endpoints, and such that for all $B \in \mathcal{F}$, the connected components of $V \cap B$ are geodesics with respect to $\mathcal{R}(k)$. In addition, if the original path U is readable in a general relator R, then V is also readable in R between the same vertices.*

Proof: If B is a member of $\mathcal{F}$ and U' is a subpath of U in B which is not a geodesic with respect to $\mathcal{R}(k)$, then pick V' to be such a geodesic. Since B is geodesic-closed with respect to $\mathcal{R}(k)$ by Lemma 11.5, there is a path reading V' in B homotopic to U' relative to its endpoints. If U is also read in a general relator R, then since by Lemma 9.8 and Axiom 1, R is collapsed and α-closed, it follows from Lemma 11.5 that the same replacement of U' with V' is permissible in R. After repeating this procedure a finite number of times, the process must stop, since the length of the path is strictly decreasing. The final path which results necessarily satisfies the statement of the lemma. □

Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \geq 3\beta + 2\epsilon$ and $\alpha \leq \frac{1}{12}$, let B be an $\mathcal{R}$-category which is α-closed with respect to $\mathcal{R}(k-1)$ and $(1-3\alpha)$-closed with respect to $\mathcal{R}_k$, let C be the local $(\alpha - 2\beta - 2\epsilon)$-closure of B, and let $\mathcal{F}$ be the set consisting of B and each of the closed cones of rank k in C which have been added to B to form C. Since C is contained in the full $(\alpha - 2\beta - 2\epsilon)$-closure of B with respect to $\mathcal{R}_k$, and by Lemma 13.5, B is embedded in that $\mathcal{R}$-category, it follows that B is also embedded in C. Clearly each of the elements of $\mathcal{F}$ are embedded in C, they are all connected $\mathcal{R}$-categories which are α-closed with respect to $\mathcal{R}(k-1)$, and their union is C. Thus Lemma 13.21 can be applied to C and $\mathcal{F}$ as follows.

Lemma 13.22 *Let G, B, C and $\mathcal{F}$ be as described above. If R, S, and T are distinct elements in $\mathcal{F}$ other than B, and X is a path in S which starts in $R \cap S$, ends in $S \cap T$ and is a geodesic with respect to $\mathcal{R}(k)$, then there is a path Y with the same endpoints as X which is contained in the intersection of $R \cup B \cup T$ with the α-closure of X in S with respect to $\mathcal{R}(k)$.*

Proof: Case 1: If R and T intersect, let v be a vertex in the intersection, let P be the shortest path from the initial vertex of X to v, and let Q be the shortest path from v to the terminal vertex of X. Since R, S, and T are geodesic-closed by Lemma 11.5, P is contained in $R \cap S$, and Q is contained in $S \cap T$. Since P and Q are in the intersection of two distinct closed cones of the same rank in a collapsed $\mathcal{R}$-category, by Axiom 2 $d_S(P) < \beta$ and $d_S(Q) < \beta$. If the winding number of the loop PQX^{-1} in S is n, then $n \leq d_S(PQX^{-1}) < d_S(X) + 2\beta$. If n is not 0 then $d_S(X) > 1 - 2\beta > \frac{1}{2} + \alpha$ and X could not be a geodesic in S by Lemma 10.3. Thus the winding number is 0, and by Lemma 7.15 there is a proof that PQX^{-1} is equivalent to 1 in $G(k-1)$. Since P, Q, and X are $(1-4\alpha)$-free paths with respect to $\mathcal{R}(k-1)$ and they form a triangle in the Cayley graph of $G(k-1)$, by Lemma 11.8 the path X itself is contained in the $R \cup T$.

Case 2: If R and T do not intersect, then let P be the shortest path from the initial vertex of X to a point in $S \cap B$, let Q be the shortest path from a vertex in $S \cap B$ to the terminal vertex of X, and let Y be the shortest path from the terminal vertex of P to the initial vertex of Q. By Lemma 13.19 both PY and YQ are $(1-3\alpha)$-free with respect to $\mathcal{R}(k-1)$. If PYQ is not $(1-3\alpha)$-free with respect to $\mathcal{R}(k-1)$ then there is a subword of PYQ containing a non-empty portion of P, all of Y and a non-empty portion of Q which has a length longer than $1 - 3\alpha$ when measured by the relator metric of some general relator in $\mathcal{R}(k-1)$. By Lemma 13.18 this would mean that either the portion in P or the portion in Q is long in the general relator. If without loss of generality it is assumed that the portion in P is long in the relator, then, since R is α-closed with respect to $\mathcal{R}(k-1)$, the entire relator is contained in R, and R and T would intersect, contradicting the initial assumption. Thus PYQ is $(1-3\alpha)$-free with respect to $\mathcal{R}(k-1)$.

By Axiom 2, $d_S(P) < \beta$, and $d_S(Q) < \beta$. Since Y is a geodesic by Lemma 10.3, $d_S(Y) < \frac{1}{2} + \alpha \leq 1 - 3\alpha$ since $\alpha \leq \frac{1}{12}$. If the loop $PYQX^{-1}$ has a non-zero winding number in S then $1 \leq d_S(PYQX^{-1}) \leq d_S(X) + 1 - 3\alpha + 2\beta \leq d_S(X) + 1 - \alpha$, by the properties of relator metrics and the constraints of the constants listed in Axiom 7. Thus $d_S(X) > \alpha$, and the α-closure of X in S with respect to $\mathcal{R}(k)$ is S itself and thus includes the path PYQ. On the other hand, if $PYQX^{-1}$ has winding number 0, then by Lemma 7.15 and Lemma 8.8 it is a loop in the Cayley category of $G(k-1)$ and thus by Lemma 8.11, X and PYQ are equivalent in $G(k-1)$. By Lemma 11.3 it follows that PYQ is contained in the α-closure of X with respect to $\mathcal{R}(k-1)$ and thus also clearly in the α-closure of X with respect to $\mathcal{R}(k)$. □

Lemma 13.23 *Let G, B, C, and $\mathcal{F}$ be as described above, and let X be a path in C. If S and T are elements of $\mathcal{F}$ which contain the initial and terminal vertices of X respectively, then there is a path with the same endpoints as X which is contained in the intersection of $S \cup B \cup T$ with the α-closure of X with respect to $\mathcal{R}(k)$. In particular, if S and T do not intersect, then the path mentioned above can be chosen to be of the form PYQ where P is the shortest path from the initial vertex of X to a vertex in the intersection of S and B, Q is the shortest path from a vertex in the intersection of B and T to the terminal vertex of X, and Y is the shortest path between the terminal vertex of P and the initial vertex of Q. If S and T do intersect, then the path can be chosen to be of the form PQ, where P is the shortest path from the initial vertex of X to a point in the intersection of S and T, and Q is the shortest path from the terminal vertex of P to the terminal vertex of X.*

Proof: Let $\mathcal{F}'$ be the smallest finite subset of $\mathcal{F}$ such that $\mathcal{F}'$ contains S, B, and T, and such that the union of the elements in $\mathcal{F}'$ contains a path X' which is accepted by the α-closure of X with respect to $\mathcal{R}(k)$. By applying Lemma 13.21 and using Lemma 13.6 it is possible to assume that X' satisfies the conclusion of Lemma 13.21. If $\mathcal{F}'$ contains an element other than S, B, and T, then applying Lemma 13.22 and Lemma 13.6 to the portion of the path X' which reads this element shows that $\mathcal{F}'$ is in fact not minimal, contradiction. Thus $\mathcal{F}'$ contains only S, B, and T. The properties of the particular paths then follow from the proof of Lemma 13.22. □

Lemma 13.24 *If G, B, C, and $\mathcal{F}$ are as described above, then C is 3α-closed with respect to $\mathcal{R}(k-1)$ and $(\alpha - 2\epsilon)$-closed with respect to $\mathcal{R}_k$.*

Proof: Let X be a path readable in C and also in a general relator $R \in \mathcal{R}(k)$ and let S and T be elements in $\mathcal{F}$ which contain the initial and terminal vertices of the path X in C respectively. If there does not exist a functor from R to C which extends the reading of X in C, then the path Y in $S \cup B \cup T$ and readable in R guaranteed by Lemma 13.23 must satisfy $d_R(X) = d_R(Y)$ by Lemma 13.20. Let $Y = Y_1 Y_2 Y_3$ be a partitioning of Y so that Y_1 lies in S, Y_2 lies in B, and Y_3 lies in T. If S and T intersect so that Y never includes an edge in B, then consider $Y_2 = \emptyset$ and $d_R(Y_2) = 0$. If R is in $\mathcal{R}_k$ then $d_R(Y_1) < \beta$, and $d_R(Y_3) < \beta$

by Axiom 2. If $d_R(Y) \geq \alpha - 2\epsilon$ then $d_R(Y_2) < \alpha - 2\beta - 2\epsilon$ and R is already included in the construction C. Thus C is $(\alpha - 2\epsilon)$-closed with respect to $\mathcal{R}_k$. If R is in $\mathcal{R}(k-1)$ then either $d_R(Y_1) < \alpha$, $d_R(Y_2) < \alpha$, and $d_R(Y_3) < \alpha$, or R is already included in S, B, or T since each of these constructions is α-closed with respect to $\mathcal{R}(k-1)$. Thus C is 3α-closed with respect to $\mathcal{R}(k-1)$. Finally, notice that by construction, every general relator in $\mathcal{F}$ other than B shares a path with B which measures at least $\alpha - 2\beta - 2\epsilon$ in the relator metric of the general relator. □

Lemma 13.25 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{12}$ and $3\beta + 2\epsilon \leq \alpha$, and let ϵ be a constant which satisfies Axiom 5. Suppose that ϵ is also an upper bound on the length of a rank i Zimin word, $i \leq k$, which is read in any general relator R having a rank strictly greater than i, as measured by the relator metric of R. If B is an $\mathcal{R}$-category which is $(1 - 3\alpha - 2\epsilon)$-closed with respect to $\mathcal{R}(k)$, then there is an $\mathcal{R}$-category C containing B which is α-closed with respect to $\mathcal{R}(k)$, and such that every vertex in C is connected to a vertex in B by a path reading a rank k Zimin word. In addition, if B is μ-closed with respect to $\mathcal{R}_j$ with $j > k$, then C is $(\mu + 2\epsilon)$-closed with respect to $\mathcal{R}_j$.*

Proof: If $k = 1$ then the result follows immediately from Lemma 13.24, so suppose that the lemma is true for all ranks strictly less than k, and $k > 1$. By applying the lemma in the previous rank, there is a finite $\mathcal{R}$-category B_1 containing B which is α-closed with respect to $\mathcal{R}(k-1)$ and $(1 - 3\alpha)$-closed with respect to $\mathcal{R}_k$. Then by applying Lemma 13.24 there is an $\mathcal{R}$-category B_2 containing B_1 such that B_2 is 3α-closed with respect to $\mathcal{R}(k-1)$ and $(\alpha - 2\epsilon)$-closed with respect to $\mathcal{R}_k$. Finally, by applying this lemma again in the previous rank, there is an $\mathcal{R}$-category C containing B_2 such that C is α-closed with respect to $\mathcal{R}(k)$. By induction every vertex in C is connected to a vertex in B_2 by a rank $(k-1)$ Zimin word. The fact that B_2 is obtained from B_1 by attaching rank k relators to B_1 shows that every vertex in B_2 is connected to a vertex in B_1 by a path readable in a rank k relator. Then, by induction again, every vertex in B_1 is connected to a vertex in B by a rank $(k-1)$ Zimin word. Combining these facts shows that every vertex in C is connected to a vertex in B by a rank k Zimin word.

Finally, let X be a path readable in C and also in a general relator $R \in \mathcal{R}_j$, $j > k$, such that there does not exist a functor from R to C which extends the reading of X in C. By repeatedly using Lemma 13.23 applied to each of the local closures, there is a word PYQ readable in both C and R where both paths are homotopic to X relative to its endpoints, and such that P and Q are rank k Zimin words and Y is a possibly empty path in the $\mathcal{R}$-category B. Moreover, by Lemma 13.20 and the fact that R is not included in C at this point, $d_R(X) = d_R(PYQ)$. By the assumptions stated in the lemma, $d_R(P) < \epsilon$ and $d_R(Q) < \epsilon$. Also, since R is not attached to C and B is μ-closed, $d_R(Y) < \mu$. Using the properties of relator metrics, the combination of these bounds shows that $d_R(X) = d_R(PYQ) < \mu + 2\epsilon$. Thus C is $(\mu + 2\epsilon)$-closed with respect to $\mathcal{R}_j$. □

Under minor restrictions it is possible to prove that the μ-closure of a finite $\mathcal{R}$-category with respect to $\mathcal{R}(k)$ is still finite. The existence of a finite μ-closure follows from the construction of a single finite μ-closed $\mathcal{R}$-category containing the original category. The first lemma shows that if this is the case, the μ-closure is also effectively constructible.

Lemma 13.26 *If B and C are finite $\mathcal{R}$-categories such that B is embedded in C, and C is μ-closed with respect to $\mathcal{R}(k)$, then the unique μ-closure of B in C with respect to $\mathcal{R}(k)$ is effectively constructible.*

Proof: Because of the existence of the $\mathcal{R}$-category C, the finiteness of the μ-closure is immediate. Also, a straightforward but inefficient procedure to find the unique minimum μ-closure of B is to list all of the $\mathcal{R}$-categories in C containing B and then to test to see which of them are μ-closed with respect to $\mathcal{R}(k)$. $\square$

Lemma 13.27 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{12}$, and $3\beta + 2\epsilon \leq \alpha$. In addition let ϵ be a constant which satisfies Axiom 5 and also such that ϵ is an upper bound on the length of a rank i Zimin word, $i \leq k$, which is read in any general relator R having a rank strictly greater than i, as measured by the relator metric of R. If B is a finite $\mathcal{R}$-category which is $(1-3\alpha-2\epsilon)$-closed with respect to $\mathcal{R}(k)$, C is the α-closure of B with respect to $\mathcal{R}(k)$, and there is a bound on the diameters of the closed cones which are in C but not in B, then C is finite and effectively constructible.*

Proof: The bound on the diameters shows that there is also a bound on the length of rank j Zimin words, $j \leq k$, if each of the component pieces which are readable in a general relator of the appropriate rank is a geodesic in that relator. As shown in the proof of Lemma 13.25, every vertex in C differs from a vertex in B by a path of this type. Since C is collapsed, and the alphabet A is finite, this shows that the number of vertices in C is also finite. Finally, the fact that C is collapsed means that the category C is also finite. $\square$

Lemma 13.28 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{12}$, and $3\beta + 2\epsilon \leq \alpha$ and $2\beta + \delta + 2\epsilon < \alpha$. In addition let ϵ be a constant which satisfies Axiom 5 and also such that ϵ is an upper bound on the length of a rank i Zimin word, $i \leq k$, which is read in any general relator R having a rank strictly greater than i, as measured by the relator metric of R. If X is a word which is $(1-3\alpha-2\epsilon)$-free with respect to $\mathcal{R}(k)$ then $str_k(X)$ is finite and effectively constructible. As a consequence, when $1 - 3\alpha - 2\epsilon \geq \frac{1}{2} + \alpha$, $str_k(X)$ is finite and effectively constructible for all words X.*

Proof: The abstract path X is an $\mathcal{R}$-category which satisfies Lemma 13.27. The condition of the bounds of the diameters is satisfied because the final result is contained in the $\mathcal{R}$-category produced by a finite number of local closures. In each local closure the added closed cones have paths measuring $\alpha - 2\beta - 2\epsilon$ in the relator metric. Without loss of generality the path can be assumed to be a geodesic and thus by Axiom 4 it has a length of at least $\alpha - 2\beta - 2\epsilon - \delta$ of the length of the relator. Since the construction so far is simply connected and finite, there is a bound on the length of possible geodesics, which when divided by $\alpha - 2\beta - 2\epsilon - \delta > 0$ gives a bound on the length of the newly attached closed cones, and this in turn leads to a finite bound on the diameters of these closed cones. The induction through the local closures shows that the final result has a bound on the diameters of the closed cones in C which are not in B. Lemma 13.27 completes the proof. Finally, in general $\mathrm{str}_k(X)$ is created using a Dehn reduction of the word X, and the inequality listed guarantees that the prior reasoning can be applied to the Dehn reduction. □

14 Finite Subgroups

The strong connection between torsion elements in G and automorphisms of general relators in $\mathcal{R}$, which was examined in Section 12, foreshadows an even stronger connection between the finite subgroups of G and the automorphism groups of the general relators in $\mathcal{R}$, when G is a general small cancellation group. In particular, under relatively mild conditions, every finite subgroup of the group is contained in the automorphism group of some general relator. The converse, that every subgroup of an automorphism group occurs as a finite subgroup, is an easy consequence of the fact that the Cayley category is proper. At the conclusion of the section Theorem A is shown.

14.1 Equivalent Conditions on Finite Subgroups

Lemma 14.4 establishes a set of four equivalent conditions on finite subgroups of general small cancellation groups. Eventually (Lemma 14.17) it will be shown that all of the finite subgroups satisfy these conditions, but for the moment, it will merely be shown that the conditions are equivalent. One of the implications in the proof is proved using a well-known fixed-point lemma for finite groups acting simplicially on finite contractible simplicial complexes, which is given below.

Lemma 14.1 *Let H be a finite group which acts simplicially on a finite contractible simplicial complex B. Then the action of H on B must have a global fixed point iff H is a finite group which is a q-group extended by a cyclic group extended by a p-group, where p and q are primes.*

Proof: See [10]. □

The following technical lemma will be needed:

Lemma 14.2 *Let $G = \langle A|\mathcal{R}\rangle$ be a finitely presented general small cancellation presentation with $\alpha \leq \frac{1}{12}$, and let R be a general relator in $\mathcal{R}$. If X is a path in R such that $d_R(X) \geq \frac{1}{2} + \delta$, then either there exists an initial segment U of X such that all paths V in R with the same endpoints as U satisfy $d_R(V) \geq 3\alpha$, or else every edge in R is long.*

Proof: Since $d_R(X) \geq \frac{1}{2} + \delta$, by Axiom 4 $|X|_R \geq \frac{1}{2}$. Since geodesic distance changes at most 1 unit when a word is shortened by a single letter, there must be an initial segment U of X such that $|U|_R$ is at most $\frac{1}{2|R|}$ from $\frac{1}{2}$. Thus $\frac{1}{2} - \frac{1}{2|R|} \leq |U|_R \leq \frac{1}{2} + \frac{1}{2|R|}$. Given any other path V in R between the same endpoints, UV^{-1} forms a loop. If the winding number of this loop is 0, then U and V are homotopic in ∂R and consequently $|V|_R \geq |U|_R$ by definition. If, on the other hand, the winding number is $k \geq 1$, then $|U|_R + |V|_R \geq k \geq 1$. In either case $|V|_R \geq \frac{1}{2} - \frac{1}{2|R|}$. If $|R| \geq 3$, then this implies that $|V|_R \geq \frac{1}{3}$, and that $d_R(V) \geq \frac{1}{3} - \delta \geq 3\alpha$ by Axiom 4 and by assumption. If, on the other hand, $|R| < 3$, then let V be a single edge. By the definition of the normalized graph metric, $|V|_R \geq \frac{1}{|R|} \geq \frac{1}{3}$, so that $d_R(V) \geq \frac{1}{3} - \delta \geq 3\alpha$ by Axiom 4 and by assumption. This shows that every edge in R is long. □

Lemma 14.3 *Let $G = \langle A|\mathcal{R}\rangle$ be a finitely presented general small cancellation presentation with $\alpha \leq \frac{1}{12}$ for which str(W) is finite and effectively constructible for all words $W \in A^*$. In addition, let H be a finite subgroup of G, let C be the Cayley category of G, and let B be a finite subcategory of C. If the natural action of H on C restricts to an action of H on B, then there is a finite, contractible, and effectively constructible subcategory B' containing B on which H acts.*

Proof: If B is not connected then the finite number of components can be connected with a finite number of simplicial paths in the 1-skeleton. In order to extend the action of the group H to the resulting $\mathcal{R}$-category, let B_1 be the union of B and the finite number of images of each of the connecting paths under the automorphisms of H. Next, consider all pairs of vertices $u, v \in B_1$. If u and v are viewed as vertices in C then there is a geodesic path X from u to v. By assumption $\mathrm{str}(X)$ exists, it is finite, and it is effectively constructible. Moreover, $\mathrm{str}(X)$ can be viewed as the α-closure of X in C with respect to $\mathcal{R}$, and by Lemma 13.12 there is a unique embedding of $\mathrm{str}(X)$ which sends the startpoint to u and the endpoint to v. The uniqueness of $\mathrm{str}(X)$ guarantees that the image of $\mathrm{str}(X)$ under $h \in H$ is the α-closure of the image of X with respect to $\mathcal{R}$ as well. Let B_2 be a subcategory of C obtained by adding to B_1 such a straightline construction for every pair of vertices u and v in B_1. Since a $\mathrm{str}(X)$ construction is added between u and v iff it is also added between $h(u)$ and $h(v)$ for all $h \in H$, the action of the group H on B_1 extends to B_2.

The key step in the proof involves showing the B_2 is $(1 - 3\alpha)$-closed with respect to $\mathcal{R}$. Consider a path V in B_2 which is readable in a general relator T with $d_T(V) \geq 1 - 3\alpha$. By Lemma 14.2 there is a subword U such that every path in T between the same endpoints as U measures at least 3α. Let u and v be the startpoint and endpoint of U respectively. Since u is in B_2 there must be a path X in B_2 which starts and ends in B_1 such that u is contained in the embedding of $\mathrm{str}(X)$ determined by the endpoints of X. Similarly, there must be a path Y in B_2 which starts and ends in B_1 such that v is contained in the embedding of $\mathrm{str}(Y)$ determined by the endpoints of Y. Let u' be one of the endpoints of X and let v' be one of the endpoints of Y. Since $\mathrm{str}(X)$ is connected and α-closed with respect to $\mathcal{R}$, there is a geodesic X' in $\mathrm{str}(X)$ from u to u'. In particular, by Lemma 13.13 str(X') embeds in str(X) and thus is contained in B_2. Similarly, since str(Y) is connected and α-closed with respect to $\mathcal{R}$, there is a geodesic Y' in str(Y) from v' to v. In particular, by Lemma 13.13 str(Y') embeds in str(Y) and thus is contained in B_2. Next, notice that u' and v' are in B_1 so that if Z is a geodesic from u' to v', str(Z) is contained in B_2 as well. Since the paths X and Y will no longer be needed, relabel X' and Y' as X and Y. The construction so far has produced a path XZY from u to v such that str(X), str(Z) and str(Y) are contained in B_2.

If XZ is not a geodesic in C, then there is a path W between the same endpoints of strictly shorter length. But then by Lemma 11.8, W is contained in the union of str(X) and str(Z) and W can be split into X' and Z' such that X' is readable in str(X) and Z' is readable in str(Z'). By Lemma 13.13, str(X') and str(Z') are embedded in str(X) and str(Z) and thus in B_2. This produces a strictly shorter word $X'Z'Y$ from u to v such that str(X'), str(Z') and str(Y) are in B_2. After this process is repeated at most a finite number of times (on XZ and on ZY alternately), and after suitable relabeling, a path XZY is obtained with str(X), str(Z), and str(Y) contained in B_2, and with the additional property that XZ and ZY are geodesics in C.

If XZY is $(1-3\alpha)$-free then by Lemma 11.3 it is readable in T, and by Lemma 14.2 $d_T(XYZ) \geq 3\alpha$. Thus one of the three subpaths, say X, is long in T and since str(X) is α-closed, T is contained in str(X) and thus in B_2. If, however, XZY is not $(1-3\alpha)$-free, then there exists a general relator R and a path P in XZY with $d_R(P) \geq 1-3\alpha$. Since XZ and ZY are geodesics and thus $(\frac{1}{2}+\alpha)$-free, it follows that P contains a portion of X and Y. Thus let $P = X_1ZY_1$. Since $\alpha \leq \frac{1}{12}$, $d_R(X_1) \geq 2\alpha$, and $d_R(Y_1) \geq 2\alpha$. In particular R is contained both in str(X) and in str(Y). This in turn implies that the path XZ lies completely in str(X). If XZ is relabeled as X, then XY is a path from u to v such that str(X) and str(Y) are contained in B_2, and both X and Y are geodesics. Applying the reduction argument described above to the path XY shows that without loss of generality, we can assume that XY is a geodesic in C. At this point, by Lemma 11.3 XY is readable in T, and by Lemma 14.2 $d_T(XY) \geq 3\alpha$. Thus one of the two subpaths, say X, is long in T and since str(X) is α-closed, T is contained in str(X) and thus in B_2. In either case T must be contained in B_2, so that B_2 is $(1-3\alpha)$-closed with respect to $\mathcal{R}$.

At this point it has been shown that B_2 is connected and that it is $(1-3\alpha)$-closed with respect to $\mathcal{R}$. Since B_2 is a subcategory of C, it is also collapsed. From Lemma 12.1, it follows that B_2 is simply connected, and then by Lemma 12.13, B_2 is contractible. Thus B_2 satisfies the conditions stated in the lemma. $\square$

Lemma 14.4 *Let $G = \langle A|\mathcal{R}\rangle$ be a finitely presented general small cancellation presentation with $\alpha \leq \frac{1}{12}$ in which str(W) is finite and effectively constructible for all words $W \in A^*$, and let C be the Cayley category of the presentation. In addition suppose that all general relators in $\mathcal{R}$ contain at least one crucial cone. If H is a finite subgroup of G then the following conditions are equivalent:*

(1) the natural action of H on C has a global fixed point

(2) H is isomorphic to a subgroup of the automorphism group of some general relator $R \in \mathcal{R}$, or the automorphism group of a labeled edge

(3) H is isomorphic to a finite group which is either a cyclic or a dihedral group extended by a 2-group

(4) H is isomorphic to a subgroup of a finite group which is a q-group extended by a cyclic group extended by a p-group, where p and q are primes.

Proof: $(1 \Rightarrow 2)$ Let $C' = \text{Chain}(C)$ be the simplicial subdivision of C. The action of H on C can then be viewed as a simplicial action of H on C' which preserves the heights of the vertices in C'. If the action of H on C' has a global fixed point, then that point must occur in the interior of a unique simplex. Since the vertices of the simplex are ordered by their height, the elements of H cannot permute them. Thus the entire simplex is fixed by H and without loss of generality the global fixed point of H can be chosen to be a vertex of C'. Call this vertex c and consider the slice category C/c. Since the action of H on C preserves height, H must act on the image of the slice category in C and by Lemma 12.3 H acts on C/c itself. Notice that if c has height 0 then H must be trivial, if c has height 1 then C/c is a labeled edge, and if c has a height at least 2 then C/c is a general relator in $\mathcal{R}$. In all three cases, the second condition is satisfied.

$(2 \Rightarrow 3)$ This follows immediately from Lemma 5.26, and the fact that the automorphism group of a labeled edge is at most cyclic order 2. $(3 \Rightarrow 4)$ Since a dihedral group can be viewed as a group of order 2 extended by a cyclic group, this step also follows immediately. $(4 \Rightarrow 1)$ By Lemma 14.3, H acts on a finite contractible subcategory B of C. Thus, by Lemma 14.1 the action of H on B has a global fixed point. Since this global fixed point is also a global fixed point of the action of H on C, the proof is complete. $\square$

Lemma 14.5 *Let $G = \langle A|\mathcal{R}\rangle$ be a finitely presented general small cancellation presentation with $\alpha \leq \frac{1}{12}$ in which str(W) is finite and effectively constructible for all words $W \in A^*$, and let H be a finite subgroup of G. In addition suppose that all general relators in $\mathcal{R}$ contain at least one crucial cone. If H is a p-group for some prime p, then H satisfies all of the equivalent conditions listed in Lemma 14.4. In particular this is true when H is a 2-group.*

Proof: The result is immediate since by assumption H satisfies condition (4) of Lemma 14.4. $\square$

Another easy application of Lemma 14.4 uses the solvability of groups of odd order to show that all odd-order subgroups of finitely presented general small cancellation groups are cyclic. Later in this section a more detailed examination of the finite subgroups of general small cancellation groups will provide another proof of this fact, so that the results collected in Theorem A remain independent of the Odd Order Theorem.

Lemma 14.6 *Let $G = \langle A|\mathcal{R}\rangle$ be a finitely presented general small cancellation presentation with $\alpha \leq \frac{1}{12}$ in which str(W) is finite and effectively constructible* *for all words $W \in A^*$, and let H be a finite subgroup of G. In addition suppose that all general relators in $\mathcal{R}$ contain at least one crucial cone. If H is a finite subgroup of G of odd order then the group H is cyclic.*

Proof: By the Odd Order Theorem, H is a solvable group and thus has a composition series whose factors are cyclic groups of prime order. The proof proceeds by induction on the length of the composition series. If the composition series has length 1 then H is cyclic and there is nothing to prove. Next, assume that the result has been shown for all finite subgroups with composition series of length k and let H have a composition series of length $k+1$. By induction, H has a cyclic normal subgroup of prime index. Since H satisfies condition (4) of Lemma 14.4, H is also isomorphic to a subgroup of a finite group which is a dihedral extended by a 2-group. In particular, there is a group homomorphism from H to a dihedral whose kernel is a 2-group. But since H has odd order, the kernel of the map is trivial and H is isomorphic to a subgroup of the dihedral group. Moreover, the absence of elements of order 2 shows that H injects into the cyclic subgroup of the dihedral and H itself is thus cyclic. This completes the induction and the proof. $\square$

14.2 Inequalities and Estimates

The following lemmas are a collection of inequalities and estimates which will be needed in the proof of Lemma 14.12.

Lemma 14.7 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{10}$ and let R be a general relator in $\mathcal{R}_k$. If U and V are words which are conjugate in $G(k-1)$ and U^n and V^n are both readable in R as loops with non-zero winding numbers, then $|d_R(U) - d_R(V)| \leq 2\gamma + 2\delta < 2\alpha$. In particular, if U and V are arbitrary representative cycles of R then $|d_R(U) - d_R(V)| \leq 2\gamma + 2\delta \leq \alpha$.*

Proof: Without loss of generality, assume that U is a geodesic in R and let W be a cycle conjugate to U in $G(k-1)$ and of minimal length subject to this restriction. First of all, it is clear that $|U| \geq |W|$ and that the cycle of W is $(\frac{1}{2} + \alpha)$-free with respect to $\mathcal{R}(k-1)$, for otherwise Lemma 10.3 would yield a contradiction to the minimality of W. Next, by Lemma 11.4 W^k is readable as a loop in R with the same winding number as U^k. Finally, notice that the automorphism of R induced by sending the initial vertex of one of the paths U to its endpoint will also send the initial vertex of each of the vertices of each of these paths W to their endpoints.

By the definition of ω_R, there is a path of length at most ω_R from the initial vertex of one of the paths U to a point in the loop W^k. Without loss of generality let W be the conjugate of W determined by the endpoint of this path. The image of this path under the automorphism determined by U sends it to another path between U^k and W^k. Since U is a geodesic it is clear that $|U| \leq |W| + 2\omega_R$. Combining these inequalities shows that $|U|$ is within ω_R of $|W| + \omega_R$. Since the word V must also satisfy these restrictions, the length of U and the length of V differ by at most $2\omega_R$, and consequently, $||U|_R - |V|_R| \leq 2\gamma$ by Axiom 3. Using Axiom 4 and Axiom 7, $|d_R(U) - d_R(V)| \leq 2\gamma + 2\delta < 2\alpha$. Finally, let U and V be representative cycles and let P be a path in R between their basepoints. Depending on the relative orientations of U and V either $PUP^{-1}V$ or $PUP^{-1}V^{-1}$ is a loop of winding number 0 and thus by Lemma 7.15 U is conjugate to either V or its inverse. The situation thereby reduces to the one described above. □

Lemma 14.8 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{12}$. If g is a non-trivial torsion element in G then there exist a word U and a general relator $R \in \mathcal{R}_k$ such that $2\alpha \leq \frac{1}{3} - 2\gamma - 2\delta \leq d_R(U) \leq \frac{1}{2} + \alpha$, the cycle of U is $(1-4\alpha)$-free with respect to $\mathcal{R}$, U represents a power of g as an automorphism of the Cayley category obtained from repeatedly squaring g, and a power of U is a non-trivial loop in R. In addition, all words which are conjugate to U in $G(k-1)$ and readable in R also satisfy the same conditions.*

Proof: Let $n > 1$ be the order of g in G. If u and v are vertices of the Cayley category of G such that the automorphism g sends u to v, then by Lemma 6.3 u and v must be distinct. Let X be a path from u to v. Since g has order n, X^n is readable as a loop in the Cayley category. Next, by Lemma 12.7 there is a word Y and a general relator R such that Y^n is readable in R as a loop with non-zero winding number, with Y cyclically reduced in the free group and $(\frac{1}{2}+\alpha)$-free with respect to $\mathcal{R}$. In particular, $d_R(Y) \leq \frac{1}{2} + \alpha \leq \frac{2}{3}$, since $\alpha \leq \frac{1}{6}$. On the other hand, since Y^n has a non-zero winding number by the properties of relator metrics $d_R(Y^n) \geq 1$, and $d_R(Y^{ni}) \geq i$. Then by properties 4 and 5, $nd_R(Y^i) \geq i$. This shows that by choosing i large enough, it is possible to make $d_R(Y^i)$ as large as desired. In particular, there is an i so that $d_R(Y^i) \geq \frac{1}{3}$. Let j be the smallest non-negative integer such that $d_R(Y^i) \geq \frac{1}{3}$ where $i = 2^j$. If $d_R(Y) \geq \frac{1}{3}$ then $j = 0$. If j is positive, then $i = 2l$ and by the properties of relator metrics $d_R(Y^i) \leq d_R(Y^l) + d_R(Y^l) \leq \frac{2}{3}$. The two instances of a path reading Y^l must have the same measure by property 6, in so far as they measure at least $\frac{1}{6} \geq \alpha$ and thus differ by an automorphism of the general relator R.

Let V be the geodesic in R between the endpoints of Y^i. It will be shown that a power of V is readable in R as a loop with a non-zero winding number. If Y^iV^{-1} is a loop in ∂R of winding number 0, then $d_R(V) = d_R(Y^i)$ by the properties of relator metrics and it is clear that a power of V is readable in R as a loop with non-zero winding number. On the other hand, if $d_R(V) \neq d_R(Y^i)$ then the winding number of Y^iV^{-1} must be at least 1. If it is more than 1 then $d_R(V)$ is at least $\frac{4}{3}$ and $|V| \geq (\frac{4}{3} - \delta)|R| > (\frac{2}{3} + \delta)|R| \geq |Y^{2^i}|$ by Axiom 4. Since this implies that V is not a geodesic, it must be the case that Y^iV^{-1} has winding number exactly 1. Next, let m be the winding number of Y^{ni}. Since $d_R(Y^{ni}) \leq nd_R(Y^i) \leq n\frac{2}{3} < n$, it follows by property 5 of relator metrics that $m < n$. Moreover, by successively removing Y^iV^{-1} it is clear that the winding number of $Y^{ni}(V^{-1})^n$ is n once it is noted that each of these Y^iV^{-1} loops must be similarly oriented. The reason for this is that if successive copies of Y^i have opposite orientations then the symmetry of the situation would require that the winding number of Y^{ni} be 0, contradiction. Finally, since the winding number $Y^{ni}(V^{-1})^n$ is n and the winding number of Y^{ni} is $m < n$, the winding number of V^n is $n - m \neq 0$, which was to be shown.

Since V is a geodesic in R, by Lemma 10.3 $d_R(V) < \frac{1}{2} + \alpha$, and by construction $d_R(V) \geq \frac{1}{3}$ either because $d_R(V) = d_R(Y^i) \geq \frac{1}{3}$ or because $d_R(V) \geq 1 - d_R(Y^i) \geq 1 - \frac{2}{3}$. Let R be a general relator in $\mathcal{R}_k$. By Lemma 10.5, V is conjugate to a word U whose cycle is $(\frac{1}{2} + \alpha)$-free with respect to $\mathcal{R}(k-1)$ and with $|V| \geq |U|$. By Lemma 11.4, U^n is readable as a loop in ∂R and it is homotopic to the loop V^n. In particular it has the same non-trivial winding number. Finally, by Lemma 14.7, $\frac{1}{3} - 2\alpha \leq d_R(U) \leq \frac{1}{2} + 3\alpha$. If $d_R(U)$ is greater than $\frac{1}{2} + \alpha$ then U can be replaced by a geodesic in R between the same endpoints and the arguments of the last two paragraphs can be repeated. Since the lengths of the words under consideration are constantly shortening, a finite number of iterations yields a word U with $\alpha \leq \frac{1}{3} - 2\alpha \leq d_R(U) \leq \frac{1}{2} + \alpha$ which satisfies the requirements of the lemma. The final statement of the lemma follows from the fact that the other words conjugate to U in $G(k-1)$ and readable in R are conjugate to Y^i as well, so that Lemma 14.7 yields the required inequalities, whereas the other conditions are immediate. □

Lemma 14.9 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation and suppose that $W = UV$ is a representative of a general relator R in $\mathcal{R}$. If U is long, negatively oriented with respect to W, and a geodesic path in its homotopy class in ∂R, then $|V| \geq |U| + (1-2\gamma)|R|$ and $d_R(V) \geq d_R(U) + 1 - 2\gamma - 2\delta$.*

Proof: Lift the word UV to R^∞, the universal cover of the boundary of R. Since U is long and negative, the ball of radius ω_R centered at the start vertex of U is strictly between the balls of radius ω_R centered at the end of U (which equals the start of V) and at the end of V. Since by the definition of ω_R these balls disconnect R^∞, the path V must contain a vertex within ω_R units of the start of U. Let P be the path from the start of U to such a vertex and let $V = V'V''$ be a split of V determined by this point. Without loss of generality, assume that U is a geodesic in R^∞. Thus $|V'| + |P| \geq |U|$. On the other hand, since PV'' is a representative of R when it is read in ∂R, it follows that $|V''| + |P| \geq |R|$. Thus $|V| + 2|P| > |U| + |R|$, and since $|P| < \gamma|R|$ by Axiom 3, $|V| > |U| + (1-2\gamma)|R|$. Using Axiom 4, $d_R(V) > d_R(U) + 1 - 2\gamma - 2\delta$. □

Lemma 14.10 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation and suppose that $W = U_1V_1U_2V_2$ is a representative of a general relator R in $\mathcal{R}$. If U_i is long, negatively oriented with respect to W, and a geodesic path in its homotopy class in ∂R for $i = 1, 2$, then $|V_1| + |V_2| \geq |U_1| + |U_2| + (1-4\gamma)|R|$ and $d_R(V_1) + d_R(V_2) \geq d_R(U_1) + d_R(U_2) + 1 - 4\gamma - 4\delta$.*

Proof: Let X and Y be the paths of shortest length in the homotopy classes of U_1V_1 and U_2V_2 in ∂R, respectively. Without loss of generality, assume that $d_R(X) \geq d_R(Y)$. Since XY is a representative, $2d_R(X) \geq d_R(X) + d_R(Y) \geq 1$, so that $d_R(X) \geq \frac{1}{2} > \alpha$, and X is long. Thus by Lemma 9.9 X is oriented. If X is negatively oriented with respect to XY, then by Lemma 14.9 $|Y| - |X| \geq$

$(1-2\gamma)|R|$. Since by assumption $d_R(X) \geq d_R(Y)$, by Axiom 4 $2\delta|R| \geq |Y| - |X| \geq (1-2\gamma)|R|$, contradiction. Thus X is positively oriented as well as long.

Next, lift the path W to R^∞, and let u_i, v_i be the starting vertex of U_i, V_i respectively, for $i = 1, 2$. The fact that X is long and positively oriented while U_1 is long and negatively oriented with respect to the cycle W in R^∞ implies that the balls of radius ω_R centered at v_1, u_1 and u_2 (in R^∞) are disjoint and that they occur in that order. Specifically, since by the definition of ω_R, these balls disconnect R^∞, the path V must pass through the ball centered at u_1. Thus there exists a path P from u_1 to a point in V with $|P| \leq \gamma|R|$. Let $V = V'V''$ be the partition of V determined by this endpoint. Since U_1 is a geodesic in R^∞, $|P| + |V'| \geq |U_1|$. Also, since the image of U_2V_2PV'' in ∂R is a representative of R which satisfies the hypotheses of Lemma 14.9 with U_2 playing the role of U, and V_2PV'' playing the role of V, it follows that $|P|+|V''|+|V_2| \geq |U_2|+(1-2\gamma)|R|$. Combining these two inequalities and using the fact that $|P| \leq \gamma|R|$ yields $|V_1| + |V_2| \geq |U_1| + |U_2| + (1-4\gamma)|R|$. Finally, applying Axiom 4 shows that $d_R(V_1)+d_R(V_2) \geq d_R(U_1)+d_R(U_2)+1-4\gamma-4\delta$. □

Lemma 14.11 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation and suppose that $W = XUYU^{-1}$ is a representative of a general relator T in $\mathcal{R}$. If both instances of U are properly oriented with respect to W, $d_T(U) \geq \alpha$, and there exists a relator R in $\mathcal{R}$, distinct from T, such that $d_R(X) \geq \alpha$ and X^n is readable as a loop in R with a non-zero winding number, then there exists a word Z such that the cycle $XZX^{-1}Z^{-1}$ is a representative of T and corresponding to the two readings of the word X in this cycle there are two instances of the relator R contained in the boundary of T. Both instances of the word X represent the same orientation-reversing automorphism of T which fixes both R_1 and R_2. Moreover, the shortest path between these two instances of R has a length of at least $(\frac{1}{2}-\beta-\delta)|T|$ units.*

Proof: First, by Axiom 6 there is a word V such that $XVYV^{-1}$ is readable as a contractible loop in ∂T. Since the reading of $XVYV^{-1}$ can be chosen so that it extends the reading of Y given by W (because the axiom is symmetric with respect to X and Y), there is another loop in ∂T which reads $XUV^{-1}X^{-1}VU^{-1}$ which is a representative since it is a combination of a loop with winding number 1 and a contractible loop. Choosing $Z = UV^{-1}$ completes the first proof. To distinguish between the two copies of Z and X, let Z_1 be the copy of Z which starts and ends at the start of the readings of X, with Z_2 being the other, and let X_1 be the reading of X which starts at the start of Z_1, with X_2 being the other. Next, by Axiom 1 the readings of X_1 and X_2 can be extended to embeddings of R in T, and since R and T are distinct, the copies of R must be contained in ∂T. Moreover, the extension can be arranged so that the reading of X in T is also in the loop X^n in R, simply by choosing this reading of X in R for use in the application of Axiom 1. Let R_1 and R_2 be the instances of R in T associated with X_1 and X_2 respectively.

Using Axiom 2 and the properties of relator metrics, it is clear that $d_T(X_i) < \beta$ for $i = 1, 2$ and that therefore $d_T(Z_i) > \frac{1}{2} - \beta$ for either $i = 1$ or $i = 2$ and thus for both $i = 1$ and $i = 2$ by another property of relator metrics and the fact that $\frac{1}{2} - \beta > \alpha$. As a technical aside, notice that the two instances of X must have the same measure only because they are long in T. As a consequence there is an automorphism g of T which sends Z_1 to Z_2, which is also the automorphism induced by either X_1 or X_2. Clearly the paths X_1 and X_2 cannot be loops themselves since this would mean that Z_1 and Z_2 correspond to the same reading in T and that at least one of the X loops would have a non-trivial winding number in T. This in turn, by a property of relator metrics and Lemma 9.10, would imply the identity of R and T contrary to the hypothesis in the statement of the lemma. Thus X_1 and X_2 are not loops and Lemma 12.5 can be applied to show that g is also an automorphism of both R_1 and R_2, so that they remain fixed under the action of g.

To prove the final claim, let P be the shortest path from R_1 to R_2, let Q_1 be a path from the start of P to the start of Z_1, and let Q_2 be a path from the end of P to the end of Z_1. Since R_1 and R_2 are stabilized by g, the image of the loop $Q_1^{-1}PQ_2Z_1^{-1}$ is a loop $Q_1^{-1}PQ_2Z_2^{-1}$ where the new Q_1 and the new Q_2 are read in R_1 and R_2 respectively. Combining these loops with the loop $X_1Z_2X_2^{-1}Z_1^{-1}$ produces a loop $Q_1X_1Q_1^{-1}PQ_2X_2^{-1}Q_2^{-1}P^{-1}$ where $Q_iX_iQ_i^{-1}$ is read in R_i, $i = 1, 2$. If this loop has a non-zero winding number then by the same reasoning as above, $d_T(Q_iX_iQ_i^{-1}) < \beta$ for $i = 1, 2$ and consequently $d_T(P) > \frac{1}{2} - \beta$. Thus by Axiom 4 $|P| > (\frac{1}{2} - \beta - \delta)|T|$. Therefore it only remains to show that the winding number of this loop is non-zero.

Suppose that one of the readings of Z, say Z_1, is negatively oriented with respect to the loop $X_1Z_2X_2^{-1}Z_1^{-1}$. Then by Lemma 14.9, $d_T(X_1) + d_T(Z_2) + d_T(X_2) \geq 1 + d_T(Z_1) - 2\gamma - 2\delta$. Since by the properties of relator metrics $d_T(Z_1) = d_T(Z_2)$ and since $d_T(X_i) < \beta, i = 1, 2$, it follows that $2\beta \geq 1 - 2\gamma - 2\delta$ which contradicts the constraints in Axiom 7. Thus both instances of Z must be positively oriented with respect to the loop. Because Z_1 and Z_2 are facing opposite directions within the loop itself, this means that the winding numbers of the loops $Q_1^{-1}PQ_2Z_1^{-1}$ and $Q_1^{-1}PQ_2Z_2^{-1}$ will cancel each other out, so that the loop $Q_1X_1Q_1^{-1}PQ_2X_2^{-1}Q_2^{-1}P^{-1}$ still has a winding number of 1. This completes the proof. □

14.3 Elements of Maximum Rank

The long proof which follows provides the key technical lemma needed to show that all finite subgroups of a general small cancellation group are contained in the automorphism group of some general relator in the presentation.

Lemma 14.12 *Let $G = \langle A|\mathcal{R}\rangle$ be a graded general small cancellation presentation with $2\beta + 2\gamma + \delta \leq \alpha \leq \frac{1}{12}$, let H be a finite subgroup of G, and let C be the Cayley category of the presentation. If $g \in H$ is an element of maximum rank (rank$(g) \geq$ rank(h) for all $h \in H$), and g rotates a general relator $R \in \mathcal{R}$, then for all $h \in H$ there is an instance of R in C which is fixed by both g and h.*

Proof: Given non-identity elements $g, h \in H$ of orders n and m respectively, then by Lemma 10.5 there are words X and Y which are cyclically Dehn-reduced with respect to $\mathcal{R}$ and which represent g and h respectively. By Lemma 12.7 the words X^n and Y^m are readable as loops with non-zero winding numbers in uniquely determined general relators R and S respectively. By assumption g has been chosen so that the rank of its associated general relator R is maximal among all of the general relators associated to the elements of H in this way. The proof below will show that each of the elements of H stabilizes an instance of R in the Cayley category which is stabilized by g. The action of g on the Cayley category of G will stabilize some instances of the general relator R in the Cayley category and possibly will not stabilize other instances. Let R_1 be an instance of R in C which is stabilized by g and let S_1 be an instance of S in C which is stabilized by h. The proof is divided into two stages. First it will be shown that R_1 and S_1 can be chosen so that they have a vertex in common. Next it will be shown that the action of h stabilizes the chosen R_1.

Claim 1: R_1 and S_1 can be chosen so that $R_1 \cap S_1 \neq \emptyset$.

Let R_1 and S_1 be chosen so that they are as close to each other in the graph metric as possible, and assume for the moment that $R_1 \cap S_1 = \emptyset$, and assume that P is the shortest possible path in the Cayley category connecting a vertex u' in R_1 to a vertex v' in S_1. By assumption, P is non-empty. Next, by Lemma 14.8 there are powers of g and h, say g^i and h^j, as well as words U and V conjugate to X^i and Y^j respectively, such that these words satisfy the conclusions of Lemma 14.8. In particular, powers of these words are readable as loops in R and S with non-zero winding numbers. Let u and $g^i(u)$ be the endpoints of a path reading U in R_1, and let Q be a path from u to u'. Then $Q^{-1}UQ$ is a path from u' to $g^i(u')$ in R_1. Let U' be a geodesic path in the boundary of R_1 between u' and $g^i(u')$ which lies in the same homotopy class as $Q^{-1}UQ$. Thus together these paths form a loop in R_1 of winding number 0, so that by Lemma 7.15 and Lemma 7.12 U and U' are conjugate in $G(k-1)$. Then, by Lemma 11.4 U^n and $(U')^n$ are both readable in R_1 as loops with the same non-zero winding number, and by Lemma 14.8 U' also satisfies the same conditions as U. Similarly choose V' as a geodesic path in the boundary of S_1 between v' and $h^j(v')$. By the same reasoning V^m and $(V')^m$ are readable in S_1 as loops with the same non-zero winding number, and V' satisfies the conclusions of Lemma 14.8. Thus for convenience U' and V' can be relabeled as U and V since they satisfy the same conditions and the original U and V will no longer be needed in the proof.

Let U, V and the two paths labeled P be called the sides of the cycle $UPV^{-1}P^{-1}$. The first task will be to show that this cycle is 3α-complement-free with respect to $\mathcal{R}$. If this were not true then there would be a word W, a word Z, and a general relator T such that W is a subword of a cyclic conjugate of $UPV^{-1}P^{-1}$, WZ is a representative of T, and $d_T(Z) < 3\alpha$. A case-by-case analysis will show that no such words W and Z can exist. The cases will be divided by the number of sides needed to contain the word W.

Case 1: First of all, the subword W cannot be completely contained in one of the sides labeled P since P is a geodesic in the Cayley category, $1-3\alpha > \frac{1}{2}+\alpha$, and by Lemma 10.3, geodesics are $(\frac{1}{2}+\alpha)$-free with respect to $\mathcal{R}$. Similarly the subword W cannot be completely contained within a side labeled U or V since this would imply that the general relator T is contained in either R or S respectively, and a similar contradiction arises due to the fact that U and V are geodesics in the boundaries of R and S.

Case 2: Next, suppose that W is a subword of two consecutive sides in the cycle $UPV^{-1}P^{-1}$. Without loss of generality assume that W is a subword of UP since the other possibilities can be handled analogously. Let $W = W_1W_2$ where W_1 is the portion of W in U and W_2 is the portion in P. If $d_T(W_1) \geq \alpha$ then by Axiom 1 the instance of the general relator T which contains the path W is contained in the instance of R containing the path U. In particular, this would mean that the path P between S_1 and R_1 would not be disjoint from R_1 except at its endpoint, contradiction. If, on the other hand, $d_T(W_1) < \alpha$ then by the properties of relator metrics, $d_T(W_2) > 1-4\alpha \geq \frac{1}{2}+\alpha$, and this contradicts the assumption that P is a geodesic.

Case 3A: Suppose that W is a subword of three consecutive sides. If the middle of these three sides is labeled P, then without loss of generality assume that W is a subword of UPV^{-1} and write $W = W_1PW_2$ in the obvious way. If either $d_T(W_1) \geq \alpha$ or $d_T(W_2) \geq \alpha$ then a contradiction to the disjointness of P from S_1 and R_1 is obtained as above. If on the other hand both measures are less than α, then $d_T(P) > 1-5\alpha \geq \frac{1}{2}+\alpha$ and P is not a geodesic in the Cayley category.

Case 3B: If the middle of the three sides is labeled either U or V then without loss of generality assume that W is a subword of $P^{-1}UP$ and write $W = W_1UW_2$ in the obvious way. Since W contains all of U and $d_R(U) \geq 2\alpha$, it follows from Axiom 1, Lemma 2.1, and Axiom 2 that R is contained in T, that the rank of R as a cone complex is less than that of T (since equality leads to a contradiction of the disjointness of P and R_1), and that $d_T(U) < \beta$. If either $d_T(W_1) < \alpha$ or $d_T(W_2) < \alpha$ then the measure of the other is more than $1-4\alpha-\beta$ and this contradicts the fact that P is a geodesic. So assume that both ends measure at least α with respect to the general relator T. If W_1 is negatively oriented with respect to the loop W_1UW_2Z, then, by applying Lemma 14.9 to this loop, $d_T(U) + d_T(W_2) + d_T(Z) \geq 1 + d_T(W_1) - 2\gamma - 2\delta$. Since $d_T(U) < \beta$, $d_T(Z) < 3\alpha$, and $d_T(W_1) \geq \alpha$, it follows that $d_T(W_2) > 1-2\alpha-\beta-2\gamma-2\delta \geq 1-4\alpha > \frac{1}{2}+\alpha$, which contradicts the fact that P, and hence W_2, is a geodesic. Thus W_1 can be assumed to be positively oriented with respect to the loop W_1UW_2Z. An analogous argument shows that W_2 must also be positively oriented with respect to the loop. Again without loss of generality assume that $|W_1| \leq |W_2|$, so that $W_2 = W_1^{-1}W_3$ for some possibly empty word W_3. Lemma 14.11 can then be applied to the representative WZ with W_1^{-1} playing the role of U, and U and W_3Z standing in for X and Y respectively. Thus there exists another instance of R, called R_2, in T which is fixed by g and which is at least $(\frac{1}{2}-\beta-\delta)|T|$ units away from R_1.

Let u be the start vertex of one of the paths U in the loop U^n in R_2. Since

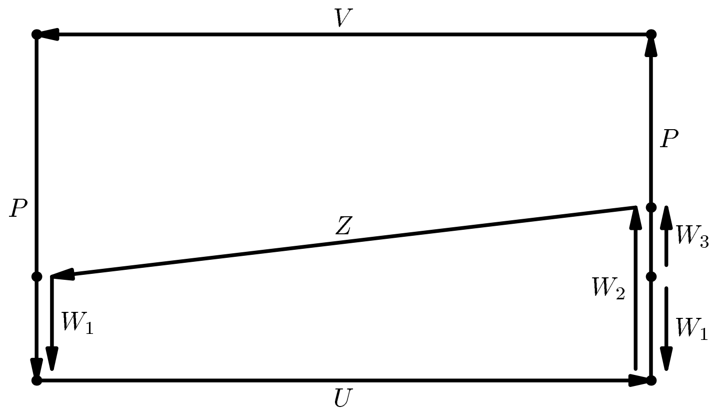

Figure 32: An illustration of Case 3B

WZ is a representative of T, by Axiom 3 there is a path from u to WZ whose length is at most $\gamma|T|$. Let Q be one such path. If the endpoint of Q lies is W then it must fall within either W_1 or W_2 and at least $(\frac{1}{2} - \beta - \gamma - \delta)|T| > \gamma|T|$ away from U. This shows that S_1 is closer to R_2 than to R_1. Even if the endpoint of Q lies in Z, the length of Z is at most $(3\alpha + \delta)|T|$ by Axiom 4 so that this endpoint lies at most $(\frac{3}{2}\alpha + \frac{1}{2}\delta + \gamma)|T| \leq 2\alpha|T|$ from a point in one of the instances of P. By Lemma 14.11, this point in P must be at least $(\frac{1}{2} - 2\alpha - \beta - \delta)|T| \geq 2\alpha|T|$ so that S_1 is closer to R_2 than to R_1, contradiction. Notice that if the middle of the three sides covered by W had been labeled V instead of U, then W would have been a subword of $P^{-1}UP$, and the conclusion would have been that there is an instance S_2 of S which is stabilized by h and R_1 is closer to S_2 than to S_1.

Case 4: The next possibility is that W is a subword of four consecutive sides. Without loss of generality, assume that W is a subword of $P^{-1}UPV^{-1}$ and write $W = W_1UPW_2$ in the obvious way. Repeating the arguments given above shows that the only possibility which avoids immediate contradictions is the one where $d_T(U) < \beta$ and $d_T(W_2) < \alpha$. If in addition $d_T(W_1) < \alpha$ then $d_T(P) > 1 - 5\alpha - \beta \geq \frac{1}{2} + 2\gamma + \delta$ which contradicts the fact that P is a geodesic. Therefore assume that $d_T(W_1) \geq \alpha$. On the other hand, since there is an initial segment of P which reads W_1^{-1}, let $P = W_1^{-1}W_3$ for some possibly empty word W_3. By property 6 of relator metrics and Axiom 1, this second instance of W_1 is also long in T. If either W_1 is negatively oriented with respect to the loop representing T, then by applying Lemma 14.9 to this loop together with the inequalities listed above, it follows that $d_T(W_3) + d_T(W_1) + \alpha + 3\alpha > 1 + d_T(W_1) - \delta - 2\gamma$. Thus $d_T(W_3) > 1 - 4\alpha - \beta - 2\gamma - \delta \geq \frac{1}{2} + \alpha$ by the restrictions assumed on the constants, which contradicts the assumption that P is a geodesic. Both instances of W_1 are thus long and positively oriented with respect to the loop representing T. Lemma 14.11 can now be applied to this

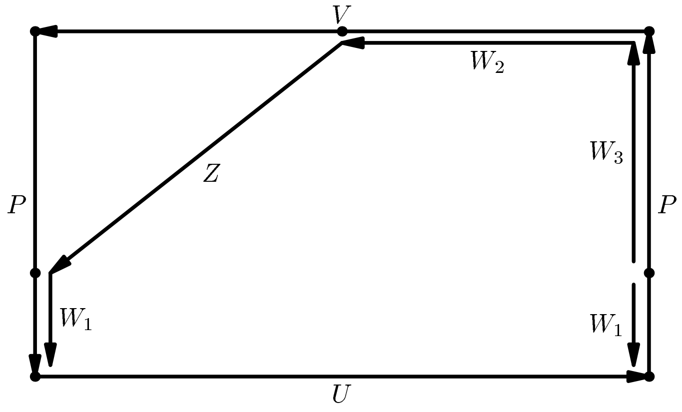

Figure 33: An illustration of Case 4

situation with W_1^{-1} playing the role of U, and U and W_3W_2Z standing in for X and Y respectively. Thus there exists another instance of R, called R_2, in T which is fixed by g and which is at least $(\frac{1}{2}-\beta-\delta)|T|$ units away from R_1.

Let u be the start vertex of one of the paths U in the loop U^n in R_2. Since WZ is a representative of T, by Axiom 3 there is a path from u to WZ whose length is at most $\gamma|T|$. Let Q be one such path. All possibilities for the endpoint of Q in WZ imply that u lies within $2\alpha|T|$ of W_2 or within $2\alpha|T|$ of one of the instances of P. To see this notice that the endpoint of Q cannot be in U (since by Lemma 14.11 the two instances of R lie at least $(\frac{1}{2}-\beta-\delta)|T|$ apart), that if the endpoint of Q lies in W_1, W_2, or P then the statement is immediate, and that if the endpoint of Q lies in Z then since $|Z| \leq (3\alpha+\delta)|T|$, it follows that u is at most $(\frac{3}{2}\alpha+\frac{1}{2}\delta+\gamma)|T| \leq 2\alpha|T|$ from either W_2 or W_1. Suppose u lies within $2\alpha|T|$ of W_2. From the estimates on the measures of the sides given above, $d_T(P)+d_T(W_1) \geq 1-4\alpha-\beta$. Thus $2|P| \geq |P|+|W_1| \geq (1-4\alpha-\beta-2\delta)|T| \geq 6\alpha|T|$, or $|P| \geq 3\alpha|T|$. In either case R_2 is stabilized by g and S_1 is closer to R_2 than to R_1. If on the other hand u lies within $2\alpha|T|$ of a point v in one of the instances of P, then since u is at least $(\frac{1}{2}-\beta-\delta)|T|$ from U, it follows that the portion of P between U and v has a length of at least $(\frac{1}{2}-2\alpha-\beta-\delta)|T| > 2\alpha|T|$. Thus R_2 is stabilized by g and S_1 is closer to R_2 than to R_1. If the four sides covered by W had been arranged differently, the conclusion would have been that there is an instance S_2 of S which is stabilized by h and R_1 is closer to S_2 than to S_1.

Case 5A: Suppose that W is a subword of five consecutive sides. This is possible because W might begin and end on the same side without overlapping. If the word W begins and ends on a side labeled U or V then without loss of generality assume that W is a subword of $V^{-1}P^{-1}UPV^{-1}$ and write $W = W_1P^{-1}UPW_2$ in the obvious way. As usual it follows fairly immediately that $d_T(W_1) < \alpha$, that $d_T(U) < \beta$, and that $d_T(W_2) < \alpha$. Thus at least one of

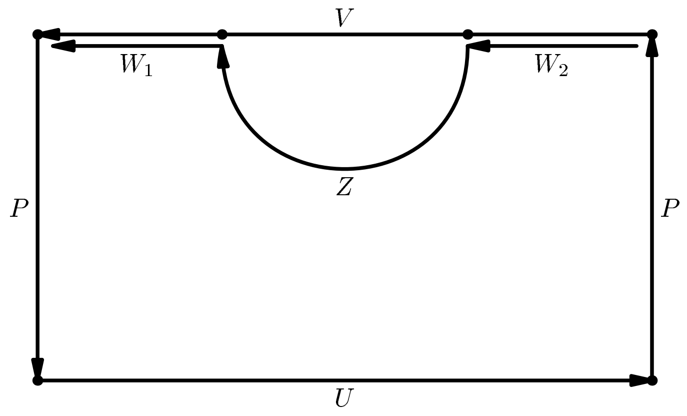

Figure 34: An illustration of Case 5A

the instances of P has $d_T(P) \geq \frac{1}{2}(1 - 5\alpha - \beta) > \alpha$. Consequently at least one of the instances of P is long, and by a property of relator metrics, both instances of P are long and have the same measure. If one of the instances is negatively oriented with respect to the loop $W_1P^{-1}UPW_2Z$ then $d_T(P) \geq 1 + d_T(P) - 4\alpha - \beta - 2\gamma - 2\delta$, or $0 \geq 1 - 4\alpha - \beta - 2\gamma - 2\delta$, contradiction. Thus both instances of P are positively oriented with respect to the loop $W_1P^{-1}UPW_2Z$. Lemma 14.11 can now be applied to this situation with P playing the role of U, and U and W_2ZW_1 standing in for X and Y respectively. Thus there exists another instance of R, called R_2, in T which is fixed by g and which is at least $(\frac{1}{2} - \beta - \delta)|T|$ units away from R_1.

As in the earlier cases, let u be the start vertex of one of the paths U in the loop U^n in R_2. There must exist a path from u to WZ whose length is at most $\gamma|T|$. Let Q be one such path. All possibilities for the endpoint of Q in WZ imply that u lies within $2\alpha|T|$ of V or within $\gamma|T|$ of one of the instances of P. To see this notice that the endpoint of Q cannot be in U (since by Lemma 14.11 the two instances of R lie at least $(\frac{1}{2} - \beta - \delta)|T|$ apart), that if the endpoint of Q lies in W_1, W_2 or one of the instances of P then the statement is immediate, and that if the endpoint of Q lies in Z then since $|Z| \leq (3\alpha + \delta)|T|$, it follows that u is at most $(\frac{3}{2}\alpha + \frac{1}{2}\delta + \gamma)|T| \leq 2\alpha|T|$ from either W_2 or W_1. Suppose u lies within $2\alpha|T|$ of W_2. From the estimates on the measures of the sides given above, $d_T(P) + d_T(P) \geq 1 - 5\alpha - \beta$. Thus $2|P| \geq (1 - 5\alpha - \beta - 2\delta)|T| \geq 5\alpha|T|$, or $|P| > 2\alpha|T|$. Thus R_2 is stabilized by g and S_1 is closer to R_2 than to R_1. If, on the other hand, u lies within $2\alpha|T|$ of a point v in one of the instances of P, then since u is at least $(\frac{1}{2} - \beta - \delta)|T|$ from U, it follows that the portion of P between U and v has a length of at least $(\frac{1}{2} - 2\alpha - \beta - \delta)|T| > 2\alpha|T|$. Thus in either case R_2 is stabilized by g and S_1 is closer to R_2 than to R_1. If the five sides covered by W had been arranged differently, the conclusion would have been that there is an instance S_2 of S which is stabilized by h and R_1 is closer

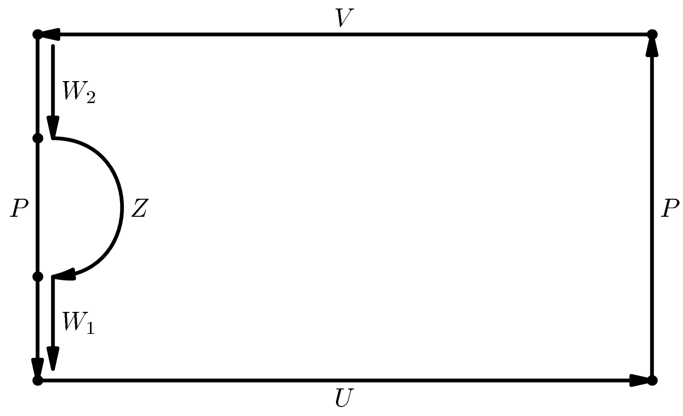

Figure 35: An illustration of Case 5B

to S_2 than to S_1.

Case 5B: Suppose that W is a subword of five consecutive sides and that W begins and ends on one of the sides labeled P. Then, without loss of generality, assume that W is a subword of $P^{-1}UPV^{-1}P^{-1}$ and write $W = W_1UPV^{-1}W_2$ in the obvious way. Moreover, let $P^{-1} = W_2W_3W_1$ for some possibly empty word W_3. Since R_1 and S_1 are chosen to be as close as possible, it follows that $|Z| \geq |W_3|$, and consequently $d_T(W_3) \leq d_T(Z) + 2\delta \leq 3\alpha + 2\delta$. From the usual argument it can be assumed that $d_T(U) < \beta$ and that $d_T(V) < \beta$. Thus $d_T(P) + d_T(W_1) + d_T(W_2) \geq 1 - 3\alpha - 2\beta$, and $d_T(W_1) + d_T(W_1) + d_T(W_2) + d_T(W_2) \geq 1 - 6\alpha - 2\beta - 2\delta \geq 4\alpha$. Consequently at least one instance of either W_1 or W_2 is long relative to T. By Axiom 1 and a property of relator metrics it follows that the other instance is also long. Without loss of generality, assume that both instances of W_1 are long relative to T.

If one of the instances of W_1 is negatively oriented with respect to the loop WZ then by Lemma 14.9 $d_T(W_1) + d_T(W_2W_3) + d_T(W_2) \geq 1 + d_T(W_1) - 3\alpha - 2\beta - 2\gamma - 2\delta$ and $d_T(W_2) + d_T(W_2) \geq 1 - 6\alpha - 2\beta - 2\gamma - 4\delta \geq 2\alpha$ so that one (and thus both) of the two instances of W_2 are long as well. If both instances of W_1 and W_2 are long and at least one instance of W_1 and one instance of W_2 are negatively oriented, then by Lemma 14.10, $d_T(W_1) + d_T(W_2) + d_T(W_3) \geq d_T(W_1) + d_T(W_2) + 1 - 3\alpha - 2\gamma - 2\delta$. Thus $d_T(W_3) \geq 1 - 3\alpha - 2\gamma - 2\delta \geq 1 - 5\alpha \geq \frac{1}{2} + \alpha$, which contradicts the assumption that P, and thus W_3, is a geodesic. Consequently at least one pair of the long W_i are positively oriented. Without loss of generality, assume that the two instances of W_1 are long and positively oriented with respect to WZ. Lemma 14.11 can now be applied to this situation with W_1^{-1} playing the role of U, and U and $W_3^{-1}W_2^{-1}V^{-1}W_2Z$ standing in for X and Y respectively. Thus there exists another instance of R, called R_2, in T which is fixed by g and which is at least $(\frac{1}{2} - \beta - \delta)|T|$ units away from R_1.

As above, let u be the start vertex of one of the paths U in the loop U^n in R_2. There must exist a path from u to WZ whose length is at most $\gamma|T|$. Let Q be one such path. All possibilities for the endpoint of Q in WZ imply that u lies either within $\gamma|T|$ of V or within $2\alpha|T|$ of one of the instances of P. To see this notice that the endpoint of Q cannot be in U (since by Lemma 14.11 the two instances of R lie at least $(\frac{1}{2}-\beta-\delta)|T|$ apart), that if the endpoint of Q lies in W_1, W_2, V, or the intact instance of P then the statement is immediate, and that if the endpoint of Q lies in Z then since $|Z| \leq (3\alpha+\delta)|T|$, it follows that u is at most $(\frac{3}{2}\alpha+\frac{1}{2}\delta+\gamma)|T| \leq 2\alpha|T|$ from either W_2 or W_1. Suppose u lies within $\gamma|T|$ of V. Then since $|P| \geq |W_1| \geq (\alpha-\delta)|T| > \gamma|T|$, R_2 is stabilized by g and S_1 is closer to R_2 than to R_1. If, on the other hand, u lies within $2\alpha|T|$ of a point v in one of the instances of P, then since u is at least $(\frac{1}{2}-\beta-\delta)|T|$ from U, it follows that the portion of P between U and v has a length of at least $(\frac{1}{2}-2\alpha-\beta-\delta)|T| > 2\alpha|T|$. Thus in either case R_2 is stabilized by g and S_1 is closer to R_2 than to R_1. If the five sides covered by W had been arranged differently, the conclusion would have been that there is an instance S_2 of S which is stabilized by h and R_1 is closer to S_2 than to S_1.

Since these six cases exhaust all of the possibilities, the cycle $UPV^{-1}P^{-1}$ must be 3α-complement-free with respect to $\mathcal{R}$. Let Z be a Dehn-reduced cycle which is conjugate to $UPV^{-1}P^{-1}$ in G. Such a cycle exists by Lemma 10.5. Then by Lemma 7.13, there is a connected and simply connected $\mathcal{R}$-diagram Δ whose boundary cycles are Z^{-1} and $UPV^{-1}P^{-1}$. Since the cycle Z is Dehn-reduced it follows that it is 4α-complement-free with respect to $\mathcal{R}$. Thus Z^p is readable as a loop with a non-trivial winding number in a general relator T. The importance of the word Z is that it is a word which represents the automorphism g^i followed by h^{-j}, so that T is the unique general relator associated to and stabilized by the automorphism g^ih^{-j}. Moreover, since the cycle $UPV^{-1}P^{-1}$ is 3α-complement-free with respect to $\mathcal{R}$, by Lemma 11.4 it follows that a power of this cycle is readable as a loop in T. Since $d_R(U) \geq 2\alpha$, by Axiom 1 there is an $\mathcal{R}$-functor from R to T. Thus the rank of T is at least as great as the rank of R. The rank of T cannot be strictly greater than that of R since this would contradict the initial choice of the automorphism g. The only possibility is for the ranks to be equal, which forces T and R to be identical by Lemma 2.1, and this in turns shows that S_1 and R_1 have a point in common, as was to be shown.

Claim 2: h fixes the instance of R which satisfies Claim 1.

This portion of the proof will be divided into two cases depending on whether the image of R_1 (chosen in Claim 1) under the automorphism h intersects with R_1 or not.

Case 1: Assume for the moment that $R_1 \cap h(R_1) = \emptyset$. This implies, in particular, that $S_1 \cap R_1$ and $S_1 \cap h(R_1)$ are disjoint. Let u be a vertex in $S_1 \cap R_1$, let U be a path in R_1 from $g^{-i}(u)$ to u, and let V be a path in S_1 from u to $h(u)$. The path $X = UV$ thus represents the automorphism hg^i. Notice that since hg^i is in H and a word representing it is read in an instance of R followed by an instance of S, there is a power of X, say X^n, which is read as a loop in C in an alternating succession of instances of R and S.

If the cycle of X is not 3α-complement-free with respect to $\mathcal{R}$, then there exist a word W, a word Z, and a general relator $T \in \mathcal{R}$, such that WZ is a representative of T, $d_T(Z) \leq 3\alpha$ and W is a subword of a cyclic conjugate of X. If W, when read in X^n, is not contained in a single selected instance of either R or S, then it is still contained in at most three instances since W is a subword of a cyclic conjugate of UV. However, since $d_T(W) \geq 1 - 3\alpha > 3\alpha$, the portion of W in one of these instances is at least α. Thus by Axiom 1, all of T is contained in this instance, but this contradicts the assumption that W is not contained in a single instance of either R or S. If W, when read in X^n, is contained in a single selected instance of R or S, then all of T is contained in the instance by Axiom 1, and thus Z^{-1} can be substituted for W in a conjugate of X to create a shorter word which represents the same automorphism, and whose n-th power is still readable in the same sequence of instances of R and S. In addition, the new cycle X' has a cyclic conjugate $U'V'$ where $U'V'$ is readable as a path in $R_1 \cup S_1$ where U' is a word in R_1 from $R_1 \cap g^{-i}(S_1)$ to $R_1 \cap S_1$, and V' is a word in S_1 from $R_1 \cap S_1$ to $h(R_1) \cap S_1$. Since the length of the representative strictly decreases, such a substitution can occur only a finite number of times before the process stops.

When the process stops, consider the relationship between the words U and U'. Both U and U' are readable as paths in R_1 from $R_1 \cap g^{-i}(S_1)$ to $R_1 \cap S_1$. Since R and S are α-closed, so are $R_1 \cap g^{-i}(S_1)$ and $R_1 \cap S_1$ by Lemma 13.1. Thus by Lemma 11.3 a path between the startpoint of U and that of U' in R_1 (which exists since R_1 is connected), is equivalent to a path in $R_1 \cap g^{-i}(S_1)$. Similarly there is a path in $R_1 \cap S_1$ from the endpoint of U to that of U'. Call these paths P and Q respectively. By Axiom 2, $d_R(P)$ and $d_R(Q)$ are at most β. If the winding number of $PU'Q^{-1}U^{-1}$ is zero, then by the properties of relator metrics, $d_R(U')+d_R(P)+d_R(Q) \geq d_R(PU'Q^{-1}) = d_R(U)$, and by Lemma 14.8, $d_R(U') \geq \frac{1}{3} - 2\beta - 2\gamma - 2\delta$ which is at least α by the restrictions assumed on the constants. If on the other hand the winding number of $PU'Q^{-1}U^{-1}$ is not zero, then the properties of relator metrics show that $d_R(U') + d_R(P) + d_R(Q) + d_R(U) \geq 1$, which by Lemma 14.8 implies that $d_R(U') \geq \frac{1}{2} - \alpha - 2\beta$ which is also greater than α. In either case $d_R(U') \geq \alpha$.

For convenience relabel the word X' as X. At this point the cycle X is 3α-complement-free with respect to $\mathcal{R}$. Let Z be a Dehn-reduced cycle which is conjugate to X in G. Such a cycle exists by Lemma 10.5. Then by Lemma 7.13, there is a connected and simply connected $\mathcal{R}$-diagram Δ whose boundary cycles are Z^{-1} and X. Since the cycle Z is Dehn-reduced it follows that it is 4α-complement-free with respect to $\mathcal{R}$. Thus Z^p is readable as a loop with a non-trivial winding number in a general relator T. The importance of the word Z is that it is a word which represents the automorphism g^ih, so that T is the unique general relator associated to and stabilized by the automorphism g^ih. Moreover, since the cycle X is 3α-complement-free with respect to $\mathcal{R}$, by Lemma 11.4 it follows that a power of this cycle is readable as a loop in T.

Since the subword U' of X^n which is contained in the original R_1 and since the measure of U' is at least α, there exists an $\mathcal{R}$-functor from R to T by

Axiom 1. Thus the rank of T is at least as great as the rank of R. The rank of T cannot be strictly greater than that of R since this would contradict the initial choice of the automorphism g. The only possibility is for the ranks to be equal, which forces T and R to be identical by Lemma 2.1 and this in turns shows that $hg^i(R_1) = h(R_1) = R_1$, as was to be shown.

Case 2: Assume that $R_1 \cap h(R_1) \neq \emptyset$. The structure of the proof is nearly identical to that of Case 1 with only minor modifications. Let u be a vertex in $R_1 \cap h(R_1)$, let U be a path in R_1 from $g^{-i}(u)$ to u, and let V be a path in $h(R_1)$ from u to $h(u)$. The path $X = UV$ thus represents the automorphism hg^i. Notice that since hg^i is in H and a word representing it is read in an instance of R, there is a power of X, say X^n, which is read as a loop in C in an succession of instances of R.

If the cycle of X is not 3α-complement-free with respect to $\mathcal{R}$, then there exist a word W, a word Z, and a general relator $T \in \mathcal{R}$ such that WZ is a representative of T, $d_T(Z) \leq 3\alpha$ and W is a subword of a cyclic conjugate of X. If W, when read in X^n, is not contained in a single selected instance of either R, then it is still contained in at most three instances since W is a subword of a cyclic conjugate of UV. However, since $d_T(W) \geq 1 - 3\alpha > 3\alpha$, the portion of W in one of these instances is at least α. Thus by Axiom 1, all of T is contained in this instance, which contradicts the assumption that W is not contained in a single instance of R. If W, when read in X^n, is contained a single selected instance of R, then all of T is contained in the instance by Axiom 1, and thus Z^{-1} can be substituted for W in a conjugate of X to create a shorter word which represents the same automorphism, and whose n-th power is still readable in the same sequence of instances of R_1. In addition, the new cycle X' has a cyclic conjugate $U'V'$ where $U'V'$ is readable as a path in $R_1 \cup h(R_1)$ with U' a word in R_1 from $R_1 \cap g^{-i}h(R_1)$ to $R_1 \cap h(R_1)$. Since the length of the representative strictly decreases, such a substitution can occur only a finite number of times before the process stops.

When the process stops, consider the relationship between the words U and U'. Both U and U' are readable as paths in R_1 from $R_1 \cap g^{-i}h(R_1)$ to $R_1 \cap h(R_1)$. Since R is α-closed, so are $R_1 \cap g^{-i}h(R_1)$ and $R_1 \cap h(R_1)$ by Lemma 13.1. Thus by Lemma 11.3 a path between the startpoint of U and that of U' in R_1 (which exists since R_1 is connected) is equivalent to a path in $R_1 \cap g^{-i}h(R_1)$. Similarly there is a path in $R_1 \cap h(R_1)$ from the endpoint of U to that of U'. Call these paths P and Q respectively. The rest of the proof of this case is word for word the same as in case 1. Thus in either case 1 or case 2, $h(R_1) = R_1$ and the proof is complete. □

Lemma 14.13 *Let $G = \langle A|\mathcal{R}\rangle$ be a finitely presented general small cancellation presentation with $2\beta + 2\gamma + \delta \leq \alpha \leq \frac{1}{12}$, let C be the Cayley category of the presentation, let H be a finite subgroup of G, and suppose that all general relators in $\mathcal{R}$ have at least one crucial cone in their boundary. If $g \in H$ has maximum rank, g rotates a general relator $R \in \mathcal{R}$, and the order of g is not a power of 2, then any instance R_1 of R in C which is fixed by g is also fixed by all of the other $h \in H$.*

Proof: In the proof of Lemma 14.12 the instances of R and S were chosen to be as close to each other as possible. The only places in the proof where the fact that R_1 is as close as possible to S_1 was used were in the later cases of Claim 1. Moreover, each invocation of this condition followed an application of Lemma 14.11. This lemma requires that g^{2^i} be an orientation-reversing automorphism of the higher-rank general relator T. But by Lemma 5.26 this would imply that the order g^{2^i} is a power of 2, and this contradicts the additional assumptions made in the statement of this lemma. Lemma 14.11, therefore, cannot be used under these circumstances, and the fact that R_1 is as close as possible to S_1 is never needed in the proof. Thus the instance of R stabilized by g can be chosen first (and arbitrarily) and then the instance of S stabilized by h can be chosen so that it is as close as possible to this fixed R_1. The proof of Lemma 14.12 then proceeds with the minor alteration described above. As a result, every element $h \in H$ must fix this arbitrarily chosen instance of R stabilized by g. $\square$

Lemma 14.14 *Let $G = \langle A|\mathcal{R}\rangle$ be a finitely presented general small cancellation presentation with $2\beta + 2\gamma + \delta \leq \alpha \leq \frac{1}{12}$, let H be a finite subgroup of G, and suppose that all general relators in $\mathcal{R}$ have at least one crucial cone in their boundary. If the order of $g \in H$ is not a power of 2, then g has maximum rank in H.*

Proof: By Lemma 14.12, g fixes the relator of maximum order, but by the structure of automorphism groups of relators (Lemma 5.26), g must be a rotation of this general relator. Thus g itself is an element of maximum rank. $\square$

Lemma 14.15 *Let $G = \langle A|\mathcal{R}\rangle$ be a finitely presented general small cancellation presentation with $2\beta + 2\gamma + \delta \leq \alpha \leq \frac{1}{12}$, let H be a finite subgroup of G, and suppose that all general relators in $\mathcal{R}$ have at least one crucial cone in their boundary. If the order of H is not a power of 2, then H is isomorphic to a subgroup of an automorphism group of a general relator in $\mathcal{R}$.*

Proof: Pick an element $g \in H$ whose order is not a power of 2, let R be the unique general relator rotated by g, and let R_1 be an instance of R_1 in C which is fixed by g. By Lemma 14.14, g is an element of maximum rank, and by Lemma 14.13 all of the $h \in H$ stabilize R_1. Thus H is a subgroup of $\text{Aut}(R_1) = \text{Aut}(R)$. $\square$

14.4 Finite Subgroups

The lemmas below complete the task of showing that all of the finite subgroups of a general small cancellation group are contained in the automorphism group of some general relator.

Lemma 14.16 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation and let C be the Cayley category of G. If H is a finite subgroup of G, then there is a finite set of general relators $\mathcal{R}'$ in $\mathcal{R}$ such that H is also a finite subgroup of the finitely presented general small cancellation presentation $G' = \langle A|\mathcal{R}'\rangle$.*

Proof: Since the general relators in $\mathcal{R}$ are thin by Lemma 9.8, it follows from Lemma 5.19 that there exists a set of standard representatives for the set $\mathcal{R}$. Once G is given a standard presentation, it is a standard exercise in combinatorial group theory to show that there exists a finitely presented subpresentation which also contains the subgroup H. The replacement of the standard representatives by the general relators $R \in \mathcal{R}$ which they represent yields a finite set of general relators $\mathcal{R}'$ which, when closed under subcones, satisfy the requirements of the lemma. □

Lemma 14.17 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation with $2\beta + 2\gamma + \delta \leq \alpha \leq \frac{1}{12}$ in which str(W) is finite and effectively constructible for all words $W \in A^*$, and such that all general relators in $\mathcal{R}$ have at least one crucial cone in their boundary, then every finite subgroup of G is a subgroup of the automorphism group of some general relator in $\mathcal{R}$.*

Proof: Let H be a finite subgroup G. By Lemma 14.16 there is a finite subset $\mathcal{R}'$ of $\mathcal{R}$ which is closed under subcones and which also contains the subgroup H. By Lemma 14.5, the lemma is satisfied whenever H is a 2-group, and since by Lemma 13.6, $\mathrm{str}(W, \mathcal{R}')$ is finite and effectively constructible, Lemma 14.15 shows that it is satisfied whenever H is not a 2-group. □

14.5 Proof of Theorem A

At this point all of the pieces needed to show Theorem A have been shown separately, so it is merely a matter of collecting the references and of showing that the assumptions listed are enough to satisfy the hypotheses of each of the quoted lemmas.

Theorem A *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation with $\alpha \leq \frac{1}{12}$, then the word and conjugacy problems for G are decidable, the Cayley graph is constructible, the Cayley category of the presentation is contractible, and G is the direct limit of hyperbolic groups. If, in addition, the presentation satisfies the hypotheses of Lemma 14.17, then every finite subgroup of G is a subgroup of the automorphism group of some general relator in $\mathcal{R}$.*

Proof: The conclusions follow from Lemma 10.21, Lemma 10.23, Lemma 10.22, Lemma 12.12, Lemma 11.10, and Lemma 14.17, respectively. Any inequality required by one of these lemmas but not listed in the statement above follows immediately from a combination of $\alpha \leq \frac{1}{12}$ and Axiom 7. □

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