# Part III
# Constructions

In Part III general presentations and various constructions over general presentations are investigated. In Section 6 $\mathcal{R}$-categories are defined and their properties are studied. Included in this section is a definition of a general version of the traditional Poincaré construction, which is used to define the group of a general presentation. Section 7 extends the combinatorial group theory notion of a planar map in the context of general relators. More flexible structures which correspond to maps on surfaces such as spheres are also described. Finally, in Section 8 the Cayley category of a general presentation is defined and its properties are discussed.

## 6 $\mathcal{R}$-Categories

This section contains a detailed study of $\mathcal{R}$-categories. After general presentations and $\mathcal{R}$-categories are defined, the notion of the collapse of an $\mathcal{R}$-category is introduced. This leads to a generalization of the usual Poincaré construction. The section concludes with a detailed description of the relationship between general presentations and traditional presentations.

### 6.1 General Presentations

Let $\mathcal{R}$ be a set of relators. A presentation is a set of generators $A$ together with a set $\mathcal{R}$ of relators labeled by $A$. The group $G$ defined by the presentation is the quotient of the free group on the alphabet $A$ by the normal closure of the relators in $\mathcal{R}$ thought of as words in $A^*$. Presentations are written $G = \langle A|\mathcal{R}\rangle$. The Cayley graph of a presentation $G = \langle A|\mathcal{R}\rangle$ is $\mathcal{C}(G, A)$. The Cayley graph of a presentation is of interest both because of the information it carries and because it is constructible if and only if the word problem for the presentation is decidable.

The familiar Poincaré construction of a presentation is a 2-dimensional cell category whose fundamental group is the group described by the presentation. Specifically, it is a 2-dimensional cell category with 1 object of dimension 0, objects of dimension 1 corresponding to the orbits of $A$, and objects of dimension 2 corresponding to the relators in $\mathcal{R}$. One way of producing the Poincaré construction of a presentation is to take the generators $A$, viewed as labeled edges, and the relators in $\mathcal{R}$, viewed as labeled 2-cells, and first identify all of the vertices and then collapse the resulting complex until a collapsed 2-dimensional cell category labeled by $A$ emerges. The operation of collapsing a category is precisely defined below.

**Example 1** The presentation $G = \langle u, v|u^2 = v^3, uv = vu\rangle$ is a presentation of the integers. The Cayley graph of $G$ is shown in Figure 15. The edges labeled $u$

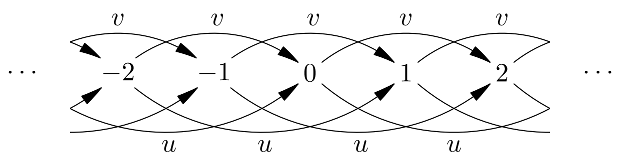


Figure 15: A presentation of $\mathbf{Z}$

and $v$ are oriented so that they start at the lesser number and end at the greater number. The Poincaré construction of the presentation is a 2-dimensional cell category homotopically equivalent to $S^1$. Let $X$ be the Poincaré construction of the presentation and let $R = \text{Cone}(X)$. Even though the cone category $R$ is not a general relator since $X$ is not a cone complex, $X$ can, if desired, be altered to produce a general relator which provides the promised counterexamples, by first subdividing the 1-skeleton into thirds, adding a new vertex in the center of each 2-cell, connecting each new vertex to each of the old vertices in its boundary, and then subdividing each of the recently added edges into a large number of pieces, say one hundred pieces each. With such an alteration it is clear that any geodesic loop in the new construction will be contained in the old 1-skeleton. For convenience the original construction $X$ will be used in the description below.

In $R$ the shortest path of winding number 1 is $uv^{-1}$ which is of length 2 and non-simple. The edge $u$ is itself a loop which has winding number 3 and the edge $v$ is a loop with winding number 2. The length of $R$ is $\frac{1}{3}$, the edge $u$ is the standard geodesic, and its width, defined below, is 2 since the removal of a vertex and all of the vertices located one edge away does not disconnect the universal cover of the boundary of the relator (see Figure 15). The construction $R$, however, is not thin, and it does not satisify the conclusions of Lemma 5.18 and Lemma 5.19.

A general presentation, denoted $\langle A|\mathcal{R}\rangle$, is defined as an alphabet $A$ of invertible generators and a set $\mathcal{R}$ of general relators labeled by $A$ which is closed under subcones. That is, if $R$ is in $\mathcal{R}$, and $S$ is a general relator contained in $R$, then $S$ is in $\mathcal{R}$. The group associated with a general presentation will not be defined until after the introduction of the general Poincaré construction later in this section. A rank function on a general presentation is a function from $A \cup \mathcal{R}$ to $\mathbf{N}$ such that the rank of each generator in $A$ is 1, and all of the general relators $R \in \mathcal{R}$ are assigned a rank of at least 2 under the restriction that whenever there is a label-preserving circular functor $f : R \to S$ between two general relators $R, S \in \mathcal{R}$, then $\text{rank}(R) \leq \text{rank}(S)$ with equality iff the $f$ is an isomorphism. Every general presentation can be assumed to be equipped with such a function, since the height function proves that such a function always exists.

The set of all general relators of rank $k$ is denoted $\mathcal{R}_k$, and the set of all general relators of rank at most $k$ is written $\mathcal{R}(k)$. Clearly, $\mathcal{R}$ is partitioned into a disjoint union of the sets $\mathcal{R}_1$, $\mathcal{R}_2$, etc., and $\mathcal{R}(k) = \mathcal{R}(k-1) \cup \mathcal{R}_k = \cup_{i=1}^{k} \mathcal{R}_i$.

The condition on the rank function guarantees that if $R$ is a general relator in $\mathcal{R}_k$, then all of the relators in its boundary are contained in $\mathcal{R}(k-1)$. Notice that $\mathcal{R}_1 = \mathcal{R}(1) = \emptyset$. A general presentation $G = \langle A|\mathcal{R}\rangle$ where $\mathcal{R}$ is a set of measured relators is called a measured presentation.

Let $\langle A|\mathcal{R}\rangle$ be a general presentation. A labeled circular category in which every slice category is either a vertex, a generator from $A$ or a general relator in $\mathcal{R}$ is called a circular category over $\langle A|\mathcal{R}\rangle$ or, more simply, an $\mathcal{R}$-category with the set of labels implicitly understood. If the underlying circular category is also a circular complex, then the result could more accurately be labeled an $\mathcal{R}$-complex. Notice that the rank function of the general presentation induces a rank function on every $\mathcal{R}$-category which assigns vertices a rank of 0, labeled edges a rank of 1, and all other objects $c$ a rank based on the rank of the general relator isomorphic to its slice category. A label-preserving circular functor between $\mathcal{R}$-categories is called an $\mathcal{R}$-functor. Notice that such a functor automatically preserves the induced ranks on its objects. An $\mathcal{R}$-functor $f : C \to B$ is called onto if every object $b \in B$ is the image of some object $c \in C$. An automorphism of an $\mathcal{R}$-category $C$ is an $\mathcal{R}$-functor from $C$ to itself which is a isomorphism. The automorphisms of an $\mathcal{R}$-category $C$ form a group called the automorphism group of $C$.

## 6.2 Collapse of an $\mathcal{R}$-Category

An $\mathcal{R}$-category $C$ is called collapsed if whenever there exist $\mathcal{R}$-functors $f, g : B \to C$ where $B$ is a connected $\mathcal{R}$-category and $f(v) = g(v)$ for a vertex $v$ in $B$, then $f$ and $g$ must be equal. Equivalently $C$ is called collapsed if its 1-skeleton is deterministic and distinct circular cones of dimension at least 2 have distinct boundaries. To say that two open cones $c_1$ and $c_2$ have the same boundaries means that $C/c_1$ and $C/c_2$ are isomorphic as labeled circular complexes and that the attaching functors agree on corresponding parts of the boundaries. Specifically, $h : C/c_1 \to C/c_2$ is an isomorphism which shows that $c_1$ and $c_2$ have the same boundaries, if the attaching functor $\phi_{c_1}$ is equal to the composition of $h$ with the attaching functor $\phi_{c_2}$. Clearly, the first definition implies the second, while Lemma 6.3 shows that the two definitions are equivalent.

A geometric description of an identification of the closed cone $C/c_1$ with the closed cone $C/c_2$ goes as follows. Because $h$ is an isomorphism between categories, its geometric realization is a homeomorphism between the polyhedra $C/c_1$ and $C/c_2$. Since attaching maps such as $\phi_{c_1}$ and $\phi_{c_2}$ are also homeomorphisms once restricted to their open cones $c_1$ and $c_2$, there is a homeomorphism $\phi_{c_2} h \phi_{c_1}^{-1}$ between points in the open cone $c_1$ and the open cone $c_2$ in the geometric circular category $C$. The collapse of $c_1$ and $c_2$ is the quotient of $C$ obtained by identifying the points in the open cones $c_1$ and $c_2$ under the above homeomorphism. The quotient map from $C$ to the quotient is a continuous map.

A categorical description of the same operation is to identify the objects $c_1$ and $c_2$ in $C$, and to identify an arrow $f : d \to c_1$ with an arrow $g : d \to c_2$ iff $h$ sends $f$ to $g$. Notice that the letters $f$ and $g$ on the righthand side represent the objects in the slice categories which correspond to the arrows $f$ and $g$ in $C$. This relation determines a unique quotient category in the sense of MacLane [11], and a unique functor from the original category onto the quotient category. These correspond to the quotient space and the quotient map described above.

The collapse of a finite $\mathcal{R}$-category $C$ can be obtained by identifying the endpoints of oriented edges which have the same initial vertex and the same label, identifying open circular cones which have the same boundaries and then repeating. Since the category is finite, and the number of objects is decreasing, the process must stop. And since the collapsing operations are confluent the result, called the collapse of $C$, is well-defined. For more general $\mathcal{R}$-categories, the usual limit construction suffices: define an equivalence relation on the objects and arrows of an $\mathcal{R}$-category $C$ where two objects (arrows) are equivalent iff there is a finite sequence of collapses after which they are equal. It only remains to check that the result of quotienting by this relation is a well-defined, collapsed $\mathcal{R}$-category $\overline{C}$ and that the obvious functor $f : C \to \overline{C}$ is a label-preserving circular functor. The category $\overline{C}$ is called the collapse of $C$ and the functor $f$ is called the collapsing functor. The collapsing functor has the additional property that every other functor from $C$ to a collapsed category $B$ factors through $f$. This proves the following lemma.

**Lemma 6.1** *If $\mathcal{R}$ is a set of general relators, and $C$ is an $\mathcal{R}$-category, then there exist a collapsed $\mathcal{R}$-category $\overline{C}$ and an onto $\mathcal{R}$-functor $f : C \to \overline{C}$ which are universal in the sense that, given any collapsed $\mathcal{R}$-category $B$ and $\mathcal{R}$-functor $g : C \to B$, there is a unique functor $h : \overline{C} \to B$ such that $hf = g$.*

The reversibility of the order in which collapses take place makes it possible to continue to collapse the edges of $C$ until a deterministic 1-skeleton emerges, and then to collapse all 2-cones until a collapsed 2-skeleton emerges, etc. Because the collapsing of higher-dimensional cones does not affect lower-dimensional skeleta, the deterministic 1-skeleton which emerges is the 1-skeleton of the final completely collapsed version, similarly for the collapsed 2-skeleton, the collasped 3-skeleton, etc. This result is summarized in the following lemma.

**Lemma 6.2** *Let $\mathcal{R}$ be a set of general relators, and let $C$ be an $\mathcal{R}$-category. For every positive integer $k$ there is an $\mathcal{R}$-category $C_k$ with a collapsed $k$-skeleton, and an $\mathcal{R}$-functor $f : C \to C_k$ which is universal in the sense that, given any $\mathcal{R}$-category $B$ with a collapsed $k$-skeleton and an $\mathcal{R}$-functor $g : C \to B$, there is a unique functor $h : C_k \to B$ such that $f$ followed by $h$ is equal to $g$. Moreover, in the particular case where $B$ is the collapse of $C$, the functor $h$ is an isomorphism when restricted to the $k$-skeletons of each category.*

If $\langle A|\mathcal{R}\rangle$ is a general presentation and every general relator $R \in \mathcal{R}$ is a collapsed $\mathcal{R}$-category in its own right, then $\mathcal{R}$ is called a collapsed set of general relators. The fact that an $\mathcal{R}$-category is collapsed restricts the number of possible $\mathcal{R}$-functors available. This is true when the collapsed category is either the domain or the range of the functor.

**Lemma 6.3** *Let $C$ be a connected $\mathcal{R}$-category, and let $B$ be an $\mathcal{R}$-category which is collapsed in the sense that it has a deterministic 1-skeleton, and distinct cones of rank at least 2 have distinct boundaries. If $f, g : C \to B$ are $\mathcal{R}$-functors with $f(v) = g(v)$ for some vertex $v$ of $C$, then $f$ and $g$ are identical. Thus, the two definitions of collapsed categories are equivalent.*

**Proof:** Since $C$ is connected, the 1-skeleton of $B$ is deterministic, and the functors $f$ and $g$ agree at a point, $f$ and $g$ must agree on the entire 1-skeleton of $C$. Next assume that $f$ and $g$ agree on the $k$-skeleton of $C$. Since the images under $f$ and $g$ of a $(k+1)$-cone agree on the boundary of the cone and $C$ is collapsed the images must be identical. Thus $f$ and $g$ agree on the $(k+1)$-skeleton of $C$, and by induction they agree completely. □

**Lemma 6.4** *If $f : B \to C$ is an $\mathcal{R}$-functor between $\mathcal{R}$-categories, and $f$ is injective on objects, then $f$ is actually an embedding of $B$ as a subcategory of $C$.*

**Proof:** If $f$ is not an embedding, then it is not injective on arrows. Since $f$ is injective on objects, however, these arrows must originate in the same hom-set in $B$. In particular, they have the same terminal object, say $b$. Next, the arrows terminating at $b$ in $B$ are in 1 to 1 correspondence with the arrows terminating at $b$ in $B/b$, which are in 1 to 1 correspondence with the arrows terminating at $f(b)$ in $C/f(b)$ since $f$ is an $\mathcal{R}$-functor, and these are in 1 to 1 correspondence with the arrows terminating at $f(b)$ in $C$, contradiction. Thus $f$ is an embedding. □

**Lemma 6.5** *If $f : B \to C$ is an $\mathcal{R}$-functor between $\mathcal{R}$-categories, $B$ is collapsed, and $f$ is injective on vertices, then $f$ is an embedding of $B$ as a subcategory of $C$.*

**Proof:** If $f$ is not an embedding, then by Lemma 6.4 it is not injective on objects. It is, by assumption, injective on vertices, so the identification must be between objects with rank at least 1. Notice that the identified objects must have the same rank since rank is preserved by $\mathcal{R}$-functors. If two distinct objects $b_1$ and $b_2$ in $B$ are chosen so that $f(b_1) = f(b_2)$, but $f$ is injective when restricted to objects of lesser rank, then the isomorphisms linking $B/b_1$ and $B/b_2$ with $C/f(b_1) = C/f(b_2)$ can be combined to show that $B/b_1$ and $B/b_2$ are isomorphic. Because of the way in which $b_1$ and $b_2$ were chosen, $b_1$ and $b_2$ must also have identical boundaries. Since $B$ is collapsed, this is impossible, and the assumption that $f$ is not an embedding must be false. □

## 6.3 Poincaré Constructions

A collapsed $\mathcal{R}$-category with a single vertex is known as a Poincaré construction. The fact that Poincaré constructions are collapsed places severe restrictions on the number of possible $\mathcal{R}$-functors as is shown above. The restrictions are in fact enough to show that Poincaré constructions are in 1 to 1 correspondence with general presentations.

**Lemma 6.6** *Let $B$ and $C$ be collapsed, connected $\mathcal{R}$-categories which have automorphism groups which act transitively on their vertex sets. If, in addition, $B$ is contained in $C$ then all of the automorphisms of $B$ extend uniquely to automorphisms of $C$.*

**Proof:** Let $u$ and $v$ be vertices in $B$. By assumption there are automorphisms $f$ and $g$ of $B$ and $C$, respectively, which send $u$ to $v$, and by Lemma 6.3 they are unique. If the range of $f$ is extended to $C$ via the embedding of $B$, and the domain of $g$ is restricted to $B$, then both are functors from $B$ to $C$ which send $u$ to $v$. Lemma 6.3 now shows that they are identical. □

**Lemma 6.7** *If $B$ and $C$ are $\mathcal{R}$-categories, $f, g : C \to B$ are $\mathcal{R}$-functors, and $B$ is a Poincaré construction, then $f$ and $g$ are identical.*

**Proof:** Since by definition $B$ has only one vertex Lemma 6.3 can be applied to each of the connected components of $C$ to show that the functors $f$ and $g$ are identical. □

**Lemma 6.8** *If $B$ and $C$ are $\mathcal{R}$-categories, $B$ is a Poincaré construction, and the generators and general relators used to construct $C$ are all used in the construction of $B$, then there is a unique $\mathcal{R}$-functor from $C$ to $B$. Consequently, distinct objects in a Poincaré construction have non-isomorphic slice categories.*

**Proof:** Once it is shown that such an $\mathcal{R}$-functor exists, Lemma 6.7 guarantees that it is unique. The existence will be shown by induction on the rank of $C$. If $C$ has rank 0 then the functor sending the objects of $C$ to the unique vertex of $B$ is clearly an $\mathcal{R}$-functor. Next suppose that the lemma is true for all $\mathcal{R}$-categories of rank less than $k$, and let $C$ have rank exactly $k$. By the induction hypothesis there is a unique functor $f$ from the $(k-1)$-skeleton of $C$ into $B$. Since the objects of rank $k$ in $C$ are terminal objects, it is enough to extend the functor to each of these objects individually. If $c$ is an object of rank $k$ in $c$ then by assumption there must be an object $b$ in $B$ such that $C/c$ is isomorphic to $B/b$ as $\mathcal{R}$-categories. Moreover, the boundary of $C/c$ mapped into $C$ and then into $B$ must agree with the image of the corresponding boundary of $B/b$ in $B$ by induction. Thus the extension of $f$ so that $c$ is sent to $b$ preserves the fact that $f$ is an $\mathcal{R}$-functor. Doing this for all objects of rank $k$ shows that the lemma is true for $\mathcal{R}$-categories of rank $k$. Thus by induction the lemma is true for all $\mathcal{R}$-categories of finite rank, and since $\mathcal{R}$-categories of infinite rank are constructed skeleton by skeleton, the lemma is also true in general. The final statement of the lemma follows from the first since distinct objects with isomorphic slice categories would have characteristic functors which would contradict the first statement. □

**Lemma 6.9** *A Poincaré construction is uniquely determined by the generators and general relators used in its construction. In particular there is a 1 to 1 correspondence between Poincaré constructions and general presentations.*

**Proof:** Let $B_1$ and $B_2$ be two Poincaré constructions for which the exact same set of generators and general relators occur as slice categories. By Lemma 6.8 there is a unique circular functor $f : B_1 \to B_2$ and again there is a unique circular functor $g : B_2 \to B_1$. By Lemma 6.8 again, the $\mathcal{R}$-functor $gf$ must be equal to the identity map on $B_1$ and the $\mathcal{R}$-functor $fg$ must be equal to the identity map on $B_2$. Thus the functors $f$ and $g$ are $\mathcal{R}$-category isomorphisms and $B_1$ and $B_2$ are isomorphic as $\mathcal{R}$-categories.

To show the second part simply notice that by starting with a general presentation $\langle A|\mathcal{R}\rangle$ it is possible to take edges labeled by the orbits of $A$, and all of the general relators in $\mathcal{R}$, and then identify all of the vertices involved, and finally collapse the result to produce an $\mathcal{R}$-category which contains precisely the generators and general relators of the general presentation. By the first part of the lemma, this Poincaré construction is uniquely determined, and the general presentation can clearly be recovered from the construction. □

In light of Lemma 6.9, define the content of a labeled circular category $C$ as the set of generators and general relators which occur in $C$ as slice categories. Lemma 6.9 can then be restated as: Poincaré constructions are uniquely determined by their content. The unique Poincaré construction corresponding to a general presentation $\langle A|\mathcal{R}\rangle$ will be called the Poincaré construction of the presentation and it will sometimes be denoted $\mathcal{P}(A, \mathcal{R})$. The fundamental group of the Poincaré construction as a topological space provides a way of assigning a group to every general presentation. If the fundamental group is $G$ then $G$ is called the group of the presentation and for simplicity this is indicated as $G = \langle A|\mathcal{R}\rangle$.

Notice that a set of generators and general relators which is not closed under subcones still creates a well-defined Poincaré construction which could be used to define the group of such a presentation. If such general presentations are allowed, however, additional care must be taken since not all slice categories of rank at least 2 would be general relators, and the statement of Lemma 6.9 would have to be altered. To avoid these situations, general presentations will always be assumed to be closed under subcones, or else it is at least tacitly understood that a general presentation refers to a set of general relators which have been 'closed' in this way.

## 6.4 Covers and Retractions

The earlier results on coverings and deformation retractions can be extended to $\mathcal{R}$-categories.

**Lemma 6.10** *If $f : C \to B$ is a covering map and $B$ is an $\mathcal{R}$-category, then $C$ can also be given an $\mathcal{R}$-category structure under which the map $f$ becomes an $\mathcal{R}$-functor. As a consequence the deck transformations of $f$ are also $\mathcal{R}$-functors.*

**Proof:** By Lemma 3.14 $C$ is the geometric realization of a cone category, and $f$ is the topological version of a cone functor between cone categories. Since a circular category is a cone category with restrictions on the types of cones, and since the slices of $C$ are isomorphic to slices of $B$ by the definition of a cone functor, $C$ is also a circular category. Finally, the induced labeling on $C$ turns $C$ into an $\mathcal{R}$-category. The last statement follows since every deck transformation is a trivial covering of $C$. $\Box$

**Lemma 6.11** *Let $\mathcal{R}$ be a set of general relators, and let $B$ be an $\mathcal{R}$-category whose fundamental group is $G$. For every normal subgroup $H$ of $G$ there is an $\mathcal{R}$-category $C$ and an $\mathcal{R}$-functor $f : C \to B$ such that $f$ is a regular covering, the fundamental group of $C$ is $H$, and the group of deck transformations of $f$ are automorphisms of $C$ which form a group isomorphic to $G/H$.*

**Proof:** From Lemma 1.6 it is clear that a regular cover exists which satisfies the conditions on the fundamental group and the group of deck transformations, and by Lemma 6.10, these can be given the structure of an $\mathcal{R}$-category and $\mathcal{R}$-functors. $\Box$

**Lemma 6.12** *Let $B$ be the Poincaré construction of a general presentation $G = \langle A|\mathcal{R}\rangle$. The regular covers of $B$ correspond to the normal subgroups of its fundamental group $G$, and the 1-skeleton of the regular cover associated with the normal subgroup $H$ is the Cayley graph $\mathcal{C}(G/H, A)$. In particular, if $C$ is the universal cover of $B$, then a path in $C$ is a loop, and thus homotopic to a point, iff the word read by the path is equivalent to 1 in the group $G$.*

**Proof:** The first assertion is a special case of Lemma 6.11. Since by the definition of a regular cover the 1-skeleton is a connected $A$-graph on which the label-preserving automorphisms act transitively on the vertices, it must be the Cayley graph of its automorphism group, by Lemma 4.4. Next, a path in $C$ is also a path in its 1-skeleton which is the Cayley graph $\mathcal{C}(G, A)$ by Lemma 6.12. By Lemma 4.5, this path is a loop iff the word read by the path is equivalent to 1 in the group $G$. Finally, since the universal cover is simply connected by definition, such a loop is homotopic to a point. $\Box$

**Lemma 6.13** *Let $G = \langle A|\mathcal{R}\rangle$ be a fixed general presentation. There is a 1 to 1 correspondence between the normal subgroups $H$ of $G$, the Cayley graphs $\mathcal{C}(G/H, A)$, the regular covers of the Poincaré construction $\mathcal{P}(A, \mathcal{R})$, and the collapses of these regular covers.*

**Proof:** The correspondence between the normal subgroups of $G$, their Cayley graphs, and the regular covers of the Poincaré construction of the presentation has already been alluded to in Lemma 6.11 and Lemma 6.12. By Lemma 6.1 the regular cover of the Poincaré construction corresponding to the normal subgroup $H$ has a unique collapse. By Lemma 6.2 the 1-skeleton of the collapsed category is identical to that of the regular cover and by Lemma 6.12 this 1-skeleton is the Cayley graph $\mathcal{C}(G/H, A)$. Since the regular cover can be reconstructed from this 1-skeleton, the correspondence between regular covers and their collapses must be 1 to 1. $\Box$

A general presentation over a set of thin general relators is closely connected with certain specific standard presentations of the same group. The connection between the two types of presentation is elucidated through the use of deformation retractions. Let $\mathcal{R}$ be a set of general relators, and let $\mathcal{R}'$ be a set of cycles in 1 to 1 correspondence with $\mathcal{R}$ such that given any general relator $R \in \mathcal{R}$, the corresponding cycle in $\mathcal{R}'$ can be read as a simple representative of $R$. The set $\mathcal{R}'$ is called a set of standard representatives for $\mathcal{R}$. Recall that if the general relators in $\mathcal{R}$ are assumed to be thin, then by Lemma 5.19 there always exists at least one set of standard representatives for $\mathcal{R}$.

**Lemma 6.14** *Let $\mathcal{R}$ be a set of general relators and let $\mathcal{R}'$ be a set of standard representatives for $\mathcal{R}$. If $C$ is an $\mathcal{R}$-category then there is a deformation retraction of $C$ onto a 2-dimensional subcomplex of the simplicial subdivision of $C$ which can be viewed as a simplicial subdivision of an $\mathcal{R}'$-category. In addition, the deformation can be chosen so that the 1-skeleton remains fixed throughout.*

**Proof:** The proof proceeds by induction on the rank of the $\mathcal{R}$-category $C$. First of all, notice that the lemma is trivially true if $C$ is an $\mathcal{R}$-category of rank 1, since in this case $C$ is simply an $A$-graph, and thus trivially an $\mathcal{R}'$-category. The deformation retraction mentioned in the lemma is the identity retraction which keeps the 1-skeleton fixed throughout. Suppose that the lemma is true for all sets of general relators $\mathcal{R}$ and $\mathcal{R}$-categories of rank at most $k$. Let $\mathcal{R}(k)$ be the subset of $\mathcal{R}$ consisting of the general relators in $\mathcal{R}$ of rank at most $k$. It should be clear that an $\mathcal{R}$-category of rank $k$ is also an $\mathcal{R}(k)$-category. Let $\mathcal{R}'(k)$ be the subset of $\mathcal{R}'$ corresponding to the general relators in $\mathcal{R}(k)$.

Consider a general relator $R$ in $\mathcal{R}$ of rank $k+1$ with a corresponding simple representative $U$ in $\mathcal{R}'$. Since the boundary of $R$, $\partial R$, is an $\mathcal{R}(k)$-category, by assumption there is a deformation retraction of $\partial R$ onto a 2-dimensional subcomplex, say $S$, of the simplicial subdivision of $\partial R$ which can be viewed as a simplicial subdivision of an $\mathcal{R}'(k)$-category. By Lemma 1.11 the deformation retraction of $\partial R$ onto $S$ can be extended to a deformation retraction of $R$ onto $\mathrm{Cone}(S)$. Since the 1-skeleton is fixed throughout, the loop $U$ is a simple loop $S$ as well as in $\partial R$, and the injection of $U$ into $S$ is a weak homotopy equivalence, and thus by Lemma 1.6 there is a deformation retraction from $S$ to $U$. By Lemma 1.12 there is a deformation retraction from $\mathrm{Cone}(S)$ onto $S \cup_U \mathrm{Cone}(U)$. Combining these two deformations, it has been shown that for all relators $R$ of rank $k+1$ there is a deformation retraction of $R$ onto $S \cup_U \mathrm{Cone}(U)$ which extends the prior retraction of $\partial R$ onto $S$. Since the 1-skeleton remains fixed throughout, $U$ is in the 1-skeleton, and $\mathrm{Cone}(U)$ is the 2-cell of a relator in $\mathcal{R}'$, the deformation from $R$ to $S \cup_U \mathrm{Cone}(U)$ satisfies the conditions of the lemma.

Next consider an arbitrary cone category $C$ of rank $k+1$. By assumption there is a deformation retraction of the $k$-skeleton which satisfies the lemma. Since this retraction of the $k$-skeleton of $C$ induces a deformation retraction of the boundary of each general relator $R$ of rank $k+1$ which is attached to the $k$-skeleton via the attaching map $\phi_R$. By the argument above, there is a deformation retraction of $R$ onto an $\mathcal{R}'$-category which extends the deformation retraction of its boundary. The image of this deformation retraction of $R$ is consistent with the earlier retraction of the $k$-skeleton, so that the deformation retractions on each of the relators of rank $k+1$ combined with the deformation on the $k$-skeleton yield a single deformation on $C$ which satisfies the lemma.

Finally, a deformation retraction on an arbitrary cone category $C$ which satisfies the requirements of the lemma can be constructed inductively via the deformation retractions on the various skeleta. $\square$

**Lemma 6.15** *If $\mathcal{R}$ is a set of thin general relators and $C$ is an $\mathcal{R}$-category, then $H_i(C) = 0$ for all $i > 2$.*

**Proof:** By the previous lemma, every $\mathcal{R}$-category has the homotopy type of a 2-dimensional cell category. Since homology groups are well-defined up to homotopy type, it is sufficient to show that the conclusion holds for 2-dimensional cell categories. But since cell categories are particular types of CW complexes, and the $n$-th homology groups of a CW complex are generated by the characteristic maps on its $n$-cells, the homology groups beyond dimension 2 must be trivial. $\square$

**Lemma 6.16** *Let $\mathcal{R}$ be a set of general relators, and let $\mathcal{R}'$ be a set of standard representatives for $\mathcal{R}$. The Poincaré construction $\mathcal{P}(A, \mathcal{R}')$ can be viewed as a connected subcomplex of the simplicial subdivision of the Poincaré construction $\mathcal{P}(A, \mathcal{R})$, and the inclusion map is a homotopy equivalence. In particular, the fundamental groups of the two constructions are the same, showing that $G = \langle A|\mathcal{R}'\rangle$ is a standard presentation of the group of the general presentation $\langle A|\mathcal{R}\rangle$.*

**Proof:** By Lemma 6.14, there is a deformation retraction of the Poincaré construction $\mathcal{P}(A, \mathcal{R})$ onto an $\mathcal{R}'$-category during which the 1-skeleton remains fixed. In particular, the $\mathcal{R}'$-category which results contains only 1 vertex, and there is a unique way of attaching every labeled 2-cell in $\mathcal{R}'$ to the 1-skeleton. Finally, since every general relator $R \in \mathcal{R}$ occurs exactly once in the Poincaré construction, then every standard representative in $\mathcal{R}'$ occurs exactly once, and the resulting deformation retraction must be already collapsed as an $\mathcal{R}'$-category. By Lemma 6.9 it is the Poincaré construction of the presentation $\langle A|\mathcal{R}'\rangle$. Since deformation retractions preserve homotopy type, the fundamental groups must be the same. $\square$

According to Lemma 6.16, if $\mathcal{R}'$ is a set of standard representatives of a set $\mathcal{R}$ of general relators, then $G = \langle A|\mathcal{R}\rangle = \langle A|\mathcal{R}'\rangle$. As a consequence, $G$ is independent of the choice of standard representatives. Given any general presentation $G = \langle A|\mathcal{R}\rangle$, there is a general presentation $G(k) = \langle A|\mathcal{R}(k)\rangle$ for each value of $k \in \mathbf{Z}^+$. If $\mathcal{R}$ has a set of standard representatives $\mathcal{R}'$, then the general presentations of $G$, $G(1)$, $G(2)$, etc., can be converted into standard presentations. In this case there are obvious group homomorphisms from $G(k)$ onto $G(k+1)$ since $G(k+1)$ is obtained from $G(k)$ by the addition of relations, and the group $G$ can itself be described as the direct limit of the groups $G(k)$. This observation is recorded below. The final lemma is an application of Lemma 6.16 which will be needed later.

**Lemma 6.17** *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation, and let $\mathcal{R}$ have a set of standard representatives. Then $G$ is the direct limit of the groups $G(k)$.*

**Lemma 6.18** *The universal cover of a deformation retraction is a deformation retraction of the universal cover. In particular, if $\mathcal{R}$ is a set of general relators and $\mathcal{R}'$ is a set of standard representatives, then $\mathcal{P}(A,\mathcal{R}')$ can be viewed as a subspace of $\mathcal{P}(A,\mathcal{R})$, and the universal cover of the former can be viewed as a subspace of the universal cover of the latter.*

**Proof:** Let $C$ be the universal cover of an $\mathcal{R}$-category $B$ with covering map $f : C \to B$, and let $h : B \times I \to B$ be a deformation retraction from $B$ onto a subcomplex. Since $C$ is simply connected, so is $C \times I$, and thus the function $g : C \times I \to B$ defined by $g(c,x) = h(f(c),x)$ must lift through $f$ to $C$ by Lemma 1.13. A routine verification shows that the lift of $g$ is a deformation retraction of $C$ onto the universal cover of the subcomplex of $B$. □

A similar result holds for regular covers of $\mathcal{R}$-categories, but it will not be needed here. The deformation retraction of a collapse, however, is not necessarily even of the same homotopy type as the collapse of the deformation retraction, as illustrated by Example 2 in Section 8.

# 7 $\mathcal{R}$-Structures

In this section the focus is on maps (in the sense of small cancellation theory), $\mathcal{R}$-diagrams, and $\mathcal{R}$-spheres. After the key concept of a map is defined, some examples are given, and a few results from Lyndon and Schupp ([9]) are quoted. In the few cases where the proofs in [9] are stronger than the statements of the lemmas to which they are attached, the statements given here have been altered to capture the full import of the proofs. The extra flexibility will be used in the general version of small cancellation theory. The section concludes with a general version of van Kampen's Lemma which is applicable in all $\mathcal{R}$-categories so long as the general relators in $\mathcal{R}$ are thin.

## 7.1 Maps

A map is an embedding of the geometric realization of a finite 2-dimensional cell category in the Euclidean plane or, equivalently, it is a bounded finite union of disjoint subsets of the plane homeomorphic to points, open intervals and open disks, and subject to those conditions implicit in the first chacterization. The implicit conditions are, namely, that the topological boundary of an open disk be the union of a finite set of points and open intervals, and that the topological boundary of an open interval be a finite set of points. The latter is the definition of a map given by Ol'shanskii in [15].

Let $N$ be a component of the complement of a connected map $M$. The boundary of $N$ is the unique reduced loop in $M$ for which there is an $\epsilon$-deformation to a loop in $N$ for all arbitrarily small $\epsilon > 0$. The boundary of

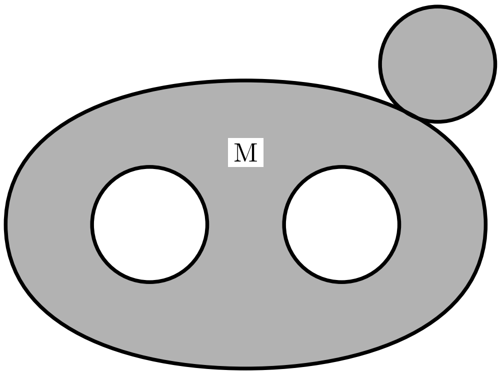


Figure 16: Non-simple boundary loops

a connected map $M$ is the set of boundaries of the components of its complement, and the boundary of a disconnected map $M$ is the union of the boundaries of its connected components. The set of loops which is the boundary of $M$ is denoted $\partial M$.

**Example 1** In Figure 16 a map $M$ is shown schematically. The shaded portion represents the union of the embedded points, lines and cells. The boundary of the map consists of three loops. The two boundary loops arising from the bounded components of the complement happen to be simple loops and the boundary loop defined using the unbounded component is not simple. This example shows how the use of $\epsilon$-deformations in the definition selects the way of traversing the non-simple loop which corresponds to our intuitive notion of a boundary.

If $M$ is connected and simply connected — the most important case — then $\partial M$ is a single loop. If $M$ is an annular map, that is, if $M$ is connected and the complement of $M$ has exactly two components, then $\partial M$ consists of two loops, which may or may not be disjoint.

**Example 2** The map sketched in Figure 17 is an example of an annular map in which the two boundary loops are not disjoint.

By a region of $M$ is meant either an open 2-cell $D$ in $M$ or a component of the complement of $M$. Every edge in $M$ is on the border of exactly two regions. The standard orientation of the plane induces an orientation of every region of $M$, and thus an orientation of the loops $\partial D$ and an orientation of the loops in $\partial M$. Every edge of $M$ is given opposite orientations by the two regions it borders. Thus every oriented edge in $M$ is a properly oriented edge of a unique boundary loop of a unique region of $M$.

Notice that any orientation of the plane induces a clockwise orientation on one of the boundary cycles of an annular diagram and a counter-clockwise ori-

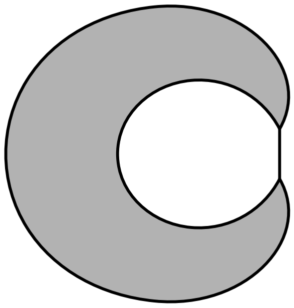

Figure 17: Non-disjoint boundary loops

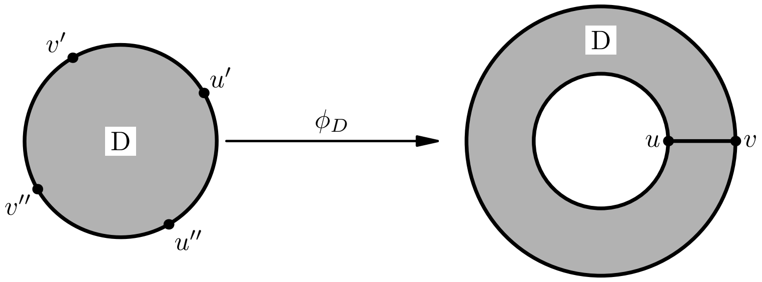


Figure 18: A self-bordering cell with attaching map

entation on the other. Since the standard orientation of the plane is counterclockwise, and boundary cycles are oriented according to the orientation of the component of the complement which they bound, the standard orientation of the boundary cycles of an annular diagram are counterclockwise on the inner cycle and clockwise on the outer cycle.

An arc of the 2-cell $D$ is a path in $\partial D$ whose image is an arc in $M$. A boundary arc of $D$ is an arc of $D$ whose image is in $\partial M$. An interior arc of $D$ is an arc of $D$ which is not a boundary arc. The number of arcs in $D$ is called the degree of $D$ and written $d(D)$. The number of interior arcs in $D$ is called the interior degree of $D$ and written $i(D)$. Notice that a single arc in $M$ might have two distinct preimages in $\partial D$ if the two orientations of the edges in the arc both lift to oriented edges in $D$, in which case it is counted twice in $d(D)$ and $i(D)$. A boundary cell is a 2-cell $D$ whose image in $M$ intersects $\partial M$, and a cell which is not a boundary cell is an interior cell. A boundary cell $D$ such that $\partial D \cap \partial M$ consists of a single boundary arc is called an exposed cell. The notation $\sum^*$ denotes a summation performed only over the exposed cells. Finally, a $C(p)$-map is a map with no vertices of degree 1 and with $i(D) \geq p$ whenever $D$ contains no boundary arcs.

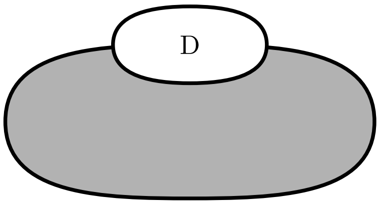


Figure 19: An exposed cell

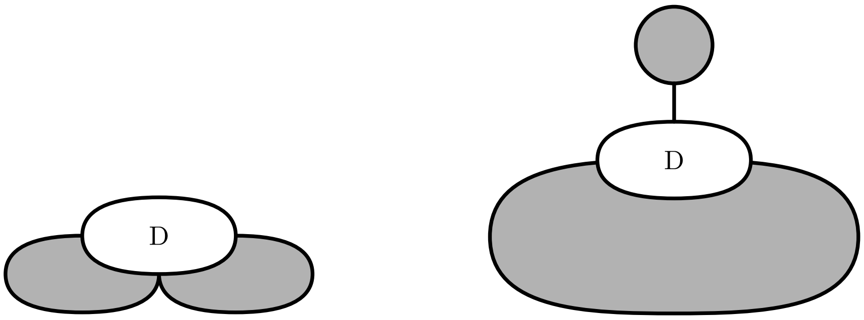


Figure 20: Two non-exposed cells

**Example 3** On the righthand side of Figure 18 is a map containing a single 2-cell which is self-bordering. It also contains two vertices and three edges. On the lefthand side is the slice of the 2-cell $D$. Notice that paths from $u'$ to $v'$ and from $u''$ to $v''$ are both sent under the attaching map $\phi_D$ to the arc between $u$ and $v$, which causes this arc to be counted twice when calculating the degrees of $D$. Specifically, the degree of $D$, $d(D)$, is 4, while the interior degree of $D$, $i(D)$, is 2, even though the map contains only 3 arcs, and only 1 interior arc. Notice that even though $\partial D$ is a simple loop, the image of $\partial D$ under $\phi_D$ is no longer simple.

**Example 4** A typical exposed cell is shown schematically in Figure 19, while Figure 20 shows two examples of 2-cells which are almost but not quite exposed cells. The first example is not exposed because the intersection of $\partial D$ and $\partial M$ is a single arc plus an isolated point on the other side of the cell. The existence of a single arc of $D$ which contains all of the edges in the overlap is a necessary but not a sufficient condition for $D$ to be exposed. In the second example, the edges in the intersection of $\partial D$ and $\partial M$ lie on a path but $D$ is not exposed since the vertex of degree 3 in the interior of the path means that the path is not a single arc.

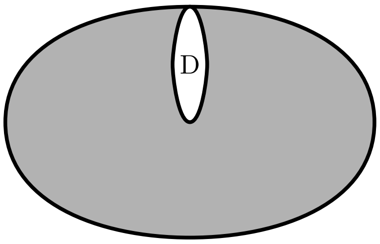


Figure 21: A boundary cell

**Example 5** In order for a cell to qualify as a boundary cell it is not necessary for it to contain an edge of $\partial M$. A vertex from $\partial M$ is sufficient. This situation is illustrated in Figure 21.

Using slightly different notation, Lyndon and Schupp [9] prove the following three lemmas.

**Lemma 7.1** *If $M$ is a connected and simply connected $C(6)$-map with more than one 2-cell, then*

$$\sum{}^{*}(4 - i(D)) \geq 6$$

.

**Lemma 7.2 (Greedlinger)** *If $M$ is a connected and simply connected $C(6)$-map with more than one 2-cell, then there exist exposed cells satisfying one of the following:*

*1)* $i(D_1), i(D_2) \leq 1$
*2)* $i(D_1), i(D_2), i(D_3) \leq 2$
*3)* $i(D_1), i(D_2) \leq 2$ *and* $i(D_3), i(D_4) \leq 3$
*4)* $i(D_1) \leq 2$ *and* $i(D_2), i(D_3), i(D_4), i(D_5) \leq 3$
*5)* $i(D_1), i(D_2), i(D_3), i(D_4), i(D_5), i(D_6) \leq 3$

Notice that Greedlinger's Lemma is more than just a statement of the consequences of the previous lemma. It states, for example, that if only two exposed cells have internal degrees less than 4 then both must internal degrees $\leq 1$, even though $i(D_1) = 0$ and $i(D_2) = 2$ satisfies the earlier inequality.

**Lemma 7.3** *Let $M$ be an annular $C(6)$-map and let $X$ and $Y$ be the boundary loops of $M$. If for every exposed cell $D$ in $M$ an edge of $X$ in $\partial D$ implies $i(D) > 4$, and an edge of $Y$ in $\partial D$ implies $i(D) > 3$, then every 2-cell in $M$ has exactly two boundary arcs, one arc in each boundary loop, and its internal degree is at most 2.*

These lemmas correspond to lemmas V.4.3, V.4.5 and a combination of lemmas V.5.3 and V.5.5 in [9]. In Lemma V.5.3, Lyndon and Schupp make use of

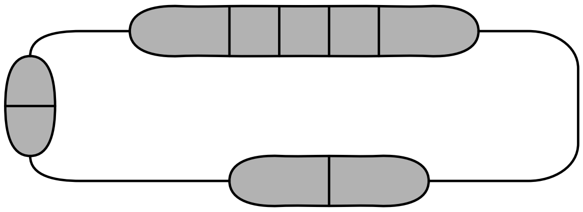

Figure 22: Islands and bridges

the additional assumption that $M$ is a $C(7)$-map, but with a little extra work, this can be reduced to the assumption that $M$ is a $C(6)$-map, given the assumption which has been added in this formulation that $i(D) > 4$ for exposed cells on one of the boundary loops. More specifically, in equation (5.3) the second and third summations are non-positive, and the first sum must be negative under the conditions listed above. This contradiction shows that hypothesis (C) in Lemma V.5.3 is false, and thus hypothesis (C) in Lemma V.5.5 is true.

In a map such as that described in Lemma 7.3, an arc shared by the two boundary loops of $M$ is called a bridge, since it looks like a bridge connecting two islands. An island is a connected component of the annular diagram once the bridges are removed. The boundary of an island consists of a path in each of the boundary cycles such that the two paths have the same endpoints but their interiors are disjoint. An internal arc in such a diagram is called a rung, and boundary arcs which are not bridges are called sides. The terminology arises from thinking of the islands as ladders.

**Example 6** The annular map shown in Figure 22 illustrates the terminology above. It contains three islands which contain one, two and five 2-cells respectively. The island with five cells contains four rungs and ten side arcs. The island with only one cell contains two side arcs and no rungs. The three islands are linked together by three bridges.

## 7.2 $\mathcal{R}$-Diagrams and $\mathcal{R}$-Spheres

A map $M$ is said to be labeled by $A$ if the embedded cell category is labeled by $A$. To accommodate later usage, the definitions given below will be stated in terms of an arbitrary set of cycles. If $\mathcal{R}$ is a set of standard relators, then $\mathcal{R}$ itself can be viewed as a set of cycles. If $\mathcal{R}$ is a set of general relators, then a set of cycles can be produced by considering all of the representatives of the general relators in $\mathcal{R}$. By Lemma 5.1 there exists at least one representative for each general relator in $\mathcal{R}$. By an abuse of notation, $\mathcal{R}$ will also be used to denote the set of cycles derived from a set of general relators.

Let $M$ be a map labeled by $A$ and let $\mathcal{R}$ be viewed as a set of cycles. If $D$ is a 2-cell of $M$ attached by the map $\phi_D : D \to M$ then the labeling of $M$ induces a labeling on the boundary of $D$. A van Kampen diagram over $\mathcal{R}$ is a labeled map $M$ such that the cycle read on the boundary of each 2-cell is in $\mathcal{R}$. Diagrams over $\mathcal{R}$ are also called $\mathcal{R}$-diagrams. More generally, if $M$ is a labeled, 2-dimensional cell category, and for every 2-cell $D$ attached to $M$ the cycle read on the boundary of $D$ is contained in $\mathcal{R}$, then $M$ is called an $\mathcal{R}$-structure. If in addition the cell category $M$ is homeomorphic to a 2-sphere, then $M$ is called an $\mathcal{R}$-sphere.

Let $M$ and $N$ be $\mathcal{R}$-diagrams and let $f : N \to M$ be a label-preserving cell functor. The $\mathcal{R}$-diagram $N$ is called a subdiagram of $M$ if the composition of $f$ with the embedding of $M$ into the plane can itself be made into an embedding by an $\epsilon$-deformation of the composition for all $\epsilon > 0$. This definition is formulated so that the simply connected diagram obtained by cutting along a simple path connecting the two boundary loops of an annular diagram is an example of a subdiagram. Similarly, if $N$ is an $\mathcal{R}$-diagram, $M$ is an $\mathcal{R}$-sphere, and $f : N \to M$ is a label-preserving cell functor such that for all $\epsilon > 0$ there is an $\epsilon$-deformation which turns $f$ into an embedding into the underlying 2-sphere, then $N$ is called a subdiagram of the $\mathcal{R}$-sphere $M$.

If every cycle $W$ is readable as a representative cycle in at most one general relator $R$ in $\mathcal{R}$, then $\mathcal{R}$ is said to have distinct representatives. When a set $\mathcal{R}$ of general relators has distinct representatives, it is possible to assign a number to each 2-cell in an $\mathcal{R}$-structure based on the rank of the general relators it represents. The rank of an $\mathcal{R}$-structure is defined to be the maximum of the ranks of its cells, and the type of an $\mathcal{R}$-structure, denoted $\mathrm{Type}(M)$, is the number of 2-cells of each rank, plus the number of edges in the structure. If $M$ is an $\mathcal{R}$-structure of rank $k$, then the type of $M$ is an ordered $k$-tuple. Types are ordered lexiconigraphically, first by rank and then by the number of cells in each rank from largest to smallest, and finally by the number of edges.

Specifically, if $M$ and $N$ are $\mathcal{R}$-structures of types $\mathrm{Type}(M)$ and $\mathrm{Type}(N)$ respectively, then $\mathrm{Type}(M) < \mathrm{Type}(N)$ if the rank of $M$ is less than that of $N$, or if they are both of rank $k$ but $M$ has fewer cells of rank $k$, or if they both have rank $k$ and they both have the same number of rank $k$ cells but $M$ has fewer cells of rank $k - 1$, etc., or if $M$ and $N$ have exactly the same number of cells in every rank and $M$ contains fewer edges. The most important fact about the ordering of types is that, because it is a lexiconigraphic ordering of noetherian orders, it is itself noetherian, meaning that all infinitely decreasing chains must stabilize. In particular, it is possible to use induction on types.

Let $\alpha$ be a fixed positive constant. A path $U$ in a measured relator $R$ is long relative to $\alpha$ iff $d_R(U) \geq \alpha$, otherwise it is said to be short relative to $\alpha$. Such paths will also be referred to simply as long and short when the constant $\alpha$ is understood from context. An arc $U$ in a diagram $\Delta$ is called a long arc if $U$ is in the boundary cycle of a 2-cell $D$ which represents a measured relator $R$, and $U$ is long under the reading of $U$ in $R$ given by composing the reading of $U$ in $\partial D$ with the reading of $\partial D$ as a representative in $R$. In small cancellation presentations different readings of the same representative in a relator differ by an automorphism of the relator and thus yield the same length on the arc $U$ by Property 6. In other situations, a particular reading of the boundary cycles of the cells in the measured relators which they represent must be chosen and

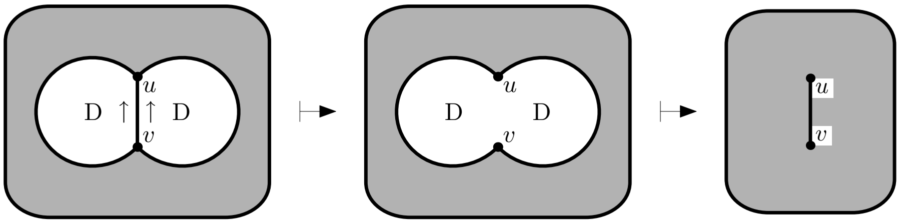


Figure 23: Removing a cancellable pair

fixed as part of the definition of a diagram over such a set of measured relators.

**Lemma 7.4** *Let $\mathcal{R}$ be a set of measured relators, and suppose that $\alpha \leq \frac{1}{p}$. If $\Delta$ is an $\mathcal{R}$-diagram with no vertices of degree 1 and no internal arcs which are long relative to $\alpha$, then $\Delta$ is a $C(p+1)$-map.*

**Proof:** If $D$ is a representative cell in $\Delta$ with no boundary arcs, then by definition it contains only short internal arcs, each of which has length strictly less than $\alpha \leq \frac{1}{p}$. Since by Lemma 5.25 the lengths must add to at least 1, there must be at least $p+1$ arcs. This shows that for every such $D$, $d(D) \geq p+1$ and completes the proof. □

Two $\mathcal{R}$-diagrams are called equivalent if they have exactly the same list of boundary cycles, even when orientations and repetitions are taken into consideration. The goal of much of small cancellation theory is to start with an $\mathcal{R}$-diagram and then systematically alter it to obtain an equivalent diagram which has additional, more managable properties. The most familiar type of alteration is the removal of what is known as a cancellable pair. The description of a cancellable pair given below is a slight generalization of the usual definition. Let $N$ be an $\mathcal{R}$-diagram with exactly two closed 2-cells which intersect in exactly one closed edge, and suppose that $N$ has no other 2-cells, edges or vertices. If the 1-skeleton of $N$ can be mapped into a general relator $R$ so that the boundary cycle of each of the 2-cells is sent to a representative of $R$, and the image of the boundary cycle of $N$ has winding number 0, then $N$ is called a cancellable pair. If $\Delta$ is an $\mathcal{R}$-diagram or an $\mathcal{R}$-sphere which contains $N$ as a subdiagram, then intuitively $N$ can be 'cut out' and the hole can be 'sewn up'. Complications can arise because of the creation of new $\mathcal{R}$-spheres. These will be treated in more detail below. An uncomplicated case is illustrated schematically in Figure 23.

It should be noted that the image of $N$ in $\Delta$ may contain many more than two boundary arcs and one internal edge. The numbers are derived from a consideration of $N$ as an $\mathcal{R}$-digram in its own right prior to its embedding in $\Delta$. In particular, even if two cells in $\Delta$ contain more than one internal edge from which these cells form a cancellable pair, the embedding of the $\mathcal{R}$-diagram effectively selects one of the edges for consideration. Finally, notice that in traditional small cancellation theory, the set $\mathcal{R}$ of possible cycles is reduced, so that the possible cancellable pairs are severely restricted. In particular, if $N$ is a cancellable pair over a traditional set of reduced cycles and $v$ is a vertex on the shared edge of $N$, the boundary cycle of one 2-cell starting at $v$, reading counter-clockwise, must be the same as the boundary cycle of the other 2-cell read clockwise starting at $v$. Thus the notion of a cancellable pair given here is an extension of the traditional concept.

## 7.3 Boundaries

Let $M$ be an $\mathcal{R}$-diagram, and let $W_1, W_2, \ldots, W_k$ be the properly oriented boundary cycles of $M$, with possible repetition. If there does not exist an $\mathcal{R}$-diagram whose properly oriented boundary cycles are a non-empty proper subset of this list then these cycles are said to be linked. The concept of an $\mathcal{R}$-diagram with linked boundaries can be reformulated in terms of $\mathcal{R}$-spheres. Let $\mathcal{R}^+$ be the union of the cycles $\mathcal{R}$ and the boundary cycles of an $\mathcal{R}$-diagram $M$. The embedding of $M$ in the plane combined with the standard embedding of the plane into the 2-sphere creates an embedding of $M$ in the 2-sphere which can then be viewed as a $\mathcal{R}^+$-sphere. From this perspective, one cell has been added to $M$ for each connected component of the complement of $M$ in the 2-sphere. Then these new cells will be called 'phantom' cells.

The boundaries of $M$ are not linked if there is an $\mathcal{R}^+$-sphere whose phantom cells can be put in 1 to 1 correspondence with some, but not all, of these phantom cells. The orientations, labels, and the number of repetitions of the boundary cycles, as well as the absence of any other phantom cells, are clearly important factors in determining whether a particular $\mathcal{R}$-sphere demonstrates that a list of cycles is not linked. Conversely, the boundary cycles of $M$ are linked if no such $\mathcal{R}^+$-sphere exists. By far the most important examples of diagrams with linked boundaries are arbitrary connected and simply connected $\mathcal{R}$-diagrams and annular $\mathcal{R}$-diagrams in which the boundary cycles do not bound connected and simply connected $\mathcal{R}$-diagrams.

The next two lemmas involve identifications of boundaries. The first is a simple topological result, while the second is a variation which takes into account the labeling on the boundary. Despite the simplicity of their proofs, these results will be extremely useful. The difference between Lemma 7.5 and Lemma 7.6 is that in the latter the identifications do not collapse edges to points. The lemmas are stated in terms of one-point products. If $B$ and $C$ are topological spaces, then a one-point product is given by selecting a point in $B$ and a point in $C$ and then identifying them. The one-point product of a finite list of spaces is simply a repetition of this operation a finite number of times. In this definition, the points selected can vary from product to product. Thus, for example, the one-point product of a finite number of edges is a finite tree, and every finite tree is the one-point product of a finite number of edges. In an arbitrary topological space the choice of points to attach is completely arbitrary. In circular categories, however, the choice of points will be restricted to the vertices in the 1-skeletons.

**Lemma 7.5** *If $M$ is a connected and simply connected map, then the identification of its entire boundary to a single point yields a structure which is a one-point product of a finite number of spheres.*

**Proof:** The identification of the boundary to a point will be done in stages. First remove all cut edges by identifying them to a point one at a time. Next proceed by induction on the number of cut vertices which exist in $M$. If none exist, then either $M$ is a single point or, since there are no cut edges, the boundary loop of $M$ is simple and its identification to a point produces a sphere. If a cut vertex exists then the inductive hypothesis can be applied to each of the connected components formed by its removal and then these structures can be reattached using a one-point product. □

**Lemma 7.6** *Let $\mathcal{R}$ be a set of cycles, and let $M$ be a connected and simply connected $\mathcal{R}$-diagram. If the boundary cycle of $M$ is a Dyck word, then there is an identification of the boundary edges so that the result is a one-point product of a finite number of $\mathcal{R}$-spheres and labeled edges.*

**Proof:** The proof proceeds by induction on the length of the boundary cycle. By Lemma 4.2 the length is even, so the minimal length is 2, and the boundary cycle must be $aa^{-1}$ for some $a \in A$. If the edges labeled $a$ and $a^{-1}$ are distinct then their identification forms a sphere both in the case where the diagram contains two vertices and where it contains only a single vertex. If these edges are not distinct then this edge is in itself the complete diagram. In either case, the lemma is true for diagrams whose boundary cycles have length 2.

Next assume that the result has been shown for all diagrams whose boundary cycles have length at most $2(k-1)$, $k > 1$, and let $M$ be a diagram with a boundary cycle of length $2k$. By Lemma 4.2 there exist a pair of adjacent oriented edges in the boundary cycle labeled $aa^{-1}$ for some $a \in A$. Let $e_1$ and $e_2$ be such a pair of adjacent edges with $v$ the vertex between them in the boundary cycle. If the initial vertex of $e_1$ and the final vertex of $e_2$ are distinct then the map can be stretched so that the edges $e_1$ and $e_2$ and their vertices are identified in a portion of the unbounded component of the complement, thus keeping the map planar. The inductive hypothesis can then be applied to this new diagram.

If, however, the initial vertex of $e_1$ and the terminal vertex of $e_2$ are one and the same vertex, say a vertex called $u$, then the subdiagram bounded by the loop $e_1e_2$ is connected, simply connected and attached to the rest of $M$ only at $u$. If this subdiagram is temporarily removed then the inductive hypothesis can be applied to each of the two connected components. Finally, it is clear that the pieces can be reattached at appropriate points using a one-point product. This completes the proof. □

Notice that since in Lemma 7.6 the only identifications are of edges, the number and the rank of the 2-cell is unchanged, and the type of the $\mathcal{R}$-structure remains constant.

## 7.4 A General Van Kampen Lemma

Lemma 7.7 and Lemma 7.8 will provide the connection between the geometry of $\mathcal{R}$-diagrams and the algebra of presentations which is needed to prove a general version of van Kampen's Lemma. For proofs see [9].

**Lemma 7.7 (van Kampen)** *Let $G = \langle A|\mathcal{R}\rangle$ be a group presentation. A word $W \in A^*$ is equivalent to 1 in $G$ iff there is a connected and simply connected $\mathcal{R}$-diagram $M$ whose boundary cycle is $W$.*

**Lemma 7.8** *Let $G = \langle A|\mathcal{R}\rangle$ be a group presentation, and let $X$ and $Y$ be words which are not equivalent to the identity in $G$. The words $X$ and $Y$ are conjugate in $G$ iff there is an annular $\mathcal{R}$-diagram $M$ with boundary cycles $X$ and $Y^{-1}$.*

These results can be extended to general presentations using the earlier results on deformation retractions.

**Lemma 7.9** *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation. If the general relators in $\mathcal{R}$ are thin, then a word $W \in A^*$ is equivalent to 1 in $G$ iff there is a connected and simply connected $\mathcal{R}$-diagram $M$ whose boundary cycle is $W$.*

**Proof:** Since the general relators are thin, there is a set of standard representatives $\mathcal{R}'$ for $\mathcal{R}$ by Lemma 5.19. By Lemma 6.16, $G = \langle A|\mathcal{R}'\rangle$ is a standard group presentation for $G$. Since all $\mathcal{R}'$-diagrams are also $\mathcal{R}$-diagrams, the result follows by Lemma 7.7. □

**Lemma 7.10** *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation, and let $X$ and $Y$ be words which are not equivalent to the identity in $G$. If the general relators in $\mathcal{R}$ are thin, then the words $X$ and $Y$ are conjugate in $G$ iff there is an annular $\mathcal{R}$-diagram $M$ with boundary cycles $X$ and $Y^{-1}$.*

**Proof:** The proof is the same as in the previous lemma. Since the general relators are thin, there is a set of standard representatives $\mathcal{R}'$ for $\mathcal{R}$, by Lemma 5.19. By Lemma 6.16 $G = \langle A|\mathcal{R}'\rangle$ is a standard group presentation for $G$. Since all $\mathcal{R}'$-diagrams are also $\mathcal{R}$-diagrams, the result follows by Lemma 7.8. □

An $\mathcal{R}$-diagram like that described in Lemma 7.9 is called a proof that $W$ equals 1 in $G$, and the $\mathcal{R}$-diagrams described in Lemma 7.10 are called conjugacy diagrams, or proofs that $X$ and $Y$ are conjugate in $G$. A more detailed analysis using the Simplicial Approximation Theorem would allow the condition that the relators are thin to be removed from the hypotheses of Lemma 7.9 and Lemma 7.10, but these lemmas will suffice for the general small cancellation theory discussed here, since all of the general relators in a general small cancellation presentation will necessarily be thin. Two useful consequences of Lemma 7.9 are given below.

**Lemma 7.11** *Let $\mathcal{R}$ be a set of thin general relators, and let $C$ be an $\mathcal{R}$-category. If $W$ is a contractible loop in $C$, then there is a connected and simply connected $\mathcal{R}$-diagram whose boundary cycle is $W$.*

**Proof:** By Lemma 1.9 there is a map from the unit disk into $C$ where the restriction of the map to the boundary recovers the reading of the loop $W$. Composing this with the unique map to the Poincaré construction of these general relators (Lemma 6.8) yields an image of a disk which shows that $W$ is contractible in the Poincaré construction. By definition this means that $W$ is equivalent to 1 in the group, and by Lemma 7.9 an $\mathcal{R}$-diagram such as the one described must exist. □

**Lemma 7.12** *Let $\mathcal{R}$ be a set of thin general relators, and let $R$ be a general relator in $\mathcal{R}$. If $W$ is a loop in $R$ of winding number 0, then there is a connected and simply connected $\mathcal{R}$-diagram $\Delta$ with boundary $W$ which contains 2-cells which represent only general relators in $\partial R$. In particular, the 2-cells in $\Delta$ represent only general relators whose height is strictly less than that of $R$.*

**Proof:** Since $W$ has winding number 0 it is contractible in $\partial R$, and thus Lemma 7.11 can be used to complete the proof. □

## 7.5 Removing Cancellable Pairs

The next several lemmas show how cancellable pairs can be systematically removed.

**Lemma 7.13** *Let $\mathcal{R}$ be a set of graded, thin general relators with distinct representatives. If $\Delta$ is a $\mathcal{R}$-sphere which contains a cancellable pair $N$ as a subdiagram, then the cancellable pair $N$ can be removed by a process which creates a one-point product of labeled edges and $\mathcal{R}$-spheres called $\Delta'$ such that Type($\Delta$) $>$ Type($\Delta'$).*

**Proof:** If the image of the two open cells of $N$ and the single internal edge are removed from $\Delta$, then the remaining $\mathcal{R}$-structure is connected and simply connected, and since it can be embedded in the plane, it is also an $\mathcal{R}$-diagram, say $\Delta'$. The boundary of $\Delta'$ is by definition a cycle $W$ which can be read as a loop of winding number 0 in the boundary of some general relator $R \in \mathcal{R}$. By Lemma 7.12 there is a connected and simply connected $\mathcal{R}$-diagram $\Delta''$ with the boundary $W$, but whose 2-cells represent general relators only in $\partial R$. The $\mathcal{R}$-diagram $\Delta''$ can be embedded in the plane upside-down, so that its boundary cycle is $W^{-1}$, and attached to $\Delta'$ at a single point producing a new connected and simply connected $\mathcal{R}$-diagram with boundary cycle $WW^{-1}$. The boundary cycle is clearly a Dyck word, and the type of this $\mathcal{R}$-diagram is strictly less than that of the $\mathcal{R}$-sphere $\Delta$ since two cells representing $R$ were removed but only cells of strictly lower rank were added. Finally, Lemma 7.6 can be used to identify the boundary edges and complete the proof. □

**Lemma 7.14** *Let $\mathcal{R}$ be a set of graded, thin general relators with distinct representatives. If $\Delta$ is a one-point product of a finite number of labeled edges and $\mathcal{R}$-spheres, then the process of repeatedly removing cancellable pairs can be used to create a one-point product of $\mathcal{R}$-spheres and labeled edges called $\Delta'$ such that Type($\Delta$) $\geq$ Type($\Delta'$) and $\Delta'$ contains no cancellable pairs.*

**Proof:** The proof proceeds by induction on the type of $\Delta$. If $\Delta$ has no cells then it is a finite tree, and the lemma is true. So suppose that $\Delta$ is a finite one-point product of labeled edges and $\mathcal{R}$-spheres and that the lemma has been shown for all such $\mathcal{R}$-structures of strictly lower type. If there does not exist a cancellable pair in $\Delta$ then $\Delta$ itself satisfies the lemma, so let $N$ be a cancellable pair in $\Delta$. Since the two 2-cells in $N$ share an edge, $N$ must actually be a subdiagram of one of the $\mathcal{R}$-spheres in the product which forms $\Delta$. Since the procedure used in Lemma 7.13 to remove a cancellable pair removes two 2-cells and an internal edge but no vertices, any vertices used for one-point attachments remain even after the removal of the cancellable pair. Thus it is possible to form the same one-point products with the other labeled edges and $\mathcal{R}$-spheres after the cancellable pair has been removed. Since this new product has strictly lower type, the induction hypothesis completes the proof. $\square$

**Lemma 7.15** *Let $\mathcal{R}$ be a set of graded, thin general relators with distinct representatives. If $\Delta$ is an $\mathcal{R}$-diagram whose boundaries are linked, then there exists an equivalent $\mathcal{R}$-diagram $\Delta'$ with no cancellable pairs, and Type$(\Delta) \geq$ Type$(\Delta')$.*

**Proof:** The proof again proceeds by induction on the type of $\Delta$. If $\Delta$ contains no 2-cells then clearly $\Delta$ has no cancellable pairs, and $\Delta$ itself satisfies the lemma. So suppose that $\Delta$ is an $\mathcal{R}$-diagram with linked boundaries and that the lemma has been shown for all such $\mathcal{R}$-diagrams of strictly lower type. If there does not exist a cancellable pair in $\Delta$ then $\Delta$ itself satisfies the lemma, so let $N$ be a cancellable pair in $\Delta$.

Adding phantom cells for each of the connected components of the complement creates an $\mathcal{R}^+$-sphere $\Delta'$. The cancellable pair $N$ is a subdiagram of $\Delta'$ as well. If the cancellable pair $N$ is removed using the procedure in Lemma 7.13, then the phantom cells persist in the result. Moreover, since they are linked, they must all be contained in the same $\mathcal{R}^+$-sphere. Ignoring the other labeled edges and $\mathcal{R}^+$-spheres in the one-point product and removing the phantom cells leaves an $\mathcal{R}$-diagram with exactly the same boundaries as $\Delta$, but of strictly lower type. The induction hypothesis can now be used to complete the proof. $\square$

# 8 Cayley Categories

The primary goal of this section is to introduce the Cayley category of a general presentation, and to prove its fundamental properties. Along the way a few results are given on regular covers and automorphism groups of $\mathcal{R}$-categories. The section concludes with a pair of key examples of these concepts.

## 8.1 Automorphisms

The next several lemmas investigate the automorphism group of regular covers of Poincaré constructions, as well as those of the collapsed versions of these spaces. Of particular interest will be the collapse of the universal cover of a Poincaré construction because of its role in the later developments.

**Lemma 8.1** *Let $C$ be a regular cover of the Poincaré construction of a general presentation. There is an isomorphism between two closed cones $C/c_1$ and $C/c_2$ iff $c_1$ and $c_2$ are lifts of the same object $c$ in the Poincaré construction iff there is a deck transformation of $C$ which sends $c_1$ to $c_2$. Moreover, such automorphisms, when they exist, are unique.*

**Proof:** The first two conditions are equivalent by Lemma 6.8, while the equivalence of the last two follows from the definition of a regular cover. The uniqueness of this automorphism is guaranteed by Lemma 1.14. □

**Lemma 8.2** *If $C$ is a regular cover of the Poincaré construction of a general presentation, then the group of label-preserving automorphisms of $\overline{C}$ is isomorphic to that of $C$. Moreover, both automorphism groups are determined by their action on a single vertex.*

**Proof:** Since the 1-skeleton of $C$ is a Cayley graph by Lemma 6.12 and Cayley graphs are deterministic, Lemma 6.2 shows that the 1-skeleton of $C$ is the same as that of $\overline{C}$. In addition the construction of $\overline{C}$ is invariant of any label-preserving automorphism, so the deck transformations of $C$ must induce label-preserving automorphisms of $\overline{C}$, and since the 1-skeleton remains the same, distinct automorphisms of $C$ must remain distinct. Thus the label-preserving automorphisms of $\overline{C}$ contain at least the deck transformations of $C$ as a subgroup. Finally, since any such automorphism must send a vertex to a vertex, and an automorphism is trivially a covering map, Lemma 1.14 shows that the two groups are identical, and that their actions are determined by a single vertex. □

**Lemma 8.3** *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation, and let $C$ be the universal cover of its Poincaré construction. If $\overline{C}$ is the collapse of $C$, then the 1-skeleton of $\overline{C}$ is the same as the 1-skeleton of $C$, and thus is the Cayley graph $\mathcal{C}(G, A)$. Moreover, the automorphism group of $\overline{C}$ is transitive on its vertices and isomorphic to $G$.*

**Proof:** The result is a special case of Lemma 8.2 using the fact that the fundamental group of $C$ is trivial and the deck transformations form the group $G$. □

**Lemma 8.4** *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation and let $C$ be the universal cover of its Poincaré construction. If $\overline{C}$ is the collapse of $C$ then $\overline{C}$ is connected and simply connected.*

**Proof:** By Lemma 8.3 the 1-skeleton of $\overline{C}$ is a Cayley graph, and thus by Lemma 4.4 the 1-skeleton is connected. By Lemma 4.8 this means that $\overline{C}$ is connected. The proof that $\overline{C}$ is also simply connected will proceed by contradiction. If $\overline{C}$ is not simply connected then it has its own universal cover, say $C'$ with a covering map $h : C' \to \overline{C}$. By Lemma 1.13, there is a lift of $f : C \to \overline{C}$ to $g : C \to C'$. Let $v$ be a vertex in $\overline{C}$ and let $u = g(f^{-1}(v))$ which is well-defined since $f$ is an isomorphism on the 1-skeleton by Lemma 6.2. If $u'$ is another vertex in $C'$ such that $h(u') = h(u) = v$, then choose a path $U$ from $u$ to $u'$. Under $h$ this creates a loop at $v$, which becomes a loop $U$ at $f^{-1}(v)$, since $f$ is isomorphic on the 1-skeletons. Thus under $g$, the path in $C'$ must have started out as a loop, and $u = u'$. This shows that $h^{-1}(v)$ contains only a single vertex. Since by Lemma 1.15 the cardinality of $h^{-1}(v)$ is the order of the fundamental group of $\overline{C}$, $\overline{C}$ is simply connected. $\Box$

The following lemma shows in more detail the relationship between the universal cover of a Poincaré construction and its collapse.

**Lemma 8.5** *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation, and let $f : C \to \overline{C}$ be the $\mathcal{R}$-functor between the universal cover of the Poincaré construction of the presentation and its collapse. If $c$ is an open cone in $\overline{C}$, then the number of open cones in $f^{-1}(c)$ is the order of the group $Aut(\overline{C}/c)$ divided by the order of the normal subgroup of automorphisms $g$ such that $\phi_c g = \phi_c$. In particular, if the attaching functor $\phi_c$ is injective on vertices then the objects in $f^{-1}(c)$ are in 1 to 1 correspondence with $Aut(\overline{C}/c)$.*

**Proof:** First of all, if $c$ is a vertex or an edge, then the lemma is trivially true, so assume that $c$ is an open cone of rank at least 2. Let $c_1$ and $c_2$ be two open cones in $f^{-1}(c)$. Since they both map to the same general relator in the Poincaré construction, by Lemma 8.1 there is a unique automorphism of $C$ which takes $c_1$ to $c_2$. This induces an isomorphism between $C/c_1$ and $C/c_2$ since the automorphism is a circular functor (Lemma 6.11 and the definition of a circular functor). Under $f$ the isomorphism becomes an automorphism of the closed cone $\overline{C}/c$ with itself.

Conversely, given any automorphism $g$ of $\overline{C}/c$ let $W$ be a path read in the 1-skeleton of $\overline{C}/c$ from a vertex $v$ to $g(v)$. Under $\phi_c$ the word $W$ is also a path read in $\overline{C}$. Since by Lemma 6.2 the 1-skeleton remains unchanged under the collapsing functor $f$, it is also read in the 1-skeleton of $C$. By Lemma 8.1 there is a unique automorphism of $C$ which sends $f^{-1}(\phi_c(v))$ to $f^{-1}(\phi_C(g(v)))$. Since the automorphism of $\overline{C}/c$ induced by $W$ fixes the 1-skeleton of $\overline{C}/c$ and the open cone $c$, the corresponding images in $\overline{C}$ are also fixed. This in turn means that this automorphism of $C$ must send an open cone in $f^{-1}(c)$ such as $c_1$ to another open cone in this set. By Lemma 4.5 the automorphism of $C$ induced by $W$ is the identity iff the path $W$ forms a loop in $C$ (or $\overline{C}$ iff the word $W$ is equivalent to 1 in $G$. $\Box$

An $\mathcal{R}$-category $C$ is called proper if it is an $\mathcal{R}$-complex. By Lemma 3.4 this is true iff the characteristic functors of the slice categories are injective on vertices. A general presentation is called proper when the universal cover of its Poincaré construction is proper. Since the $\mathcal{R}$-functor from the universal cover of a Poincaré construction is an isomorphism when restricted to the 1-skeleton by Lemma 8.3, the universal cover is proper iff its collapse is as well. Thus Lemma 8.5 has the following corollary.

**Corollary 8.6** *If $G = \langle A|\mathcal{R}\rangle$ is a proper general presentation, then the universal cover of its Poincaré construction is collapsed iff there are no non-trivial automorphisms of the general relators $R \in \mathcal{R}$.*

## 8.2 Cayley Categories

A Cayley category is a collapsed, connected, simply connected $\mathcal{R}$-category whose group of label-preserving automorphisms acts transitively on its vertex set. By Lemma 8.3 and Lemma 8.4, the collapse of the universal cover of a Poincaré construction is an example of a Cayley category. The goal of the next few lemmas is to show that these are actually the only examples.

**Lemma 8.7** *Let $B$ be a Cayley category and let $v$ be a vertex of $B$. If $R$ is a general relator used in the construction of $B$ and $u$ is a vertex in $R$ then there exists a unique $\mathcal{R}$-functor from $R$ to $B$ which sends $u$ to $v$. Similarly, if $a$ is the label on one of the oriented edges of $B$ then there exists a unique oriented edge with $v$ as its initial vertex which is labeled by $a$.*

**Proof:** The existence portion of these two statements follows from the facts that they occur somewhere in $B$ and that by definition the automorphism group of $B$ acts transitively on its vertices. Since generators and general relators are connected and $B$ is collapsed, Lemma 6.3 guarantees the uniqueness of each. □

**Lemma 8.8** *Let $B$ be a Cayley category, and let $\Delta$ be a connected and simply connected $\mathcal{R}$-diagram with boundary cycle $W$. If $B$ contains all of the generators used and general relators represented in $\Delta$, then the path $W$ is always read as a loop in $B$.*

**Proof:** The proof is by induction on the number of 2-cells in $\Delta$. If $\Delta$ contains no cells, then it is a finite tree. Start by sending the base of the loop $W$ to a vertex in $B$. The partial functor into the 1-skeleton of $B$ can be extended one edge at a time. By Lemma 8.7 there is always an edge in $B$ which extends the functor while preserving the label on the edge. The completed functor shows that $W$ is read as a loop in $B$. If $\Delta$ contains exactly one cell, then the boundary of this cell can be read in the unique copy of the general relator it represents attached to the appropriate point in $B$. This partial map can then be extended to the finite trees which might be attached to the boundary of the single cell one edge at a time, exactly as before. Finally, suppose that the lemma has been shown for all $\mathcal{R}$-diagrams with fewer than $k$ cells and let $\Delta$ have exactly $k$ cells, $k > 1$. Using a cell which contains an edge in the boundary, $\Delta$ can easily be split into two subdiagrams which have strictly fewer cells and whose boundary cycles once combined yield a word which reduces to $W$ in the free group. Since by induction both of these words label loops in $B$, so does their concatenation. Moreover, since the 1-skeleton of $B$ is deterministic, the reductions in the free group necessary to produce $W$ can be accomplished in $B$ as well, showing that $W$ is read as a loop. □

**Lemma 8.9** *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation in which all of the general relators are thin, and let $B$ be a Cayley category whose content is identical with that of $G$. If $W$ is a word which is equivalent to 1 in $G$ then the path $W$ always forms a loop when read in $B$.*

**Proof:** By Lemma 7.9 there is a connected and simply connected $\mathcal{R}$-diagram whose boundary cycle is $W$, and then Lemma 8.8 completes the proof. □

**Lemma 8.10** *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation in which all of the general relators are thin. If $B_1$ and $B_2$ are Cayley categories whose content is identical with that of $G$, then they have isomorphic 1-skeletons. In particular the 1-skeleton of a Cayley category which contains generators $A$ and general relators $\mathcal{R}$ is the Cayley graph $\mathcal{C}(G, A)$ where $G$ is the group of the general presentation $G = \langle A|\mathcal{R}\rangle$.*

**Proof:** Let $B$ be either Cayley category. If a word $W$ is equivalent to 1 in $G$ then by Lemma 8.9 it is read as a loop in $B$. Conversely let $W$ be read as a loop in $B$. Since $B$ is simply connected, this loop is also contractible. Thus, by Lemma 7.11 there is a connected and simply connected $\mathcal{R}$-diagram with boundary cycle $W$, and by Lemma 7.9 $W$ is equivalent to 1 in $G$. By definition, the 1-skeleton of $B$ is deterministic and the automorphism group acts transitively on its vertices. Since by Lemma 4.8 it is also connected, Lemma 4.4 shows that it is a Cayley graph. Since it is now known that $W$ is read as a loop in $B$ iff $W$ is equivalent to 1 in $G$, the Cayley graph in question must be $\mathcal{C}(G, A)$. Since all Cayley categories with this content have $\mathcal{C}(G, A)$ as their 1-skeleton, the proof is complete. □

**Lemma 8.11** *Let $G = \langle A|\mathcal{R}\rangle$ be a general presentation, and let $B$ be a Cayley category whose content is identical with that of $G$. A path in $B$ forms a loop iff the word read by the path is equivalent to 1 in $G$. In particular, given words $X$ and $Y$, $XY^{-1}$ is equivalent to 1 in $G$ iff $X$ and $Y$ are equivalent in $G$ iff $X$ and $Y$ can be read in the Cayley category as paths which start and end at the same vertices.*

**Proof:** The first statement is a combination of Lemma 8.10 and Lemma 4.5 while the second is an immediate consequence of the first. □

**Lemma 8.12** *All Cayley categories with the same content are isomorphic. In particular Cayley categories are in 1 to 1 correspondence with general presentations and Poincaré constructions, and all Cayley categories are the collapse of the universal cover of a Poincaré construction of some general presentation.*

**Proof:** Let $B_1$ and $B_2$ be two Cayley categories which have the same content. By Lemma 8.10 there is an isomorphism between the 1-skeletons of $B_1$ and $B_2$. Suppose that there is an $\mathcal{R}$-isomorphism from the $(k-1)$-skeleton of $B_1$ to the $(k-1)$-skeleton of $B_2$. If $b$ is an object of $B_1$ of rank exactly $k$, then $b$ is a terminal object and extension of the functor to $b$ will have no effect on the other elements of the $k$-skeleton, and thus can be chosen independently. By assumption the image of the boundary of $b$ is already set in correspondence with objects and arrows in $B_2$. In particular, if a vertex in the image of $B_1/b$ is considered, there is a particular vertex in $B_2$ to which it corresponds. By Lemma 8.7 there exists an $\mathcal{R}$-functor from $B_1/b$ extending this correspondence. Since by Lemma 6.3 this agrees on the boundary of $b$ with the existing identification of the $(k-1)$-skeletons, there is a coherent extension of the functor to the object $b$. When this is done for all objects of rank $k$, an $\mathcal{R}$-functor has been produced from the $k$-skeleton of $B_1$ to that of $B_2$. Since by Lemma 8.7 the choices were forced, and since the same procedure could be applied from $B_2$ to $B_1$ with the identical correspondence, the $\mathcal{R}$-functor between the $k$-skeletons must be an isomorphism. Continuing in this way produces the desired isomorphism between $B_1$ and $B_2$. As mentioned above, the existence of a Cayley category whose content is the same as that of a given general presentation has already been shown by Lemma 8.3 and Lemma 8.4. This completes the proof. □

In [9] Lyndon and Schupp define a similar construction for traditional 2-dimensional presentations over reduced sets of relators, which they call the Cayley complex of the presentation. The construction defined in [9] is collapsed and it agrees with the definition given here of the Cayley category of a presentation in the cases where both are defined.

## 8.3 Cyclics and Dihedrals

The two examples given below are good illustrations of the constructions described in the previous sections.

**Example 1** Let $\mathbf{Z}_n = \langle a | a^n \rangle$ be the cyclic group of order $n$. The Cayley graph $\mathcal{C}(\mathbf{Z}_n, a)$ is the abstract loop $\mathrm{L}_n$ with all positively oriented edges labeled by $a$. The cone over this loop is the labeled 2-cell corresponding to the cycle $a^n$. If a single copy of this 2-cell is attached to the Cayley graph the result is the Cayley category of the presentation. If, on the other hand, $n$ distinct copies of the relator are attached to the loop which is the Cayley graph, then the resulting construction is the universal cover of the Poincaré construction of the presentation. Finally, the Poincaré construction can be obtained by identifying the vertices either of the Cayley category or of the universal cover, and then collapsing the result. The Poincaré construction consists of a single vertex, a single loop edge labeled $a$, with a single copy of the 2-cell attached.

Example 1 highlights the distinction between the universal cover of the Poincaré construction and the Cayley category of a presentation. The universal cover has many convenient topological properties which are not shared by the Cayley category. However, if the primary intent of an investigation is to prove results about the van Kampen diagrams over the presentation, then the Cayley category is a more appropriate construction to study.

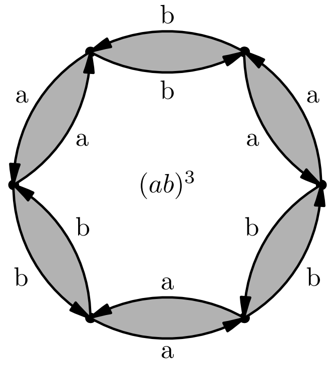


Figure 24: The dihedral group $\mathbf{D}_6$

**Example 2** Let $\mathbf{D}_{2n} = \langle a, b | a^2, b^2, (ab)^n \rangle$. The Cayley graph $\mathcal{C}(\mathbf{D}_{2n}, \{a, b\})$ consists of a vertex set $\cong \mathbf{Z}_{2n}$ with two edges between every pair of consecutive vertices. Between the vertices $2i$ and $2i+1$ there are two positively oriented edges, one starting at each vertex and ending at the other, both labeled $a$. Similarly, between $2i+1$ and $2i+2$ there are two positively oriented edges, one starting at each vertex and ending at the other, both labeled $b$. In Figure 24 the Cayley graph of $\mathbf{D}_6$ is represented. The arrows are meant to indicate that starting at the vertex 0 and reading clockwise around the outside of the circles produces the word $(ba)^3$, and that reading counter-clockwise around the inside of the circles produces the word $(ab)^3$.

Let $S$ be the Cayley graph $\mathcal{C}(\mathbf{D}_{2n}, A)$ in which between each pair of vertices a 2-cell labeled either $a^2$ or $b^2$ is attached depending on the labeled edges available. The Cayley category of the presentation is given by attaching two copies of the 2-cell labeled $(ab)^n$ onto the construction $S$ so that their edges are disjoint. Topologically, the construction is a 2-sphere. It is simply connected but not contractible. The universal cover is given by attaching $n$ distinct copies of the 2-cell labeled $(ab)^n$ to both the top and the bottom, and 2 distinct copies of the 2-cells labeled $a^2$ and $b^2$ whenever they occur.

Let $R = \mathrm{Cone}(S)$. The general presentation $\langle A | a^2, b^2, R \rangle$ is an alternative presentation of $\mathbf{D}_{2n}$. Since the relators $a^2$ and $b^2$ are contained in $R$, the Poincaré construction of this presentation is given by identifying the vertices of $R$ and collapsing the result. The universal cover of the Poincaré construction is more easily visualized as $2n$ copies of the general relator $R$ attached to a single copy of the boundary $\partial R = S$. The Cayley category is isomorphic to the general relator $R$ itself.

Since the relators in the standard presentation are simple representatives of the general relators in the general presentation, by Lemma 6.16 there is a deformation retraction of the Poincaré construction of the general presentation onto the Poincaré construction of the standard presentation. By Lemma 6.18, this deformation lifts to a deformation retraction between the universal covers.

That there is no deformation retraction between the Cayley categories is clear since in the case of the general presentation the Cayley category is contractible while in the case of the standard presentation it is not. Finally, for large $n$ the general presentation given here is a general small cancellation group as defined in Section 9.

Because 2-dimensional relators are sufficient to describe any group, geometric group theory is usually done with 2-dimensional cell categories. But by allowing more general, higher-dimensional relators, most of the desirable properties of ordinary relators are preserved, and in addition, more groups fall within the purview of small cancelation theory. In particular, general relators allow the Burnside groups for sufficiently large exponents to be described as generalized twelfth groups. Moreover, in Section 12 it will be shown that the Cayley category of a general small cancellation presentation is contractible even in cases such as the presentation of the dihedral groups as given above, where the Cayley complex of a retracted presentation of the same group is not contractible. The contractibility of this canonical geometric construction is indicative of the benefits, both geometrically and algebraically, to the researcher who uses general relators.