Part IV

Small Cancellation Theory

In Part IV the focus is narrowed once again to examine those general presentations which satisfy a version of small cancellation theory. In Section 9 the general small cancellation theory alluded to in the title of the article is presented along with its most immediate consequences. In Section 10 it is shown how Dehn's algorithm can be applied to $\mathcal{R}$-diagrams over general small cancellation presentations to prove the decidability of the word and conjugacy problems for these groups. Finally, in Section 11 finitely presented general small cancellation groups are shown to be Gromov hyperbolic groups, and arbitrary general small cancellation groups are shown to be the direct limit of hyperbolic groups.

9 General Small Cancellation Theory

This section begins by discussing the traditional small cancellation theory and the ways it can be generalized. Then the axioms used to define general small cancellation groups and presentations are given. After discussing the axioms, a few of the more immediate consequences are shown.

9.1 Traditional Small Cancellation Theory

Let $\mathcal{R}$ be a set of standard relators. In traditional small cancellation theory the relators in $\mathcal{R}$ are thought of as a set of cyclically reduced words closed under inverse and cyclic conjugates. If R and S are distinct words in $\mathcal{R}$, and $R = UV_1$ and $S = UV_2$, then U is called a piece relative to $\mathcal{R}$. Alternatively, a piece can be thought of as a possible label on an internal arc in an $\mathcal{R}$-diagram which has no cancellable pairs. In the first definition it is important to realize that the words R and S may be cyclic conjugates and thus belong to the same cycle even though they are distinct words. Similarly, in the second definition the 2-cells on either side of the internal arc may represent the same relator.

A third definition uses $\mathcal{R}$-functors. Let the relators in $\mathcal{R}$ be viewed as closed cones over labeled abstract loops reading cycles which are reduced in the free group. A word U is not a piece relative to $\mathcal{R}$ if whenever U is read in relators R and S by $\mathcal{R}$-functors f and g respectively, there always exists an $\mathcal{R}$-functor $h : R \to S$ such that $hf = g$. By Lemma 2.1, the functor h must be an isomorphism (since in the traditional theory all relators have the same rank), thus showing that this definition is equivalent to the first two definitions.

The traditional theory focuses on two types of hypotheses called $C(p)$ and $C'(\alpha)$. The first is stated in terms of pieces and the second uses the normalized graph metric. A set of standard relators is said to satisfy $C(p)$ if the relators are cyclically reduced and no word in $\mathcal{R}$ is the product of fewer than p pieces. The notions of a $C(p)$ set of relators and a $C(p)$-map are closely related.

Lemma 9.1 *If $\mathcal{R}$ is a $C(p)$ set of standard relators, and Δ is an $\mathcal{R}$-diagram with cyclically reduced boundaries and no cancellable pairs, then the underlying map of Δ is a $C(p)$-map.*

Proof: To prove the result it is sufficient to show that there are no vertices of degree 1, and that all internal arcs are pieces. Since the boundary cycles of Δ are cyclically reduced, and since $\mathcal{R}$ is a $C(p)$ set of relators, the boundary cycle of every region is cyclically reduced. If Δ contained a vertex v of degree 1, then a small disk around v would be strictly contained in some region and the boundary cycle of this region would not be cyclically reduced, contradiction. If an internal arc is not a piece of a relator, then the label on the arc is the initial segment of a unique word in $\mathcal{R}$ and either the arc is on the border of two distinct 2-cells, or the arc is read in the boundary of a single 2-cell in two distinct ways. In the former case the cells must form a cancellable pair, contradicting the assumption about Δ. And in this latter case, the two readings yield opposite orientations of the boundary, forcing the boundary of the 2-cell to be a word in the free group which is conjugate to its own inverse. By Lemma 4.2 the only word for which this is true is the empty word. Thus every internal arc must be a piece and the proof is complete. □

Lemma 9.2 *Let $\mathcal{R}$ be a $C(p)$ set of standard relators. If Δ is an $\mathcal{R}$-diagram whose boundary cycles are cyclically reduced and linked, then there exists an equivalent $\mathcal{R}$-diagram Δ' over a $C(p)$-map with Type$(\Delta) \geq$ Type(Δ').*

Proof: Since a collection of reduced cycles clearly has distinct representatives and a rank function which assigns every relator the rank of 2, the result follows immediately from Lemma 7.9 and Lemma 9.1 □

A set of cyclically reduced standard relators $\mathcal{R}$ is said to satisfy $C'(\alpha)$ if every piece of every relator has a length strictly less than α in the normalized graph metric. More specifically, let $\mathcal{R}$ be viewed as a set of measured relators by defining $d_R(U)$ as the length of the path U in the normalized graph metric on R for all paths U in R. The condition $C'(\alpha)$ is satisfied by $\mathcal{R}$ if whenever U is a piece relative to $\mathcal{R}$ and U is readable in R, then $d_R(U) < \alpha$. In other words, $\mathcal{R}$ satisfies the condition $C'(\alpha)$ if whenever $d_R(U) \geq \alpha$ for some path U in R, then U is not a piece relative to $\mathcal{R}$. The third description of a piece, given above, can be used to state the condition $C'(\alpha)$ in the form in which it appears in the general theory. A set of standard relators $\mathcal{R}$ is said to satisfy $C'(\alpha)$ if whenever U is read in relators R and S by $\mathcal{R}$-functors f and g respectively and $d_R(U) \geq \alpha$, there exists an $\mathcal{R}$-functor h such that $hf = g$.

The above discussion motivates the following definition. Let $\mathcal{R}$ be a set of measured relators, let C be an $\mathcal{R}$-category, and let μ be a non-negative real. The $\mathcal{R}$-category C is called μ-closed if whenever a word U is read in C by an $\mathcal{R}$-functor g and U is also read in a general relator R by a functor f with $d_R(U) \geq \mu$, there exists a unique functor $h : R \to C$ such that $hf = g$. As an example, Cayley categories are constructed so that every general relator is attached to every vertex in every possible way, so Cayley categories are 0-closed.

Another example is the condition $C'(\alpha)$. A set of standard relators $\mathcal{R}$ satisfies $C'(\alpha)$ iff every relator $R \in \mathcal{R}$ is α-closed with respect to $\mathcal{R}$. The notion of being μ-closed is related to the earlier notions of being μ-reduced, μ-free, or μ-complement-free. If a labeled abstract path or loop is μ-closed, then the word or cycle with which it is labeled is μ-free. Conversely, because of the inclusion or exclusion of equality, abstract paths and loops labeled by μ-free words and cycles are structures which are λ-closed for all $\lambda > \mu$, but not necessarily structures which are μ-closed. For example, in the traditional theory, where the normalized graph metric is the relator metric used, it is clear that a word is μ-reduced iff it is μ-free, and it is also clear that a word can contain exactly one-half of a relator so that it is $\frac{1}{2}$-reduced (Dehn-reduced) but not necessarily $\frac{1}{2}$-closed.

Lemma 9.2 can be combined with the lemmas from Lyndon and Schupp to yield the results below. The results are stated in terms of k-remnants. A word V is called a k-remnant if some word $R \in \mathcal{R}$ has the form $R = U_1U_2U_3 \dots U_kV$ where $U_1, \dots, U_k$ are pieces.

Lemma 9.3 *Let $\mathcal{R}$ be a set of standard relators satisfying $C(6)$. If W is a cyclically reduced word equivalent to 1 in the group $G = \langle A|\mathcal{R}\rangle$ then either W is equal to 1, W is a relator, or a cyclic conjugate of W has the form $U_1V_1 \dots U_nV_n$ where each V_k is an $i(V_k)$-remnant such that the number n of the V_k and the numbers $i(V_k)$ satisfy the relation*

$$\sum_{k=1}^{n}(4 - i(V_k)) \geq 6$$

Proof: Consider a reduced proof-diagram whose boundary reads W which must exist by Lemma 7.10 and Lemma 9.2. If W is not equal to 1 or to a relator in $\mathcal{R}$, then the reduced diagram must contain more than one 2-cell. The result follows from Lemma 7.1 by simply letting the words V_k represent the label on the unique boundary arcs of the exposed cells of the diagram. □

Lemma 9.4 *Let $\mathcal{R}$ be a set of standard relators satisfying $C'(\alpha)$ for some $\alpha \leq \frac{1}{8}$. If X and Y are cyclically Dehn-reduced words which are conjugate in the group $G = \langle A|\mathcal{R}\rangle$ but not equivalent to 1 in G, then there exists an annular $\mathcal{R}$-diagram Δ with boundary cycles X and Y^{-1} such that Δ contains no cancellable pairs, and every 2-cell in Δ has internal degree of at most 2. Moreover, every 2-cell contains exactly one arc in each boundary cycle, and the length of the boundary arcs is always more than $\frac{1}{4}$ and no more than $\frac{1}{2}$ of the length of the relator represented by the cell.*

Proof: Since $\mathcal{R}$ is a $C(6)$ set of relators, the diagram Δ is a $C(6)$-map by Lemma 9.1. Let D be an arbitrary exposed cell. Without loss of generality, let the unique boundary arc of D be part of the boundary cycle labeled X. Since X is Dehn-reduced at most half of the boundary of D is contained in X, which means that at least half of the boundary of D is internal. Since the internal arcs are pieces (Lemma 9.1), and by assumption each piece is strictly less than $\frac{1}{8}$ of the boundary, the internal degree must be greater than 4. Thus every exposed cell D in Δ has $i(D) > 4$, and Lemma 7.3 can be applied to complete the proof. The lower bound on the length the boundary arcs follows from the observation that both rungs if they exist are strictly less than $\frac{1}{8}$ of the boundary and a side is at most $\frac{1}{2}$, so the other side must contain more than $\frac{1}{4}$ of the boundary. $\square$

Lemma 9.4 implies in particular that there are cyclic conjugates X_1 of X and Y_1 of Y and a piece U of some relator R such that $UX_1 = Y_1U$. Moreover, more than $\frac{1}{4}$ of the length of R is contained in both X_1 and Y_1, significantly restricting the number of relators which can take on this role. The above lemmas quickly lead to the solution to the word problem and the conjugacy problem for small cancellation groups using a procedure known as Dehn's algorithm. Since the traditional versions are well-known and the general versions will be covered in depth later, no further details will be given here. One application will be given here since it is not as familiar, and since it is also the basis of the construction used in Section 13 to create α-closures of words and cycles. The following lemma is a consequence of the details given in the proof of Lemma 9.4.

Lemma 9.5 *Let $\mathcal{R}$ be a set of standard relators satisfying $C'(\alpha)$ for some $\alpha \leq \frac{1}{8}$. If X is a cyclically Dehn-reduced word which is not equivalent to 1 in the group $G = \langle A|\mathcal{R}\rangle$, then there exists a finite $\mathcal{R}$-structure which contains every cyclically Dehn-reduced word conjugate to X in the group G read as a loop in its 1-skeleton, homotopic to the loop reading X.*

Proof: First find the necessarily finite set of all paths in the abstract loop X which read more than one-fourth of a relator R in $\mathcal{R}$, whose length is at most $|X|$. Next, simply attach all of the appropriate relators R, thought of here as labeled 2-cells, to these paths in the loop X so that the intersection of the loop X and ∂R contains the particular path in question. If the resulting structure is collapsed then a finite $\mathcal{R}$-structure is formed which satisfies the conclusion of the lemma.

To see that this is true, let Y be any non-trivial cyclically Dehn-reduced word conjugate to X in G, and let Δ be a reduced annular diagram proving that X and Y are conjugate. The structure of Δ is given by Lemma 9.4. The result follows from the observation that the cell category of Δ maps into the construction. Begin by sending the boundary cycle of Δ labeled X to the image of the loop X under the collapsing functor. Then extend this map to the cells of Δ. The structure of Δ described in Lemma 9.4 guarantees that all of the cells needed for the extension have already been added to the construction, and the collapsed nature of the construction guarantees that all identifications of boundary edges in Δ are also identified in the construction. Once Δ is mapped into the construction it is clear that the construction must contain a loop reading Y and the image of Δ provides the homotopy to the loop reading X. $\square$

One thing to be aware of is that the $\mathcal{R}$-structure constructed in Corollary 9.5 may not be planar, although for 2-dimensional complexes at worst it is homeomorphic to a Möbius band. Example 1 illustrates this worst-case scenario.

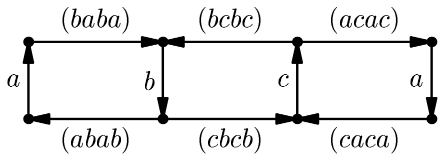

Figure 25: A Möbius strip which contains all reduced conjugates of U

Example 1 Consider the set of relators $\mathcal{R} = \{(ab)^5, (bc)^5, (ca)^5\}$ and the word $X = ababa(bcbc)^{-1}acac$. If the left and right edges labeled a in Figure 25 are identified with the proper orientation, then the resulting non-planar $\mathcal{R}$-structure is the construction described in Corollary 9.5 applied to X. In this instance it produces a Möbius band. In reading the figure, the reader should keep in mind that the parentheses indicate the order in which the label is written (left to right) may conflict with the order in which the letters occur in the arc (indicated by the direction of the arrow). For instance, the arc at the bottom of the leftmost rectangle is labeled $abab$, which means that the letters a, b, a, b are encountered when reading the arc from right to left. From an examination of the construction, it becomes clear that there are exactly 2 cycles, other than X itself, which are non-trivial, Dehn-reduced, and conjugate with X in the group $G = \langle a, b, c | \mathcal{R} \rangle$.

9.2 General Small Cancellation Axioms

The description of the axioms of a general small cancellation presentation involves five constants. The constant α is analogous to the constant used in traditional small cancellation theory in that it measures the degree to which one relator can be contained in another without being subsumed. The other constants are more or less specific to general relators. The axioms are as follows:

Axiom 1 *There is a constant α such that every general relator $R \in \mathcal{R}$ is α-closed with respect to $\mathcal{R}$. In particular, if U is a word readable in general relators R and S via $\mathcal{R}$-functors f and g respectively, and $d_R(U) \geq \alpha$, then, since S is α-closed, there exists a unique $\mathcal{R}$-functor $h : R \to S$ such that $hf = g$.*

Axiom 2 *There is a constant β such that whenever a word U is readable in general relators $R, S \in \mathcal{R}$ by $\mathcal{R}$-functors f and g respectively, and either rank$(R) <$ rank(S) or rank$(R) =$ rank(S) but there does not exist an $\mathcal{R}$-functor $h : R \to S$ with $hf = g$, then $d_S(U) < \beta$.*

Axiom 3 *There is a constant γ such that for all general relators $R \in \mathcal{R}$, $\omega_R \leq \gamma|R|$.*

Axiom 4 *There is a constant δ such that the length of a path U in a general relator $R \in \mathcal{R}$ in the relator metric d_R is within δ of its length in the normalized graph metric on the boundary of R. Specifically $|d_R(U) - |U|_R| \leq \delta$.*

Axiom 5 *There is a constant ϵ such that whenever U is the shortest possible path from a vertex in a general relator $R \in \mathcal{R}$ to a loop with non-zero winding number in R, the length of U in the relator metric is at most ϵ. That is, $d_R(U) \leq \epsilon$.*

Axiom 6 *If $W = XUYU^{-1}$ is a representative of a general relator R in which both instances of U are properly oriented with respect to W, and $d_R(U) \geq \alpha$, then there exists a word V such that the cycle $XVYV^{-1}$ is readable as a contractible loop in ∂R extending the reading of X given by W. The cycle thus bounds a connected and simply connected $\mathcal{R}$-diagram Δ with rank(Δ) < rank(R).*

Axiom 7 *The constants α, β, γ, δ, and ϵ satisfy the following constraints: $\beta \leq \alpha$, and $\gamma, \delta, \epsilon < \alpha$, and $2\gamma + \delta \leq \alpha \leq \frac{1}{6}$.*

A general small cancellation presentation $G = \langle A | \mathcal{R} \rangle$ is a measured presentation which satisfies the axioms listed above. A group which possesses a general small cancellation presentation is called a general small cancellation group.

9.3 Basic Consequences

As stated above, the constant α corresponds to the constant used in the traditional theory. The other four constants provide bounds on various aspects of general relators. The constant β bounds the length of a path in a general relator of lower rank when it is measured by the relator metric of a general relator of higher rank. The constant γ bounds the ratio of the width of the general relator to its length. The constant δ bounds the difference between the normalized graph metric and the relator metric on the boundary of a general relator. And the constant ϵ bounds the length of geodesics going across general relators as measured by their relator metrics.

The general small cancellation theory presented here claims to be an extension of the more traditional theory in the following sense. Every traditional small cancellation presentation which satisfies the condition $C'(\alpha)$ for some $\alpha \leq \frac{1}{6}$ is also a general small cancellation for the same value of α, with $\beta = \alpha$, and with $\gamma = \delta = \epsilon = 0$. The constants γ, δ, and ϵ are zero because traditional relators have width 0 and the metric used is the normalized graph metric. The comparison can also be made more explicit. The hypothesis $C'(\alpha)$ is formulated in terms of pieces. Axiom 1 states that all pieces measure less than α in the relator metric, which in this case is the same as the normalized graph metric on R. To see this, view the relators R and S as closed cones of height 2. Then by Lemma 2.1 the functor mentioned in the axiom must be an isomorphism. Thus, words which contain more than α of the boundary of a relator are readable in only one relator in $\mathcal{R}$. The fact that Axiom 2 is true when $\beta = \alpha$ is true more generally, and this is the content of Lemma 9.6. In the traditional theory, the relators are cyclically reduced, which makes the loops labeled by these cycles deterministic with width 0. Thus Axiom 3 is satisfied. By using the normalized graph metric, the lefthand side of Axiom 4 is always 0, and thus the inequality is true. Again, since the width of the relators is 0, Axiom 5 is immediate. And by Lemma 4.2, Axiom 6 is predicated on a situation which never occurs in the traditional theory. The constraints in Axiom 7 are immediate.

In the general case, Axiom 1 is a rough way of saying that either two general relators have a small overlap or one of them is contained in the boundary of the other, Axiom 3 guarantees the 'niceness' of the relators, and Axiom 4 guarantees the 'niceness' of the relator metrics. Axiom 6 provides a reduction in a situation which does not occur in the 2-dimensional relators. The other two axioms are derivable from those already listed. In Lemma 9.6 it is shown that $\beta = \alpha$ satisfies Axiom 2, and in Lemma 9.7 it is shown that $\epsilon = \gamma + \delta$ satisfies Axiom 5.

Lemma 9.6 *If $G = \langle A|\mathcal{R}\rangle$ is a measured presentation satisfying Axiom 1, then Axiom 2 is also satisfied for the constant $\beta = \alpha$.*

Proof: Suppose that $d_S(U) \geq \alpha$ in one of the situations described by Axiom 2. By Axiom 1 there is a functor $h' : S \to R$ with $h'g = f$. But then Lemma 2.1 combined with the assumptions of Axiom 2 shows that R and S have the same rank and that h' is an isomorphism. If $h : R \to S$ is the inverse of the isomorphism h', then $hf = g$, which is contrary to the initial assumption. $\square$

Lemma 9.7 *If $G = \langle A|\mathcal{R}\rangle$ is a measured presentation satisfying Axiom 3 and Axiom 4, then Axiom 5 is also satisfied for the constant $\epsilon = \gamma + \delta$.*

Proof: If U is the geodesic described in Axiom 5, then by the definition of the width of a general relator and Axiom 3 respectively, $|U| < \omega_R \leq \gamma|R|$. Thus, by Axiom 4, $d_R(U) < \gamma + \delta$. $\square$

The reason for the redundancy is to introduce all five of the constants at the same time and in the same place. The constants are retained because certain constructions, such as the α-closure of an $\mathcal{R}$-category, are only possible if these constants satisfy stricter inequalities than the easy bounds mentioned above. The only constraints which have been included in Axiom 7 are those needed to reproduce the most basic results of small cancellation theory. Other useful restrictions will be added as needed in specific situations. The remaining results in this section will show that general small cancellation presentations satisfy the special conditions needed to apply some of the lemmas developed earlier. Specifically, the general relators in a general small cancellation presentation are thin and collapsed, all long arcs are oriented, and the set $\mathcal{R}$ has distinct representatives.

Lemma 9.8 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. If R is a general relator in $\mathcal{R}_k$, then R is thin and collapsed, and ∂R is α-closed with respect to $\mathcal{R}(k-1)$.*

Proof: By Axiom 7, $\gamma \leq \frac{1}{12} < \frac{1}{4}$. Thus $4\omega_R < |R|$ and R is by definition thin. Next, notice that the 1-skeleton of R is deterministic by the definition of a general relator. Let S be a general relator in R. By Lemma 5.1, the boundary of S includes a representative cycle U, and by one of the properties of relator metrics the word U has $d_S(U) \geq 1$. Thus by Axiom 1 there is a unique functor from S to R which includes this reading of the word U in R. In particular, no two distinct closed cones in R have identical boundaries. Thus R is collapsed. Finally, the fact that ∂R is α-closed with respect to $\mathcal{R}(k-1)$ is an immediate consequence of Axiom 1 and the fact that $\mathcal{R}$-functors from lower-ranked general relators into R must have their image in ∂R. □

Lemma 9.9 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. If U is a path in a general relator R, and U is long in R, then U is also oriented with respect to R. More specifically, whenever $d_R(U) \geq \alpha$, $|U| \geq 2\omega_R$.*

Proof: Since by definition $d_R(U)$ is a function on the endpoints of the path U lifted to R^∞, without loss of generality U can be assumed to be a geodesic in R^∞. Then by Axiom 4, Axiom 7, and Axiom 3 respectively, $|U| \geq (\alpha - \delta)|R| \geq 2\gamma|R| \geq 2\omega_R$. □

Lemma 9.10 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation, and a word W is readable in relators R_1 and R_2 with $d_{R_1}(W) \geq \beta$ and $d_{R_2}(W) \geq \beta$, then R_1 and R_2 are isomorphic as $\mathcal{R}$-categories, and the isomorphism can be chosen so that the paths reading W are identified. In particular, for each word W there is at most one general relator R in which $d_R(W) \geq \beta$.*

Proof: Without loss of generality, assume that $\text{rank}(R_1) \leq \text{rank}(R_2)$. Then by Axiom 2, $\text{rank}(R_1) = \text{rank}(R_2)$ and there is a functor from R_1 to R_2 sending the reading of W in R_1 to the reading of W in R_2. By Lemma 2.1, this functor is an isomorphism. □

Lemma 9.11 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation, and a cycle W is readable as a loop with non-zero winding number in relators R_1 and R_2, then R_1 and R_2 are isomorphic as $\mathcal{R}$-categories, and the isomorphism can be chosen so that the loops reading W are identified. In particular, the set $\mathcal{R}$ has distinct representatives.*

Proof: By the definition of relator metrics, $d_{R_1}(W) \geq 1$ and $d_{R_2}(W) \geq 1$. Thus Lemma 9.10 can be used to complete the proof. □

Lemma 9.12 *Let $G = \langle A|\mathcal{R}\rangle$ be a graded, measured presentation, and let $\alpha, \beta, \gamma, \delta$, and ϵ be fixed constants. The presentation of G is a general small cancellation presentation iff for all $k \in$ integers $G(k) = \langle A|\mathcal{R}(k)\rangle$ is a general small cancellation presentation. In particular, if sets of general relators $\mathcal{R}_k$ are created so that $G(k) = \langle A|\mathcal{R}(k)\rangle$ is a general small cancellation presentation for the same set of constants then $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation.*

Proof: Notice that a counterexample to either Axiom 1 or Axiom 2 involves only two general relators, that a counterexample to Axiom 3, Axiom 4, or

Axiom 5 involves only one general relator, that the ranks of the general relators needed to produce a counterexample to Axiom 6 are bounded by the rank of R, and that a counterexample to Axiom 7 does not involve any of the general relators. Thus any counterexample proving that $G = \langle A|\mathcal{R}\rangle$ is not a general small cancellation presentation can also be used to show that the presentation of $G(k)$ is not a general small cancellation presentation for some positive integer k. $\square$

10 Dehn's Algorithm

This section begins with a proof that certain $\mathcal{R}$-diagrams can be altered to produce equivalent $\mathcal{R}$-diagrams over $C(p)$-maps. Once the proof is completed, it will be possible to apply the known results on $C(p)$-maps to diagrams over general small cancellation presentations. These results are then used to prove that the word and conjugacy problems are decidable using a version of Dehn's algorithm. Another use of Dehn's algorithm is to show that the Cayley category of a general small cancellation presentation is proper. This result is given at the end of the section.

10.1 Dehn-reduced Words and Cycles

Recall that a word W is Dehn-reduced with respect to $\mathcal{R}$ if W is reduced in the free group and there do not exist a word U, a word V, and a general relator $R \in \mathcal{R}$ such that U is a subword of W, UV^{-1} is a representative of R and $|U| > |V|$. If such a situation does arise then it is clear that replacing U with V in W produces a strictly shorter word which is equivalent to W in the group $G = \langle A|\mathcal{R}\rangle$.

Lemma 10.1 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation and $\mathcal{R}'$ is a subset of $\mathcal{R}$, then it is decidable whether a word W is Dehn-reduced with respect to $\mathcal{R}'$.*

Proof: If W is not reduced in the free group, then this fact is easy to discover in finite time. If there exists a word U, a word V and a general relator $R \in \mathcal{R}'$ showing that W is not Dehn-reduced, then this can also be discovered in finite time, since $2|W| \geq 2|U| > |U| + |V| \geq |R|$, showing that there is a bound on the length of the general relators which need to be considered. Since by Lemma 9.11 general relators have distinct representatives, and the alphabet A is finite, this bound of the length of the general relator R guarantees that only a finite number of general relators exist which can satisfy this condition. Finally, since the list of general relators of interest is finite and is derived solely from the length of W, it is easy to exhaustively check whether there exist words U and V in each general relator in the list which together satisfy the requirements. $\square$

Lemma 10.2 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. For every word W there exists an effectively constructible word of shorter length which is Dehn-reduced with respect to $\mathcal{R}'$ and equivalent to W in G.*

Proof: First of all, by Lemma 10.1 it is decidable whether a given word W is Dehn-reduced with respect to $\mathcal{R}'$ or not. If W is not Dehn-reduced then it can be reduced either by removing the subword of the form aa^{-1} or by replacing the subword U with the word V to create a strictly shorter word which is equivalent to W in G. Since these reductions shorten the length of the word, only a finite number of such reductions can occur before the word is Dehn-reduced. □

Recall that a word W is called μ-free with respect to a set of general relators $\mathcal{R}$ if W is reduced in the free group and there do not exist a subword U of W and a general relator $R \in \mathcal{R}$ such that $d_R(U) > \mu$.

Lemma 10.3 *If U is a word readable in a general relator R and $d_R(U) > \frac{1}{2} + 2\gamma + \delta$ then there exist a subword U' in U and a word V such that $U'V^{-1}$ is a representative of R and $|U'| > |V|$. In particular, words which are Dehn-reduced with respect to a set of general relators $\mathcal{R}$ must also be $(\frac{1}{2} + 2\gamma + \delta)$-free and thus $(\frac{1}{2} + \alpha)$-free with respect to $\mathcal{R}$.*

Proof: Since $d_R(U) \geq \alpha$, the path U is oriented, by Lemma 9.9. Lift U to a path in R^∞ and call the initial vertex v_0 and the final vertex u. Pick v_1 as the unique vertex such that the path from v_0 to v_1 is a representative and v_1 and u are on the same side of v_0, meaning that both vertices are in the same connected component of R^∞ when $\text{Ball}(v_0, \omega_R)$ is removed. The proof is divided into two cases.

Case 1: If there is a vertex u' of the lifted path U contained in $\text{Ball}(v_1, 2\omega_R)$ then define V to be a geodesic from v_1 to u' and define U' as the initial segment of U which ends at u'. By definition $|V| < 2\omega_R \leq 2\gamma|R|$ by Axiom 3. Also, by construction $U'V^{-1}$ is a representative of R, so that $|U'| + |V| \geq |R|$ by the definition of $|R|$, and $|U'| > (1 - 2\gamma)|R|$. Finally, $1 - 2\gamma > 2\gamma$ by Axiom 7, so $|U'| > (1 - 2\gamma)|R| > 2\gamma|R| > |V|$.

Case 2: If there is not such vertex, then the balls of radius ω_R centered at v_0, v_1, and u are disjoint. More specifically, since these balls disconnect R^∞, the ball centered at u must be between those centered at v_0 and v_1. Define V to be a geodesic path from v_1 to u, and define $U' = U$. The word UV^{-1} is clearly a representative of R. It only remains to show that $|U| > |V|$. If, without loss of generality, it is assumed that U is also geodesic, then since UV^{-1} is a geodesic 2-gon which represents R, Lemma 5.21 shows that $|U| + |V| < |R| + 4\omega_R \leq (1 + 4\gamma)|R|$. Next, since $d_R(U) > \frac{1}{2} + 2\gamma + \delta$, Axiom 4 implies that $|U| > (\frac{1}{2} + 2\gamma)|R|$. Combining these inequalities shows that $(\frac{1}{2} + 2\gamma)|R| > |V|$, and thus $|U| > |V|$. Finally, the fact that Dehn-reduced words are also $(\frac{1}{2} + \alpha)$-free follows from Axiom 7. □

Lemma 10.4 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, let B be an $\mathcal{R}$-category, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If B is μ-closed with respect to $\mathcal{R}'$ for some $\mu \geq \frac{1}{2} + 2\gamma + \delta$, then given any path X in B, there is a path Y between the same two vertices which is μ-free with respect to $\mathcal{R}'$, homotopic to X relative to its endpoints, and equivalent to X in $G' = \langle A|\mathcal{R}'\rangle$. In addition $|X| \geq |Y|$.*

Proof: If X is not itself μ-free with respect to $\mathcal{R}'$ then there exist a word U and a general relator $R \in \mathcal{R}'$ such that U is a subword of X and $d_R(U) > \mu$. By Lemma 10.3 there are words U' and V such that U' is a subword of U and thus of X, and the replacement of U' with V creates a shorter word between the same two vertices, homotopic to the original path X. Since the length of the word is strictly descreasing, such replacements can occur only a finite number of times before they stop at a word which satisfies the conditions of the lemma. $\square$

Recall that a cycle W is called Dehn-reduced iff all of the cyclic conjugates of the word W are Dehn-reduced, and similarly, that the cycle W is called μ-free iff all of the cyclic conjugates of the word W are μ-free. Notice that even if a cycle W is Dehn-reduced this does not imply that the cycle W^2 is Dehn-reduced since there may exist a subword U of length slightly longer than W which leads to a reduction. A similar comment applies to μ-free cycles.

Lemma 10.5 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. For every cycle W there exists an effectively constructible cycle of shorter length which is Dehn-reduced with respect to $\mathcal{R}'$ and conjugate to W in G. Moreover, the resulting Dehn-reduced cycle is $(\frac{1}{2}+2\gamma+\delta)$-free with respect to $\mathcal{R}'$.*

Proof: By Lemma 10.2 the word W can be Dehn-reduced with respect to $\mathcal{R}'$. If one of the cyclic conjugates of the result is not Dehn-reduced then Dehn-reduce the cyclic conjugate and repeat this process. Since the reductions shorten the length of the cycle and since cyclic conjugation leaves the length unchanged, the process must stop after a finite number of steps. The result is clearly conjugate to W in G, and Dehn-reduced with respect to $\mathcal{R}'$. Finally, since all of the cyclic conjugates are Dehn-reduced with respect to $\mathcal{R}'$, all of the cyclic conjugates are also $(\frac{1}{2} + 2\gamma + \delta)$-free with respect to $\mathcal{R}'$ by Lemma 10.3. Thus the resulting cycle is $(\frac{1}{2} + 2\gamma + \delta)$-free with respect to $\mathcal{R}'$. $\square$

Lemma 10.6 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, let B be an $\mathcal{R}$-category, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If B is μ-closed with respect to $\mathcal{R}'$ for some $\mu \geq \frac{1}{2} + \delta + 2\gamma$, then given any loop X in B, there is a loop Y which is μ-free with respect to $\mathcal{R}'$, homotopic to X, and conjugate to X in $G' = \langle A|\mathcal{R}'\rangle$. In addition $|X| \geq |Y|$.*

Proof: The result follows by merely applying Lemma 10.4 repeatedly to the cyclic conjugates of the loop obtained so far. Since the length of the loop is strictly decreasing, the process stops after a finite number of steps at a loop which satisfies the conditions of the lemma. $\square$

Although it is immediate that a word W is Dehn-reduced whenever the cycle W is Dehn-reduced, the reverse is not always true. The lemmas below illustrates some conditions under which it is possible to conclude that a cycle is Dehn-reduced from the fact that a word is Dehn-reduced.

Lemma 10.7 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If the word W^{n+1} is Dehn-reduced with respect to $\mathcal{R}'$, then the cycle W^n is Dehn-reduced with respect to $\mathcal{R}'$.*

Proof: The result follows from the observation that all words readable in the cycle of W^n with a length less than $|W^n|$ are also readable in the word W^{n+1}. Thus, if the cycle W^n is not Dehn-reduced, then the word W^{n+1} is also not Dehn-reduced. □

If a path is read in a collapsed $\mathcal{R}$-category and there is an automorphism of this $\mathcal{R}$-category which sends the start vertex of the path to its end vertex, then this automorphism is called the automorphism represented by the path. Notice that the definition of a collapsed $\mathcal{R}$-category guarantees that this automorphism is unique whenever it exists, thereby justifying the use of the definite article.

Lemma 10.8 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. In addition, let W be a word, and let R be a general relator which does not contain any power of W as a loop. If U is a W-periodic word which is readable in R with $d_R(U) > 2\beta$, then $|U| < 2|W|$.*

Proof: Assume $|U| \geq 2|W|$, and without loss of generality, assume that W is an initial segment of U, so that $U = WV$ and $|V| \geq |W|$. If $d_R(V) \geq \beta$, then by Axiom 2, the readings of V as initial and final segments of U and thus in R differ by an automorphism of R. Since W is a path in R between the initial vertices of these readings, W represents this automorphism of R. As a consequence, W^n is a loop in R for some power of W, contradicting the assumptions stated in the lemma. Thus $d_R(V)$ must be strictly less than β. Similarly, if $d_R(W) \geq \beta$, then the readings of W as initial segments of U and of V lead to an automorphism of R which can be represented by W, which again leads to power of W being a loop in R. Thus $d_R(W)$ must also be strictly less than β. But in this case, by the properties of relator metrics, $d_R(U) \leq d_R(W) + d_R(V) \leq 2\beta$. This final contraction shows that the initial assumption that $|U| \geq 2|W|$ must itself be false. □

Lemma 10.9 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If W is a word such that no power of W is ever readable as a loop with nontrivial winding number in any of the general relators in $\mathcal{R}'$, and either the word W^3 or the cycle W^2 is Dehn-reduced with respect to $\mathcal{R}'$, then the cycle W^i is Dehn-reduced with respect to $\mathcal{R}'$ for all $i > 1$.*

Proof: Suppose that the cycle of W^i is not Dehn-reduced with respect to $\mathcal{R}'$. Then there exists a W-periodic word U and a general relator $R \in \mathcal{R}'$ with a path U readable in R such that $|U| \geq \frac{1}{2}|R|$. By Axiom 4, $d_R(U) \geq \frac{1}{2} - \delta > 2\beta$ by Axiom 7. Thus by Lemma 10.8, the length of U is less than twice that of W, so that U is readable in both the word W^3 and the cycle W^2. The path U and the relator R prove that the word W^3 and the cycle W^2 are not Dehn-reduced, contradiction. □

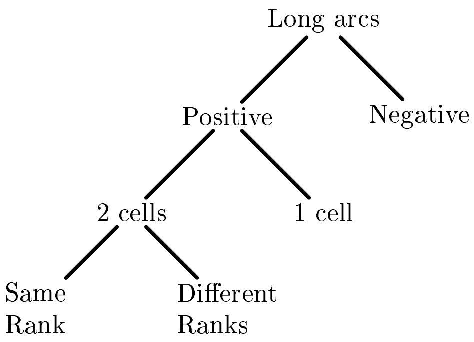

Figure 26: The four varieties of long arcs

10.2 Long Arcs and Ladders

The first step in applying Dehn's algorithm to diagrams over general small cancellation presentations is to show that the key diagrams can be modified so that the underlying maps are $C(6)$-maps. This is shown in Lemma 10.16. Once this is established, the results quoted from [8] can be applied. The proof proceeds by showing that each of four kinds of long internal arcs in a diagram can be removed in such a way that the type of the diagram is always strictly decreasing. The four kinds of long internal arcs are long negative arcs, long positive arcs between a single cell and itself, long positive arcs between 2 distinct cells of different ranks, and long positive arcs between 2 distinct cells of the same rank. The relations between these kinds of arcs are depicted in Figure 26. In each of the four cases it will be shown that the diagram can be altered to produce an equivalent diagram of lower type. Since diagram types satisfy the descending chain condition, it is clear that every sequence of reductions must terminate in a finite number of steps in a diagram in which none of the above situations occur.

Lemma 10.10 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. If $W = UV$ is a representative of a general relator $R \in \mathcal{R}$ such that $d_R(U) \geq \alpha$ and the orientation of U is the opposite of that of W, then there exist paths V_1, V_2, and U' such that $V = V_1V_2$, the loop $U'V_2$ is a representative of R, and $|U'| < |U|$.*

Proof: Figure 27 schematically illustrates the argument below. First lift W from R to R^∞ and let v_0 and v_1 be its initial and terminal vertices, respectively. Furthermore, let u be the terminal vertex of the path U in W. Next, draw balls of radius ω_R around v_0, v_1, and u. Since U and W have opposite orientations, the ball centered at v_0 must be strictly between the balls centered at v_1 and u. In particular, since the path V starts at u and ends at v_1 there must exist a vertex u' in V which lies strictly within ω_R units of v_0. Use the vertex u' to define paths V_1 and V_2 such that $V = V_1V_2$, and define U' to be a geodesic

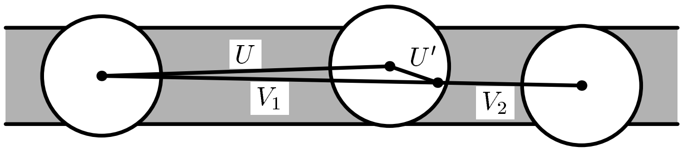

Figure 27: Shortening long negative arcs

path from v_0 to u'. Clearly, $U'V_2$ is a representative of R, and by Lemma 9.9, $|U| \geq 2\omega_R$ while by construction $|U'| < \omega_R$. $\square$

Lemma 10.11 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let Δ be an $\mathcal{R}$-diagram whose boundary cycles are linked. If D is a 2-cell in Δ and U is an internal arc which is both long and negative in D, then there is an $\mathcal{R}$-diagram Δ' which is equivalent to Δ but with Type$(\Delta) >$ Type(Δ').*

Proof: Let $W = UV$ be the boundary cycle of D. By Lemma 10.10 there are words V_1, V_2, and U' such that $V = V_1V_2$, the loop $U'V_2$ is a representative of R, and $|U'| < |U|$. The proof divides into three cases.

Case 1: If U' is non-empty, then simply remove the edges and vertices in the interior of the arc U along with the interior of the cell or cells on either side of U. Then add a new arc corresponding to the word U' which starts at the initial vertex of U and ends at the terminal vertex of V_1. A new cell or two new cells can then be attached to the new holes. By construction $U'V_2$ is a representative of R. Since the cell on the other side of the arc U' originally contained U in its boundary, by Axiom 1 the relator S which the cell represents contains a complete copy of R. Since $U'V_1^{-1}U^{-1}$ is a loop of winding number 0 in R, the same is true as a loop in S. Thus the replacement of U with $U'(V_1)^{-1}$ creates a new boundary cycle which is also a representative of S. Notice that this argument remains unchanged even in the case where the arc U borders the same cell in two distinct ways. Since the new $\mathcal{R}$-diagram contains exactly the same number of cells in each rank but fewer edges, the type of the new $\mathcal{R}$-diagram is strictly less than that of Δ.

Case 2: If U' is empty and the initial vertex of U is distinct from the terminal vertex of V_1 then the construction proceeds as in Case 1, except that once the interior of U and the interior of the cell(s) on either side of U have been removed, the map can be stretched so that these vertices are identified in the space where the cell(s) used to be. The rest of the proof is as before.

Case 3: If U' is empty, and the initial vertex v of U is identical with the terminal vertex of V_1, then first attach phantom cells to Δ to form an $\mathcal{R}^+$-sphere, and then remove the interior of the arc U along with the interior of the cells on either side of U. The fact that these vertices are identical forces the conclusion that the cells on either side of U are distinct. Let D' be the other cell, and let UV' be its boundary. The result of the removals is a connected and simply connected $\mathcal{R}^+$-diagram with v as a cut point. This $\mathcal{R}^+$-diagram can be viewed as the one-point product of two connected and simply connected $\mathcal{R}^+$-diagrams with boundary cycles $U'V_2 = V_2$ and $U'V_1^{-1}V' = V_1^{-1}V'$. Each of these will be considered separately. Since $U'V_2$ is a representative of R, a cell can be added to form an $\mathcal{R}^+$-sphere. By the same argument as in Case 1, the cycle $U'V_1^{-1}V'$ is a representative of the same general relator as the original boundary of D'. Thus a cell can be added to this $\mathcal{R}^+$-diagram so that it becomes an $\mathcal{R}^+$-sphere as well. Consider the location of the phantom cells. Since the boundary cycles are linked they must all be in one or the other of the $\mathcal{R}^+$-spheres. From this sphere, remove the phantom cells and the result will be an $\mathcal{R}$-diagram of strictly lower type, which is equivalent to Δ. □

Lemma 10.12 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let Δ be an $\mathcal{R}$-diagram whose boundary cycles are linked. If D is a labeled 2-cell in Δ, with boundary cycle W, and U is an internal arc of D in two distinct ways which is long in D, then there is an $\mathcal{R}$-diagram Δ' which is equivalent to Δ but with Type$(\Delta) >$ Type(Δ').*

Proof: If either instance of U in the boundary ∂D is negatively oriented then Lemma 10.11 can be applied to complete the proof, so assume that both instances are positively oriented with respect to ∂D. The cycle read by the boundary of D has the form $XUYU^{-1}$ for some non-empty words X and Y. The words X and Y must be non-empty since both instances of U are positively oriented. If, say, X was empty then since R is deterministic, the two paths labeled U would have the same image in R, showing that they have opposite orientations with respect to ∂D.

If the interior of D and the interior of the arc U are removed, the result is an annular hole with boundary cycles labeled X and Y. Let u be the vertex in the boundary cycle from which X can be read, and let u' be the vertex in the other boundary cycle from which Y can be read. The $\mathcal{R}$-diagram of strictly lower rank with boundary cycle $XVYV^{-1}$, whose existence is guaranteed by Axiom 6, can be placed in the annular hole so that the initial vertex of the first instance of V is attached to u, and the terminal vertex of this instance of V is attached to u'. See Figure 28 for an illustration. The boundary of the hole is a Dyck word, and since all of the cells added have strictly lower rank than the one which was removed the result is an $\mathcal{R}$-diagram of strictly lower type.

At this point the diagram contains all of the original boundary cycles plus a region bounded by a Dyck word. If phantom cells are added for each of the boundary cycles of Δ then the result is a connected and simply connected $\mathcal{R}^+$ diagram whose boundary is a Dyck word. By Lemma 7.6 the boundary edges can be identified and since the boundary cycles of Δ are linked, all of the phantom cells end up in the same $\mathcal{R}$-sphere of the one-point product. Ignoring the other items in the one-point product and removing the phantom cells leaves an $\mathcal{R}$-diagram Δ' which is equivalent to Δ and of strictly lower type. □

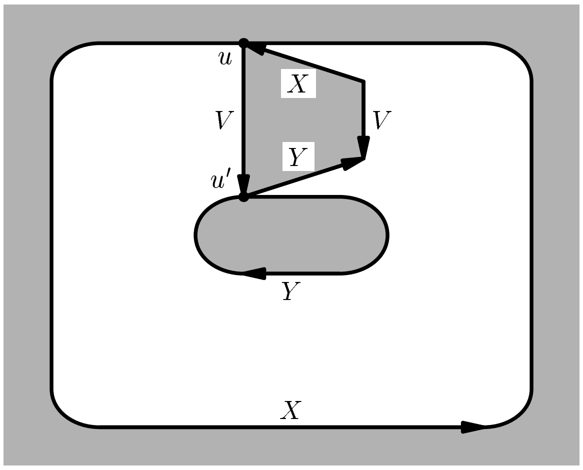

Figure 28: An illustration of Lemma 10.12

Lemma 10.13 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let Δ be an $\mathcal{R}$-diagram whose boundary cycles are linked. If D_1 and D_2 are distinct 2-cells in Δ with the same rank, U is an internal arc which borders both D_1 and D_2, and U is long in D_1, then D_1 and D_2 form a cancellable pair. In consequence there is an $\mathcal{R}$-diagram Δ' which is equivalent to Δ but with Type$(\Delta) >$ Type(Δ').*

Proof: Let UV_1 be the representative of D_1 and UV_2 be the representative of D_2. Since U is long in D_1 and contained in the boundary of D_2, by Axiom 1 the entire relator R_1 maps into R_2. Since by assumption R_1 and R_2 have the same rank, by Lemma 2.1 R_1 and R_2 are isomorphic and will hereafter be referred to as R, and the functor from R_1 to R_2 can then be described as an automorphism of R. Since the arc U is long in D_1, U is long in D_2 as well. By Lemma 10.11 both instances of U can be assumed to be positively oriented. The cycle $V_1(V_2)^{-1}$ is then a loop which is readable in R and its winding number is 0. Thus the 2-cells D_1 and D_2 form a cancellable pair. Since by Lemma 9.11 $\mathcal{R}$ has distinct representatives, Lemma 7.9 yields the desired result. □

Lemma 10.14 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let Δ be an $\mathcal{R}$-diagram whose boundary cycles are linked. If D_1 and D_2 are distinct 2-cells in Δ of different ranks, and U is an internal arc which borders both D_1 and D_2 with U long in D_1, then there exists an $\mathcal{R}$-diagram Δ' which is equivalent to Δ but with Type$(\Delta) >$ Type(Δ').*

Proof: Let UV_1 be the representative of D_1 and UV_2 be the representative of D_2. Since U is long in D_1, and contained in the boundary of D_2, by Axiom 1 the entire relator R_1 maps into R_2. By Lemma 2.1 this means that the rank of R_1 is at most that of R_2. Since the ranks are assumed to be distinct, R_1

must be of strictly lower rank, and the functor from R_1 to R_2 sends R_1 into the boundary of R_2. Thus the loop read by the representative of R_1 is a loop with winding number 0 in R_2, and $V_1(V_2)^{-1}$ is also a representative of R_2. Thus the diagram Δ' obtained by simply erasing the interior of the arc U is also an $\mathcal{R}$-diagram and one of strictly lower type. □

Lemma 10.15 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. If Δ is an $\mathcal{R}$-diagram whose boundary cycles are linked, then there exists an effectively constructible $\mathcal{R}$-diagram Δ' which is equivalent to Δ, contains no long internal arcs, and for which Type$(\Delta) \geq$ Type(Δ').*

Proof: If Δ has no long internal arcs then there is nothing to prove, so assume that such an arc exists. Depending on which kind of long internal arc it is, one of the previous four lemmas can be applied to produce an equivalent diagram of strictly lower type. If the new diagram is also not reduced then the same process can be repeated. Since diagram types satisfy the descending chain condition, the process must terminate after a finite number of steps at an $\mathcal{R}$-diagram equivalent to Δ which contains no long internal arcs. □

Lemma 10.16 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \leq \frac{1}{p}$. If Δ is an $\mathcal{R}$-diagram whose boundary cycles are linked and cyclically reduced, then there is an $\mathcal{R}$-diagram Δ' which is equivalent to Δ and such that the underlying map of Δ' is a $C(p+1)$-map. In particular, since all general small cancellation presentations satisfy $\alpha \leq \frac{1}{6}$, the $\mathcal{R}$-diagram Δ' is over a $C(7)$-map.*

Proof: The result follows immediately from Lemma 10.15 and Lemma 7.4. □

The next several lemmas apply the results quoted about $C(6)$-maps to diagrams over general small cancellation presentations.

Lemma 10.17 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. If W is a cyclically reduced word which is equivalent to 1 in G, and Δ is a connected and simply connected $\mathcal{R}$-diagram with boundary cycle W over a $C(7)$-map, then Δ contains at least two exposed cells with internal degree at most 3. Moreover, the labels on the boundary arcs of these exposed cells show that the cycle W contains at least two disjoint subwords U for which there is a general relator R such that $d_R(U) > 1 - 3\alpha$.*

Proof: The combination of Lemma 10.16 and Lemma 7.2 produces the two 2-cells, while Lemma 5.25 furnishes the measurement on the boundary arc. □

Lemma 10.18 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. If a non-trivial word W is equivalent to 1 in G, then either W is not reduced in the free group, or there is a subword U in W and a general relator $R \in \mathcal{R}$ such that $d_R(U) > 1-3\alpha$. As a consequence, if $3\alpha + 2\gamma + \delta \leq \frac{1}{2}$, then the only Dehn-reduced word which is equivalent to 1 in G is the identity itself. In particular, these results are true when $\alpha \leq \frac{1}{6}$ and $\gamma = \delta = 0$ or whenever $\alpha \leq \frac{1}{8}$.*

Proof: Assume that W is equivalent to 1 in G and reduced in the free group. The only way that cycle W could not also be cyclically reduced in the free group is if the first and the last letters of W are inverses of each other. Removing these letters and repeating the process if necessary shows that there are words V and W' such that $W = VW'V^{-1}$ and W' is cyclically reduced in the free group. If W' is the empty word, then W itself is not reduced, contradicting the initial assumption. Thus W' is non-trivial and by Lemma 10.17 there is a connected and simply connected $\mathcal{R}$-diagram Δ over a $C(7)$-map with boundary cycle W' which contains at least two exposed cells with internal degree at most 3. The $\mathcal{R}$-diagram Δ can be altered to form a new $\mathcal{R}$-diagram Δ' with boundary W by adding a path reading V attached to Δ at the appropriate vertex. This addition potentially occurs in the middle of the unique boundary arc of one of the exposed cells, but this leaves at least one exposed cell D with its boundary arc uninterrupted. The unique boundary arc U of D and the general relator R which it represents complete the proof. $\square$

Lemma 10.19 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let X and Y be cycles which are conjugate but not equivalent to the identity in G. If X is 4α-complement-free and Y is 3α-complement-free, then there exists an annular $\mathcal{R}$-diagram Δ with boundary cycles X and Y^{-1} in which every 2-cell D has exactly two boundary arcs, one in the loop X and the other in Y, and with internal degree at most 2. Moreover, both boundary arcs are long relative to D. In particular, if $4\alpha + 2\gamma + \delta \leq \frac{1}{2}$, then any two Dehn-reduced cycles in G are the boundary cycles of an annular $\mathcal{R}$-diagram such as Δ. This is true, more specifically, when $\alpha \leq \frac{1}{8}$ and $\gamma = \delta = 0$, and whenever $\alpha \leq \frac{1}{10}$.*

Proof: Since X and Y are conjugate and non-trivial, by Lemma 7.13 there is an annular $\mathcal{R}$-diagram Δ with oriented boundary cycles X and the inverse of Y. By Lemma 10.16, it can be assumed that Δ is over a $C(7)$-map. An application of Lemma 7.3 shows that Δ has the desired structure. The fact that both boundary arcs are long in D follows from Lemma 5.25. Finally, the inequality listed in the statement of the lemma was chosen to guarantee that Dehn-reduced cycles, which are $(\frac{1}{2} + 2\gamma + \delta)$-free by Lemma 10.5, would also be $(1 - 4\alpha)$-free, and thus 4α-complement-free. The specific examples clearly satisfy this inequality because of the restriction on the choices of the constants. $\square$

Lemma 10.20 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let X and Y be words which are equivalent but not equivalent to the identity in G. If X is 4α-complement-free and Y is 3α-complement-free, then there exists a connected and simply connected $\mathcal{R}$-diagram Δ with boundary cycle XY^{-1} in which every 2-cell D has either exactly two boundary arcs, one in the loop X and the other in Y, and with internal degree at most 2, or one boundary arc which contains an endpoint of X in its interior, and with internal degree at most 1. Moreover, both boundary arcs are long relative to D. In particular, if $4\alpha + 2\gamma + \delta \leq \frac{1}{2}$, then any two equivalent Dehn-reduced words in G form* *a boundary cycle of an $\mathcal{R}$-diagram such as Δ. This is true, more specifically, when $\alpha \leq \frac{1}{8}$ and $\gamma = \delta = 0$, and whenever $\alpha \leq \frac{1}{10}$.*

Proof: If X and Y have the same first letter or the same last letter, then remove the common letter from both words, and repeat this process until it no longer occurs. The removed letters will be added in afterwards. If either the new X or the new Y is equivalent to 1 in G then they both are, and since they are Dehn-reduced, Lemma 10.18 shows that they are empty words. If letters have been removed which need to be added back in, then the original X and Y were equal and the loop X satisfies the conditions of the lemma. On the other hand if the new X and the new Y are not equivalent to 1 in G, then XY^{-1} is the boundary of a connected and simply connected $\mathcal{R}$-diagram Δ by Lemma 7.12. By Lemma 10.16, it can be assumed that Δ is over a $C(7)$-map.

Since X and Y are 4α-complement-free and 3α-complement-free words, every exposed cell D of Δ either contains an edge of X and has $i(D) > 4$, or it contains an edge of Y and has $i(D) > 3$, or it contains one of the endpoints of X in the interior of its unique boundary arc. Since by Lemma 7.12 the endpoints of the path X are distinct, the map can be stretched and these points identified to form an annular $\mathcal{R}$-diagram Δ' with boundary cycles X and Y^{-1}. If letters have been removed from the original words then a path reading these removed letters can be added instead between the endpoints of X. The map remains $C(7)$ under this procedure, and all exposed cells D now satisfy the conditions necessary to apply Lemma 7.3. This shows that the original $\mathcal{R}$-diagram Δ has the desired structure. The fact that all boundary arcs are long in X and in Y follows from Lemma 5.25. Finally, the inequality listed in the statement of the lemma was chosen to guarantee that Dehn-reduced words, which are $(\frac{1}{2} + 2\gamma + \delta)$-free by Lemma 10.2, would also be $(1 - 4\alpha)$-free, and thus 4α-complement-free. The specific examples clearly satisfy this inequality because of the restriction on the choices of the constants.□

10.3 The Word and Conjugacy Problems

The following lemma shows that under mild additional restrictions on the choice of constants the word problem for general small cancellation groups is always decidable.

Lemma 10.21 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation with $3\alpha + 2\gamma + \delta \leq \frac{1}{2}$, then the group G has a decidable word problem. In particular, these results are true when $\alpha \leq \frac{1}{6}$ and $\gamma = \delta = 0$ or whenever $\alpha \leq \frac{1}{8}$.*

Proof: Since words X and Y are equivalent in G iff the word $W = XY^{-1}$ is equivalent to 1, it is sufficient to show that it is decidable whether words are equal to 1. By Lemma 10.2 given any word W there is a word Z which is effectively constructible, Dehn-reduced with respect to $\mathcal{R}$, and equivalent to W in the group G. If Z is the empty word, then by construction W is equivalent to 1 in G. Conversely, if Z is not the empty word, then W cannot be equivalent to 1 in G, since the existence of such a word Z would contradict Lemma 10.18. $\square$

One of the consequences of a decidable word problem is that the Cayley graph of the group is effectively constructible.

Lemma 10.22 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation with $3\alpha + 2\gamma + \delta \leq \frac{1}{2}$, then the Cayley graph $\mathcal{C}(G, A)$ is effectively constructible.*

Proof: Since by Lemma 10.21 it is decidable whether any two words are equal, it is possible to test all pairs of words of length less than n, and from this information to construct the ball $\mathrm{Ball}(v_0, n)$ in the Cayley graph. This can be done successively for larger and larger n, to construct as much of the Cayley graph as is desired. $\square$

Under a slightly stronger restriction on the choice of constants, the conjugacy problem is also decidable.

Lemma 10.23 *If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation with $4\alpha + 2\gamma + \delta \leq \frac{1}{2}$, then the group G has a decidable conjugacy problem. In particular, this is true when $\alpha \leq \frac{1}{8}$ and $\gamma = \delta = 0$ or whenever $\alpha \leq \frac{1}{10}$.*

Proof: Let X and Y be words, and by Lemma 10.5 assume without loss of generality that the cycles of X and Y are Dehn-reduced. A word is conjugate to 1 in G iff it is equivalent to 1 iff its Dehn reduction is the empty word by Lemma 10.18 and Lemma 10.5. Thus it is decidable whether a word is conjugate to 1 in G. If neither word is conjugate to 1 but they are conjugate to each other then there exists an $\mathcal{R}$-diagram Δ which satisfies the conclusion of Lemma 10.19. The crux of the problem is to decide in a finite amount of time whether such a diagram Δ exists.

First, start with the Dehn-reduced abstract loop X, and find all subwords U in X and general relators R with $|U| \leq |X|$, U readable in R, and $d_R(U) \geq \alpha$. Since by Axiom 4 $|X| \geq |U| \geq (\alpha - \delta)|R| > 0$, there is an a priori bound on the length of the relator R. By Lemma 9.11 this in turn shows that only a finite number of possible general relators need to be considered in the search for an $\mathcal{R}$-diagram having the properties of Δ. Also, since every cell in Δ would have to contain an edge of X, there is a bound on the number of cells of potential diagrams. And from the bounds on the number of relators and their lengths comes a bound on the number of edges in Δ as well. At this point it is possible simply to exhaustively search among the possible diagrams with the right structure; the outcome of the search decides the conjugacy problem for the words X and Y. $\square$

11 Gromov Hyperbolic Groups

This section derives a number of results on geodesic words which are then used to demonstrate how close general small cancellation presentations are to hyperbolic groups in the sense of M. Gromov [7]. The main results are that finitely presented general small cancellation groups are hyperbolic, and more generally, general small cancellation groups are the direct limit of the hyperbolic groups obtained by choosing finite subsets of the general relators.

11.1 Geodesics

Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. If Y is a geodesic in the Cayley category of the presentation then Y is called a geodesic with respect to $\mathcal{R}$. If Y is a geodesic with respect to $\mathcal{R}$ which is equivalent to X in G, then Y is called a geodesic of X in G. Let B be an $\mathcal{R}$-category. Suppose that whenever X is a path in B and Y is a geodesic of the word X in G, there is another path in B reading Y which is homotopic to X relative to its endpoints. In this case B is said to be geodesic-closed with respect to $\mathcal{R}$. If $G = \langle A|\mathcal{R}\rangle$ is a general small cancellation presentation and B is an $\mathcal{R}$-category, then under certain conditions it is possible to show that every path in B is homotopic to a path which reads a geodesic in G.

Lemma 11.1 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, let $\mathcal{R}'$ be a subset of $\mathcal{R}$, and let B be an $\mathcal{R}$-category which is $(\frac{1}{2}-\delta)$-closed with respect to $\mathcal{R}'$. If X is a path in B, and Y is a Dehn reduction of X with respect to $\mathcal{R}'$, then there is a path reading Y in B between the same two vertices which is homotopic to X relative to its endpoints. Similarly, if X is a loop in B, and Y is a Dehn reduction of X with respect to $\mathcal{R}'$, then there is a loop reading Y in B which is homotopic to the loop X. These results are true in particular whenever B is α-closed with respect to $\mathcal{R}'$.*

Proof: Let Z be a word or cycle obtained from X by a single Dehn reduction. In particular, suppose that there exist words U and V and a general relator R such that UV^{-1} is a representative of R, U is a subword of X, $|V| < |U|$, and Z is the result of replacing the subword U in X with V. Since $2|U| > |U|+|V| \geq |UV|$ and since $|UV| \geq |R|$ by the definition of $|R|$, it follows that $|U| > \frac{1}{2}|R|$. Thus by Axiom 4, $d_R(U) > \frac{1}{2} - \delta$. Because B is $(\frac{1}{2}-\delta)$-closed, there is a functor from R to B which sends the reading of U in R to the reading of U in B. In particular this shows that the word V is readable in B between the same two vertices. Thus Z is readable in B, and in the case where X and Z are words they have the same endpoints. Moreover, since R is contractible, the readings of U and V are homotopic in R relative to their endpoints, and the homotopy thus constructed is sent under the functor into B to a homotopy between the readings of U and V in B relative to their endpoints. And finally, this shows that X and Z are homotopic in B. If X and Z are words, then the homotopy can be performed relative to the endpoints of X. Since Y is obtained from X by a finite sequence of such reductions, the results follow. $\square$

Notice that Y in the statement of the lemma is a Dehn reduction of X and not an arbitrary Dehn-reduced word equivalent to X in G. There is no guarantee, and in fact it is not true, that arbitrary Dehn-reduced equivalent words can always be obtained from a given word by a sequence of Dehn reductions. The next two lemmas address the existence of such paths. The inclusion of such paths in an arbitrary category B requires a strong hypothesis, such as that B is α-closed, not merely that it is $(\frac{1}{2}-\delta)$-closed.

Lemma 11.2 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation. If $XUYV$ is a representative of a general relator $R \in \mathcal{R}$ and X is $(1-3\alpha)$-free, while U and V are short with respect to R, then $d_R(Y) > \alpha$. As a consequence, if $\alpha \le \frac{1}{8}$ and X is Dehn-reduced then Y is long.*

Proof: Since U and V are short, $d_R(U) < \alpha$, and $d_R(V) < \alpha$. Combining these inequalities with the properties defining relator metrics yields $1 \le d_R(XUYV) \le d_R(X) + d_R(U) + d_R(Y) + d_R(V) < d_R(Y) + 1 - \alpha$. Thus $d_R(Y) > \alpha$. Finally, if X is Dehn-reduced with respect to R and $\alpha \le \frac{1}{8}$, then by Lemma 10.3, $d_R(X) < \frac{1}{2} + \alpha \le 1 - 3\alpha$. □

Lemma 11.3 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \le \frac{1}{8}$, let $\mathcal{R}'$ be a subset of $\mathcal{R}$, and let B be an $\mathcal{R}$-category which is collapsed and α-closed with respect to $\mathcal{R}'$. If X is a path in B and Y is a word which is 3α-complement-free with respect to $\mathcal{R}'$ and equivalent to X in $G' = \langle A|\mathcal{R}'\rangle$, then there is a path reading Y in B which is homotopic to the path X relative to its endpoints. Similarly, if X is a loop in B and Y is a cycle which is 3α-complement-free with respect to $\mathcal{R}'$ and conjugate to X in G', then there is a loop reading Y in B which is homotopic to the loop X.*

Proof: By Lemma 11.1 it is enough to show the result when X is Dehn-reduced. In this case Lemma 10.20 and Lemma 10.19 construct $\mathcal{R}$-diagrams which have very particular structures. Because each cell in the diagram has a boundary arc in Y which is 3α-complement-free and rungs which are short, it follows immediately that the boundary arc in X has length at least α relative to the relator metric of the general relator which the cell represents. Since B is α-closed, the entire boundary of the cell is readable in B as a loop extending the reading of the portion of the boundary in the path X. Next since B is collapsed, the labeling of the 1-skeleton is deterministic so that the word or cycle Y is readable as a path or loop in B. Clearly it is homotopic to the path or loop X. In the case where X and Y are words, the path Y is read between the same vertices as X, and the homotopy can be performed relative to the endpoints of X. □

Lemma 11.4 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \le \frac{1}{8}$, let $\mathcal{R}'$ be a subset of $\mathcal{R}$, and let B be an $\mathcal{R}$-category which is collapsed and α-closed with respect to $\mathcal{R}'$. If X^n is a loop in B and Y is a cycle which is 3α-complement-free with respect to $\mathcal{R}'$ and conjugate to X in G', then there is a loop reading Y^n in B which is homotopic to the loop X^n. In particular, if X^n is a loop in a general relator $R \in \mathcal{R}_k$, and Y is a cycle which is 3α-complement-free with respect to $\mathcal{R}(k-1)$ and conjugate to X in $G(k-1)$, then there is a loop reading Y^n in R which is homotopic to the loop X^n.*

Proof: The proof is nearly identical to the proof of Lemma 11.3 except that after creating a conjugacy diagram between the cycles X and Y it is the n-fold cover of this diagram which is used to create the loop Y^n in B. By Lemma 9.8, the above reasoning is applicable to the situation mentioned in the final sentence. $\square$

Lemma 11.5 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $\alpha \le \frac{1}{8}$, and let $\mathcal{R}'$ be a subset of $\mathcal{R}$. If B is an $\mathcal{R}$-category which is collapsed and α-closed with respect to $\mathcal{R}'$, then B is geodesic-closed with respect to $\mathcal{R}'$.*

Proof: Since words which are geodesic with respect to $\mathcal{R}'$ are clearly Dehn-reduced with respect to $\mathcal{R}'$, by Lemma 10.3 they are also $(\frac{1}{2}+\alpha)$-free with respect to $\mathcal{R}'$. Because $\alpha \le \frac{1}{8}$, they are also $(1-3\alpha)$-free with respect to $\mathcal{R}'$, and the result follows from Lemma 11.3. $\square$

11.2 Hyperbolic Groups

Let XYZ be a geodesic triangle in the Cayley graph of a group G. If every vertex in Z is within τ units of a vertex of either X or Y as measured by the graph metric, then it is called τ-thin with respect to the side Z. If it is τ-thin with respect to each of the three sides then the triangle itself is said to be τ-thin. If all geodesic triangles in the Cayley graph of G are τ-thin for some fixed non-negative real constant τ then the group is called τ-hyperbolic, and a group which is τ-hyperbolic for some non-negative τ is simply called a hyperbolic group, or sometimes a Gromov hyperbolic group. The next several lemmas show that subject to certain restrictions on the constants, general small cancellation presentations satisfy a variety of the thin triangle condition.

Lemma 11.6 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation, and let C be its Cayley category. If X, Y, and Z are paths such that the cycle XYZ is readable as a loop in C, then there exist words U, V, and W, such that UV^{-1}, VW^{-1}, and WU^{-1} are $\frac{2}{3}$-reduced and are equivalent to X, Y and Z respectively. In addition, U, V, and W can be chosen so that $|X|+|Y| \ge |U|+|V|+|W|$.*

Proof: Start with any set of words U, V, W, such that UV^{-1}, VW^{-1}, and WU^{-1} are equivalent to X, Y, and Z, such as $U = X$, $W = Y^{-1}$, and $V = \emptyset$. The word $WU^{-1} = Y^{-1}X^{-1}$ is equivalent to Z in G because XYZ is equivalent to 1 in G by Lemma 8.11. To complete the proof, the words U, V, and W will be altered so that the sum $|U|+|V|+|W|$, will always be decreasing.

First, replace U, V, and W with words which are geodesic in $\mathcal{R}$. Next, if the word, say UV^{-1}, contains more than two-thirds of a representative, then since U and V are geodesics, the subword must contain the vertex between U and V^{-1}. Let $U = U_1U_2$ and $V = V_1V_2$ so that $U_2V_2^{-1}$ is the subword in question. The remaining portion of the representative must be a word P from the initial vertex of U_2 to the initial vertex of V_2, and it must be strictly shorter than either U_2 or V_2. Without loss of generality, assume it is shorter than U_2. Then U_2 can be replaced with P and the words U, V, and W relabeled accordingly. Specifically,

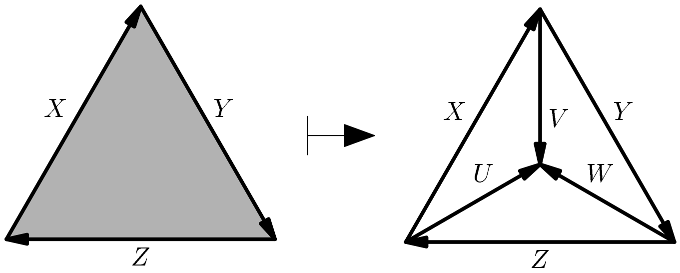

Figure 29: An illustration of Lemma 11.6

the new U is U_1P, the new V is V_1, and the new W is WV_2^{-1}. Notice that the sum $|U| + |V| + |W|$ strictly decreases under this procedure. Since the sum is a positive integer, this process can be carried out only a finite number of times before it ceases to be possible. When the process stops, the words U, V, and W satisfy the conditions of the lemma. The final statement follows from the observations that $|X| + |Y|$ is the initial value of the sum $|U| + |V| + |W|$, and that this sum only decreases throughout the procedure. □

Lemma 11.7 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $3\alpha + \delta \leq \frac{1}{3}$, and let C be its Cayley category. If X, Y, and Z are $(1 - 4\alpha)$-free paths in C which form a triangle, then every vertex v in Z is connected to a vertex in either X or Y by a path PQ where P and Q are each readable in some general relator in $\mathcal{R}$.*

Proof: Let U, V, and W be the paths created by Lemma 11.6, and let Δ be a connected and simply connected $\mathcal{R}$-diagram with boundary cycle XVU^{-1}. By Lemma 10.15 Δ can be assumed to have only short internal arcs. Suppose there is an exposed cell D with $i(D) \leq 3$, which represents a general relator R, and whose boundary arc is contained in the subword VU^{-1}. Then by Axiom 4, the geodesic connecting the adjoining internal arcs of D has a length of less than $(3\alpha + \delta)|R|$. Since by assumption this is less than $\frac{1}{3}|R|$, then the word read by the boundary arc must be more than two-thirds of a representative of R, contradicting the choice of U, V, and W. Thus Δ has no such exposed cells. On the other hand, since X is $(1 - 4\alpha)$-free, Δ satisfies all of the conditions necessary to apply Lemma 10.20. And as a consequence, every vertex of X is joined to a vertex in UV^{-1} by a path which is readable in a general relator in $\mathcal{R}$, and vice versa. The same is true of Y and Z as well. Thus every vertex in Z is connected to a vertex in either U or W by a path Q which is readable in some general relator in $\mathcal{R}$. If the vertex is in U then this vertex is connected to a vertex in X by a path readable in a general relator, and if the vertex is in W, then it is connected to a vertex in Y by a path readable in a general relator. □

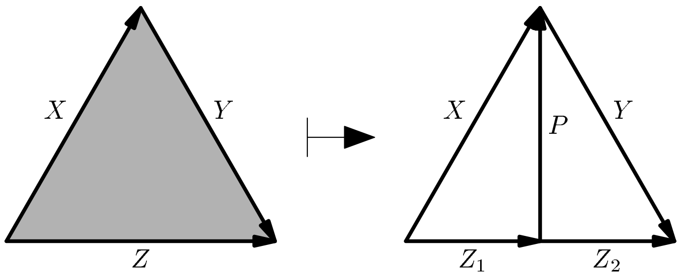

Figure 30: An illustration of Lemma 11.8

Lemma 11.8 *Let $G = \langle A|\mathcal{R}\rangle$ be a general small cancellation presentation with $3\alpha + \delta \le \frac{1}{3}$ and $\alpha \le \frac{1}{10}$, and let X, Y, and Z be $(1-4\alpha)$-free paths in its Cayley category which form a triangle. If B and C are α-closed categories in the Cayley category which contain the paths X and Y respectively, then there exist words Z_1, Z_2 and P such that Z_1P is read in B between the same vertices as X, $P^{-1}Z_2$ is read in C between the same vertices as Y, and $Z_1Z_2 = Z$. In particular Z is contained in the union of B and C.*

Proof: By Lemma 11.6 there exist words U, V and W satisfying the conditions of that lemma. Since UV^{-1} is equivalent to X in G and $(1-3\alpha)$-free, by Lemma 11.3 it is contained in B. Similarly, there is a path reading VW^{-1} in C between the same vertices as the path Y. Since the Cayley category is deterministic, the reading of V in both instances is the same. Thus the vertex v which is the terminal vertex of V is contained in both B and C. Let Δ be an $\mathcal{R}$-diagram with no long internal arcs, which shows that WU^{-1} is equivalent to Z in G. Since Z is $(1-4\alpha)$-free, by Lemma 11.2 every cell in Δ is long in WU^{-1}. If it is long in U then since U is contained in B, the entire general relator represented by the cell is contained in B. Similarly, if the cell is long in W then since W is contained in C the entire general relator represented by the cell is contained in C. The only cell in question is the cell D which contains the vertex v in its boundary. Since $\alpha \le \frac{1}{10}$, $\frac{1}{2} - 3\alpha$ is at least 2α. Thus either the boundary of D in U is long or the boundary of D in W is long. In either case, Z is contained in the union of B and C and if P is a path in Δ to the crossover point, then Z is divided into Z_1 and Z_2 which satisfies the lemma. □

Lemma 11.9 *If $G = \langle A|\mathcal{R}\rangle$ is a finitely presented general small cancellation presentation with $3\alpha+\delta \le \frac{1}{3}$ and $\alpha \le \frac{1}{10}$, then the group G is Gromov-hyperbolic. In particular, the result is true when $\alpha = \frac{1}{10}$ and $\gamma = \delta = 0$, or whenever $\alpha \le \frac{1}{12}$.*

Proof: Since $\mathcal{R}$ is finite, there is a finite bound on the diameters of all of the general relators in $\mathcal{R}$. Call this bound τ. By Lemma 11.7 every geodesic triangle in the Cayley graph is 2τ-thin. Since the choice of τ is independent of the triangle involved, the group G is 2τ-hyperbolic, and G is therefore a hyperbolic group. $\square$

Since every finitely presented general small cancellation group is hyperbolic, the results of this well-developed theory are available for application. In the other direction, the axioms of general small cancellation theory provide an efficient way to construct examples which are necessarily hyperbolic. Potential applications of hyperbolic group theory to specific presentations hinge on the ability to show that the presentation is indeed hyperbolic, using one of the many equivalent definitions of hyperbolicity such as the thin triangle condition. It is precisely at this point where most of the practical difficulties arise and where general small cancellation theory is of the most use. General small cancellation groups provide an array of accessible examples of hyperbolic groups, and the hyperbolicity in this instance is easy to check since all that is needed is to check the axioms of general small cancellation theory.

Lemma 11.10 *If $G = \langle A|\mathcal{R}\rangle$ is a finitely generated general small cancellation presentation with $3\alpha + \delta \leq \frac{1}{3}$ and $\alpha \leq \frac{1}{10}$, then the group G is the direct limit of hyperbolic groups. In particular, the result is true when $\alpha = \frac{1}{10}$ and $\gamma = \delta = 0$, or whenever $\alpha \leq \frac{1}{12}$.*

Proof: Recall that by definition general small cancellation groups are finitely generated. Since A is finite, the number of relators must be countable. Using the previous lemma, add the relators in one at a time, together with any others which need to be added in order to add in that particular relator. The number of relators so far will always be finite, so that by Lemma 11.9 the group defined so far will always be hyperbolic. Also, since the relators were arranged so that all of them are used in the limit, the direct limit of the groups must satisfy all of these relators. Finally, since the group itself satisfies no other relations, the group G must be equal to the limit. $\square$

Since it is known that the Burnside groups are not automatic and thus not hyperbolic (see [5]), one application of Lemma 11.9 is that the Burnside groups cannot be finitely presented using general small cancellation theory. It will be shown that the Burnside groups of sufficiently large exponent do, however, possess a general small cancellation presentation. It follows by Lemma 11.10 that the Burnside groups of sufficiently large exponent are the limit of hyperbolic groups.