Part I

Cones

In this part the theory of structures iteratively built out of cones is developed. Section 1 briefly reviews the necessary topological preliminaries, before describing the construction of conical CW complexes. Sections 2 and 3 are dedicated to the special cases of these conical CW complexes which can be described by a partially ordered set or a category, respectively. These cases are the ones which will be used in the remaining parts of the article.

1 Conical CW Complexes

Traditional CW complexes are constructed inductively by attaching $(n+1)$-dimensional balls along their spherical boundaries to an already constructed n-skeleton. Conical CW complexes are a variation on this procedure in which topological cones are attached along their bases to an already constructed n-skeleton. As in the traditional case there is both an inductive construction and an internal description of the completed construction which complements the inductive version. The section concludes with some results regarding deformation retractions and coverings of such spaces.

1.1 Topological Preliminaries

The necessary results and definitions from algebraic topology are given below.

Quotient Spaces If X is a topological space, $f : X \to Y$ is a surjection and Y is a set, then there is a unique largest topology on Y for which f is continuous. This topology, called the quotient topology on Y induced by f, is defined by letting a subset $V \subset Y$ be an open set iff $f^{-1}(V)$ is open in X. When the set Y is given this topology the continuous function f is called a quotient map, and Y is called a quotient space. The word 'map' is used in this article both to refer to a continuous function between topological spaces as is done here, and to refer to a particular planar construction used in traditional small cancellation theory. The usage intended should be clear from context. Let X and Y be spaces with $A \subset X$ a closed subset. If f is a map from A to Y, then the space $Y \cup_f X$ is defined as the quotient space of the disjoint union of X and Y by the equivalence relation $\sim$ generated by the relation $a \sim f(a)$ for all $a \in A$. Lemmas 1.1, 1.2, and 1.3 are standard. Proofs can be found in reference works such as [2]. Lemma 1.1 is illustrated in Figure 1.

Lemma 1.1 *An onto map $f : X \to Y$ is a quotient map if and only if it is true that for all functions $g : Y \to Z$ (the composite $h = g \circ f$ is continuous $\Leftrightarrow$ the function g is continuous).*

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "h"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Figure 1: Quotient maps

Lemma 1.2 *Let $f : X \to Y$ be a quotient map. If Z is a space and $h : X \to Z$ is a map that is constant on each set $f^{-1}(y)$, for all $y \in Y$ then g induces a map $g : Y \to Z$ such that $g \circ f = h$.*

Lemma 1.3 *The canonical map $Y \to Y \cup_f X$ is an embedding onto a closed subspace. The canonical map $X \setminus A \to Y \cup_f X$ is an embedding onto an open subspace.*

Topological Cones A cone over a topological space X is the quotient space of $X \times [0, 1]$ obtained by identifying the subspace $X \times \{0\}$ to a point, together with the quotient map. The cone over X will be denoted $\mathrm{Cone}(X)$. The quotient map $f : X \times [0, 1] \to \mathrm{Cone}(X)$ is contained in the definition because the second coordinate provides a useful partitioning of $\mathrm{Cone}(X)$ into layers. The point $f(x, 0)$ is called the vertex of the cone, the subset $f(X \times [0, 1))$ is called the interior of the cone and the subset $f(X \times \{1\})$ is called the base or the boundary of the cone. Since by Lemma 1.3 the subset $X \times (0, 1]$ is embedded in $\mathrm{Cone}(X)$ by the map f, the base of the cone is canonically homeomorphic to X. When the original space X is viewed as identical with the base of $\mathrm{Cone}(X)$, the cone over X is seen to be an extension of X. If a topological space C is given the structure of a topological cone $\mathrm{Cone}(X)$ for some X then the base of C will be denoted ∂C. The Lemma 1.4 is a special case of Lemma 1.2.

Lemma 1.4 *Let X be a topological space, and let $f : X \times [0, 1] \to Cone(X)$ be the quotient map described above. If $g : X \times [0, 1] \to Y$ is a continuous map, and $g(x, 0) = g(x', 0)$ for all $x, x' \in X$, then there is a unique map $h : Cone(X) \to Y$ for which $h \circ f = g$.*

Lemma 1.5 *If X and Y are topological spaces and $f : X \to Y$ is a continuous map between them, then there is a canonical continuous map $g : Cone(X) \to Cone(Y)$ which is identical to f when restricted to the base of each cone. Moreover, the map g is injective iff f is injective, g is surjective iff f is surjective, and g is a homeomorphism iff f is a homeomorphism.*

Proof: Let $h_X : X \times [0, 1] \to \mathrm{Cone}(X)$ and $h_Y : Y \times [0, 1] \to \mathrm{Cone}(Y)$ be the respective quotient maps. By Lemma 1.4, there is a unique map $g : \mathrm{Cone}(X) \to \mathrm{Cone}(Y)$ such that $h_Y(f(x), t) = g(h_X(x, t))$ for all $x \in X$ and $t \in [0, 1]$. See

$$
\begin{tikzcd}
X \times [0,1] \arrow[r, "f\times \mathrm{id}"] \arrow[d, "h_X"'] & Y \times [0,1] \arrow[d, "h_Y"'] \\
\mathrm{Cone}(X) \arrow[r, "g"] & \mathrm{Cone}(Y)
\end{tikzcd}
$$

Figure 2: Extending maps over cones

Figure 2. The correspondences between the properties of f and g are straightforward and are left to the reader. $\square$

The particular cones of interest in this article are cones over compact subsets of a Euclidean space such as $\mathbf{R}^n$. If X is a compact subset of $\mathbf{R}^n$ and $\mathbf{R}^n$ is embedded in $\mathbf{R}^n \times \mathbf{R}$, then there is a standard construction of $\mathrm{Cone}(X)$ given by the set of points $(tx, 1-t)$ for $t \in [0,1]$ and $x \in X$. The notation tx is meant to indicate the scalar multiplication of the coordinates of the point x in $\mathbf{R}^n$ by the real number t. These points form a subspace of $\mathbf{R}^{n+1}$ which is homeomorphic to $\mathrm{Cone}(X)$. Indeed the function which sends (x,t) to $(tx, 1-t)$ is a function from $X \times [0,1]$ to the standard construction of $\mathrm{Cone}(X)$ which factors through the quotient and provides the homeomorphism. Alternatively, the standard construction of $\mathrm{Cone}(X)$ can be described as the set of all points $(\lambda x, \mu 1)$ with $x \in X$, $\lambda, \mu \geq 0$, and $\lambda + \mu = 1$.

Homotopy Theory and Deformation Retractions If $h : B \times [0,1] \to C$ and $f, g : B \to C$ are maps such that $f(b) = h(b,0)$ and $g(b) = h(b,1)$ for all $b \in B$ then h is called a homotopy between f and g. If A is a subspace of B and $h(a,t) = h(a,t')$ for all $a \in A$ and $t, t' \in [0,1]$ then f and g are said to be homotopic relative to A. Let C be a topological space with B a subspace of C. A deformation retraction from C to B is a homotopy between the identity map on C and a map from C to $B \subset C$ which remains constant on B during the homotopy. More specifically, a deformation retraction is a continuous function from $h : C \times [0,1] \to C$ such that $h(c,0) = c$ for all $c \in C$, $h(b,t) = b$ for all $b \in B$ and $t \in [0,1]$ and $h(c,1) \in B$ for all $c \in C$. A map $f : C \to D$ is called a homotopy equivalence iff there exists a map $g : D \to C$ such that the composition $f \circ g$ is homotopic to the identity map on D and the composition $g \circ f$ is homotopic to the identity map on C. The map f is called a weak homotopy equivalence iff the induced homomorphisms $f_i : \pi_i(C) \to \pi_i(D)$ between the ith homotopy groups are isomorphisms for all $i \geq 0$. The relative space (C, B) is called ∞-connected if the relative homotopy groups $\pi_i(C, B)$ are trivial for all i. For traditional complexes these concepts are equivalent in the following sense.

Lemma 1.6 *If C is a connected CW complex and B is a connected subcomplex, then the following conditions are equivalent:*

1) there is a deformation retraction from C to B
2) the inclusion map $i : B \to C$ is a homotopy equivalence
3) the inclusion map $i : B \to C$ is a weak homotopy equivalence
4) the relative CW complex (C, B) is ∞-connected.

Proof: The implications $1 \Rightarrow 2 \Rightarrow 3$ are well-known to be true in general topological spaces. The equivalence of 3 and 4 follows from the fact that C and B are path-connected and from the long exact sequence of homotopy groups. Finally $4 \Rightarrow 1$ is shown in [13]. □

A space C is called contractible if there is a deformation retraction of C to a point, and it is called weakly contractible if all of its homotopy groups are trivial. The next two lemmas are easy consequences of Lemma 1.6. A useful connection between homotopy theory and cones is given in Lemma 1.9.

Lemma 1.7 *A CW complex is contractible iff it is weakly contractible.*

Lemma 1.8 *If B is a subcomplex of a contractible CW complex C then B is contractible iff there is a deformation retraction of C onto B.*

Lemma 1.9 *Let $f : B \to C$ be a continuous map. The map f is homotopic to a constant map iff there is a map from Cone(B) to C whose restriction to its boundary B is the map f. As a special case, notice that a topological loop in a topological space C is homotopic to a point iff there is a map from the unit disk into C so that the restriction of this map to the unit circle is the topological loop.*

Proof: Let $h : B \times [0, 1] \to C$ be a homotopy showing that the map f is homotopic to a constant map, and let $g : B \times [0, 1] \to \text{Cone}(B)$ be the quotient in the definition of Cone(B). By Lemma 1.2 there exists a map $h' : \text{Cone}(B) \to C$ with $h = h'g$, and the map h' satisfies the lemma. Conversely, given a map $h' : \text{Cone}(B) \to C$ as described, the composition $h = h'g$ is a homotopy proving that the restriction of h' to its boundary is homotopic to a constant map. □

The following lemmas are being recorded for later use.

Lemma 1.10 *If C is a topological space, then Cone(C) is contractible.*

Proof: Let $f : C \times [0, 1] \to \text{Cone}(C)$ be the quotient map. The function $h(f(c, t), s) = f(c, ts)$ with $c \in C$, and $s, t \in [0, 1]$ provides a well-defined deformation retraction from the topological cone Cone(C) to its vertex. □

Lemma 1.11 *If there is a deformation retraction of a CW complex C onto a subcomplex B then there is a deformation retraction of Cone(C) onto Cone(B) which extends this deformation.*

Proof: Let $f : C \times [0, 1] \to \text{Cone}(C)$ be the usual quotient map, and let $h : C \times [0, 1] \to C$ be a deformation retraction of C onto B. Then the map $g : \text{Cone}(C) \times [0, 1] \to \text{Cone}(C)$ defined by $g(f(c, t), s) = (h(c, s), t)$ is a well-defined deformation retraction of Cone(C) onto Cone(B) which extends h. □

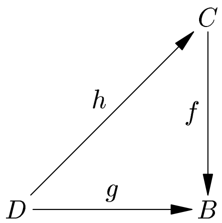

Figure 3: Lifting maps

If B is a subspace of C, then let $f : B \to B$ be the identity map from B viewed as the base of Cone(B) to B viewed as a subspace of C. By the notation $C \cup_B \mathrm{Cone}(B)$ is meant the union of the cone Cone(B) and the space C identified along the subspace B, or more specifically, the space $C \cup_f \mathrm{Cone}(B)$. Notice that $C \cup_B \mathrm{Cone}(B)$ is canonically situated as a subspace of Cone(C).

Lemma 1.12 *If B is a subcomplex of a connected CW complex C and there exists a deformation retraction from C to B, then $C \cup_B$ Cone(B) is contractible, and there is a deformation retraction from Cone(C) onto $C \cup_B$ Cone(B).*

Proof: The deformation retraction h from C to B can be combined with the identity deformation on Cone(B) which leaves every point in Cone(B) fixed. The extension is well-defined since the two deformations agree on their overlap, B. Together they yield a deformation retraction from $C \cup_B \mathrm{Cone}(B)$ to Cone(B). Since by Lemma 1.10 Cone(B) is contractible, $C \cup_B \mathrm{Cone}(B)$ is also contractible, and by Lemma 1.8 there exists a deformation retraction from the space Cone(C) to $C \cup_B \mathrm{Cone}(B)$. □

Covering Spaces An onto map $f : C \to B$ between connected CW complexes is called a covering map if for every point $b \in B$ there is a connected open set U containing b such that $f^{-1}(U)$ is a nonempty disjoint union of sets $U_i \subset C$ where f restricted to each U_i is a homeomorphism onto U for all i. If $f : C \to B$ is a covering map with $f(c_0) = b_0$ then the group homomorphism $f_1 : \pi_1(C, c_0) \to \pi_1(B, b_0)$ induced by f is an injection. The topological space B is called the base space and C is the covering space or simply a cover of B. A covering space which is simply connected is called a universal cover. Let $g : D \to B$ and $h : D \to C$ be maps between connected spaces and let $f : C \to B$ be a covering map. If $f \circ h = g$, then h is said to be a lift of g. When $g : D \to B$ is understood from context, and there exists such a map $h : D \to C$, then D is said to lift to C. See Figure 3. Lemmas 1.13 and 1.14 are special cases of the Lifting Theorem in algebraic topology.

Lemma 1.13 *Let $f : C \to B$ and $g : D \to B$ be maps between connected CW complexes, where f is a covering map. If D is simply connected and b_0, c_0, and d_0 are points in these spaces such that $f(c_0) = b_0 = g(d_0)$, then there is a lift*

$h : D \to C$ of g with $h(d_0) = c_0$. More generally, such a lift exists whenever $f_*(\pi_1(D)) = 0$.

Lemma 1.14 *Let B, C, and D be connected CW complexes, let $f : C \to B$ be a covering map, and let $g : D \to B$ be a continuous map. If h_1 and h_2 are two lifts of g and $h_1(d) = h_2(d)$ for some $d \in D$, then $h_1(d) = h_2(d)$ for all $d \in D$. In other words a lift is uniquely determined by the image of a single point.*

If $f : C \to B$ is a covering map and $g : C \to C$ is a homeomorphism of C with itself such that $f \circ g = f$, then g is called a deck transformation. The set of all deck transformations forms a group. If the group acts transitively on the set $f^{-1}(b_0)$ then C is a regular cover of B and f is a regular covering map.

Lemma 1.15 *Let B be a connected CW complex with fundamental group G. There is a 1 to 1 correspondence between the regular covers of B and the normal subgroups of G. In particular, the regular cover C of B corresponding to the normal subgroup H of G is the unique regular cover, up to homeomorphism, whose fundamental group C is isomorphic to H. Moreover, the group of deck transformations of the covering map $f : C \to B$ is isomorphic to G/H.*

The key fact is that such a regular cover exists for every normal subgroup of G. To illustrate the lemma, notice that the regular cover corresponding to group G is B itself, and the regular cover corresponding to the trivial subgroup is the universal cover of B.

1.2 Inductive Construction

To establish notation, let $\mathbf{N}$ denote the set of natural numbers $0, 1, 2, \ldots$, and let the symbol $\sqcup$ indicate a disjoint union. A set I which is the disjoint union of sets $I_n, n \in \mathbf{N}$ will be called a graded index set. Equivalently, a graded index set can be defined by means of a rank function $r : I \to \mathbf{N}$, with $r(i) = n \iff i \in I_n$. In this case n is called the rank of the element i. If $J = \sqcup_{n \in \mathbf{N}} J_n$ is another graded index set and $f : I \to J$ is a function which preserves the index, meaning that $f(I_n) \subset J_n$ for all $n \in \mathbf{N}$, then f is called a graded index function.

A conical CW complex C is constructed inductively from spaces C_i and maps $\partial\phi_i$ indexed by some graded index set I subject to certain restrictions on the spaces and the maps. The restrictions on the spaces C_i are as follows: the space C_i must be a single point for all i of rank 0, while for all other $i \in I$, C_i must be the topological cone over some nonempty compact subset of a Euclidean space. The base, or boundary, of the cone is called ∂C_i so that $C_i = \text{Cone}(\partial C_i)$ for all $i \in I_n$ with $n \geq 1$. Note that the spaces ∂C_i need not be connected. For convenience, this notation is extended to I_0 by defining the unique point in C_i to be its vertex, defining $\partial C_i = \emptyset$ for each $i \in I_0$, and setting $\text{Cone}(\emptyset) = C_i$, even though technically a point is not a topological cone over the empty set. For all $i \in I_n$, the space C_i is called an n-cone or a cone of rank n, ∂C_i is called its base, and $C_i \setminus \partial C_i$ is called its interior. Again, the terminology is used even in the case $i \in I_0$ although the 0-cones are not technically cones at all. These conventions provide a coherent notation throughout the induction. Finally, for each $n \in \mathbf{N}$ let $X_n = \sqcup_{i \in I_n} C_i$ and $\partial X_n = \sqcup_{i \in I_n} \partial C_i$. Furthermore let $X = \sqcup_{n \in \mathbf{N}} X_n$ and $\partial X = \sqcup_{n \in \mathbf{N}} \partial X_n$.

The construction of a conical CW complex consists of inductively defining spaces $C^{(n)}$, called the n-skeletons of C, with $C^{(n)} \subset C^{(n+1)}$ for all $n \in \mathbf{N}$ and $C = \cup_{n \in \mathbf{N}} C^{(n)}$. The construction begins by defining $C^{(0)} = X_0$. The space $C^{(0)}$ is a discrete set of points in 1 to 1 correspondence with the set I_0 which is given the discrete topology. Next assume that $C^{(n-1)}$ has already been defined for some $n \geq 1$. The restriction on the maps $\partial \phi_i$ mentioned earlier is that for each $i \in I_n$, $\partial \phi_i$ must be a map from ∂C_i to the $(n-1)$-skeleton. If $\{\partial \phi_i : \partial C_i \to C^{(n-1)} | i \in I_n\}$ is a set of such maps, then piecing these together forms a map $\partial \phi_n : \partial X_n \to C^{(n-1)}$ which agrees with $\partial \phi_i$ when ∂X_n is restricted to ∂C_i. The n-skeleton can then be defined as $C^{(n)} = C^{(n-1)} \cup_{\partial \phi_n} X_n$. By Lemma 1.3, $C^{(n-1)}$ is embedded in $C^{(n)}$ as a closed subspace and there is a canonical map $\phi_n : X_n \to C^{(n)}$ which is a homeomorphism when restricted to $X_n \setminus \partial X_n$. More specifically, if the restriction of ϕ_n to the space C_i is called ϕ_i, then the map ϕ_i agrees with $\partial \phi_i$ on ∂C_i and by Lemma 1.3 ϕ_i is a homeomorphism when restricted to $C_i \setminus \partial C_i$. The map ϕ_i is called the characteristic map of the cone C_i and the map $\partial \phi_i$ is called the attaching map of the cone C_i. If C_i is an n-cone then the homeomorphic sets $C_i \setminus \partial C_i$ and $\phi_i(C_i \setminus \partial C_i)$ are called open n-cones. By Lemma 1.3 $\phi_i(C_i \setminus \partial C_i)$ is an open subset of the n-skeleton $C^{(n)}$.

Once the n-skeletons $C^{(n)}$ have been constructed for all n, the space C is defined to be the union of these nested spaces, $C = \cup C^{(n)}$, under the weak topology, the one which specifies that a subset U is open (or equivalently closed) in C iff $U \cap C^{(n)}$ is open (closed) in $C^{(n)}$ for all $n \in \mathbf{N}$. When convenient the characteristic maps ϕ_i and the attaching maps $\partial \phi_i$ will be considered as maps into C instead of into one of the skeletons. Maps $\phi : X \to C$ and $\partial \phi : \partial X \to C$ can be defined by piecing together the maps ϕ_i and $\partial \phi_i$ with $i \in I$, respectively. As is the case with CW complexes, an open cone of height n is only guaranteed to be an open subset of the n-skeleton $C^{(n)}$, and is not necessarily an open subset of the final complex C.

Lemma 1.16 *If C is a conical CW complex then, using the notations established above, C is the disjoint union of its open cones, the n-skeleton is closed in C, and the map $\phi : X \to C$ is a quotient map. Additionally, a subset V of C is open (closed) iff $\phi_i^{-1}(V)$ is open (closed) for all $i \in I$.*

Proof: The fact that C is the disjoint union of open cones and the fact that the n-skeleton is closed in C follow inductively from Lemma 1.3 and the definition of the weak topology, respectively. Let V be a subset of C. The weak topology on C guarantees that V is open (closed) in C iff $V \cap C^{(n)}$ is open (closed) for all $n \in \mathbf{N}$. The set $V \cap C^{(n)}$ is open (closed) iff $V \cap C^{(n-1)}$ is open (closed) and the sets $\phi_i^{-1}(V)$ are open (closed) in C_i for all $i \in I_n$ since $C^{(n)}$ is defined by a quotient map. By an easy induction it is clear that $V \cap C^{(n)}$ is open (closed) iff $\phi_i^{-1}(V)$ is open (closed) in C_i for all $i \in \cup_{j=0}^{n} I_j$. Putting this all together, V is open (closed) in C iff $\phi_i^{-1}(V)$ is open (closed) in C_i for all $i \in I$ iff $\phi^{-1}(V)$ is open (closed) in X. Thus by virtue of the definition, ϕ is a quotient map. $\square$

As a consequence of Lemma 1.16, every point c in C can be assigned a rank based on the unique i such that c is in $\phi_i(C_i \setminus \partial C_i)$. Many of the properties and proofs for traditional CW complexes carry over unchanged to conical CW complexes. The following lemma is an example. The proof given in [2] carries over verbatim.

Lemma 1.17 *Let C be a conical CW complex with the notation defined above. If U is a subset of C which has no two points in the same open cone, then U is closed and discrete. If V is a compact subset of C then V is contained in a finite union of open cones. As a consequence, $\phi_i(C_i)$, and indeed every compact subset of C, is contained in a finite subcomplex of C.*

Just as open and closed subsets of conical CW complexes can be determined 'locally', continuous maps from conical CW complexes can also be described 'locally'. This fact is recorded in the following lemma.

Lemma 1.18 *Let C be a conical CW complex with characteristic maps $\phi_i : C_i \to C$, with $i \in I$, and let D be an arbitrary topological space. A function $f : C \to D$ is continuous iff all of the composites $f \circ \phi_i$ are continuous for all $i \in I$. More specifically, if f is a bijection such that f restricted to $\phi_i(C_i)$ is a homeomorphism, and subsets V of D are open in D iff $V \cap f(\phi_i(C_i))$ is open in $f(\phi_i(C_i))$ for all $i \in I$, then f is a homeomorphism.*

Proof: By Lemma 1.16, the map $\phi : X \to C$ is a quotient map. The first statement is then an immediate consequence of Lemma 1.1. The second assertion follows since $V \subset D$ is open iff $V \cap f(\phi_i(C_i))$ is open in $f(\phi_i(C_i))$ for all $i \in I$ iff $f^{-1}(V \cap f(\phi_i(C_i)))$ is open in $\phi_i(C_i)$ for all $i \in I$ iff $f^{-1}(V)$ is open in C by Lemma 1.16. $\Box$

1.3 Internal Description

In order to completely describe a conical CW complex it is sufficient to give the cones C_i, their bases ∂C_i, the topological space C, and the characteristic maps $\phi_i : C_i \to C$, all indexed by some graded index set I. The above list is sufficient since the attaching maps $\partial\phi_i$ and the n-skeletons $C^{(n)}$ can be reconstructed from the given information. Conversely, given a set of topological spaces C_i with subspaces ∂C_i, a topological space C, and a set of maps $\phi_i : C_i \to C$, all indexed by some graded index set I, and satisfying certain minimal conditions, it is possible to show that the space C is homeomorphic to a space arrived at following the inductive procedure described above. This static description of a conical CW complex is called an internal description to distinguish it from the constructive approach taken above. Lemma 1.18 will then be used below to show the equivalence of the constructive and the internal descriptions of conical CW complexes.

Let C_i be a collection of topological spaces with subsets ∂C_i, and let $\phi_i : C_i \to C$ be a set of maps to a topological space C, all indexed by some graded index set I. Assume that C_i is a single point and $\partial C_i = \emptyset$ for all $i \in I_0$ and that $C_i = \text{Cone}(\partial C_i)$ with ∂C_i a nonempty compact subset of some finite-dimensional Euclidean space for all other $i \in I$.

Assume also that C is the disjoint union of the sets $\phi_i(C_i \backslash \partial C_i)$, that $\phi_i(C_i) \subset \partial\phi_j(\partial C_j)$ for some $i \in I_m$ and $j \in I_n$ implies $m < n$, and that C has the weak topology with respect to the maps ϕ_i, $i \in I$. This latter condition means that a subset U is open in C iff $\phi_i^{-1}(U \cap \phi(C_i))$ is open in C_i for all $i \in I$. When all of these conditions are satisfied, the spaces $C_i, \partial C_i$, and the maps ϕ_i are said to provide C with the structure of a conical CW complex. When these spaces and maps are understood from context, C is said to have a conical CW complex structure.

Lemma 1.19 *A topological space C has a conical CW complex structure iff it is homeomorphic to a space arrived at by following the inductive construction described above.*

Proof: If C is a topological space which has a conical CW complex structure, then the spaces C_i, ∂C_i, and the maps ϕ_i restricted to the bases ∂C_i satisfy all of the conditions needed for them to be used to inductively construct a conical CW complex. Lemma 1.18 is then sufficient to show that the constructed complex is homeomorphic with the original topological space C in the obvious way. The converse is immediate from the inductive construction. $\square$

A cone map between two conical CW complexes is a map which, when composed with a characteristic map of the domain, yields a characteristic map of the range. More specifically, let B be a conical CW complex with characteristic maps $\phi_i : B_i \to B$ indexed by a graded index set I, and let C be another conical CW complex with characteristic maps $\phi_j : C_j \to C$ indexed by a graded index set J. A cone map between B and C is a map $f : B \to C$ together with a graded index function $p : I \to J$ and a set of homeomorphisms $h_i : B_i \to C_{p(i)}$ such that $f \circ \phi_i = \phi_{p(i)} \circ h_i$, for all $i \in I$. If $X = \sqcup B_i$, and $Y = \sqcup C_j$, then the maps ϕ_i, ϕ_j, and h_i can be pieced together to form maps ϕ_X, ϕ_Y and h respectively. Using this notation the cone map condition becomes $f \circ \phi_X = \phi_Y \circ h$. See Figure 4. Notice that since p is a graded index function, the homeomorphisms h_i must send cones of rank n in B to cones of rank n in C, and similarly the map f must send points of rank n in B to points of rank n in C.

Lemma 1.20 *The composition of cone maps is a cone map, and bijective cone maps are homeomorphisms.*

Proof: The first assertion is immediate from the definition, since each of the necessary conditions is well-behaved under composition. For the second assertion let $f : B \to C$ be a bijective cone map between conical CW complexes B and C. Since attaching maps are homeomorphisms when restricted to open cones, f is a bijective cone map, and the interior of every cone B_i is nonempty, the function between the graded index sets must be injective. Also, since f is onto and C is the disjoint union of the nonempty sets $C_i \setminus \partial C_i$, the function between the graded index sets must be surjective, and thus bijective. Thus,

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"'] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

Figure 4: A cone map between conical CW complexes

without loss of generality, the same set I can represent the graded index set of both B and C.

Next let X, ϕ_X, Y, ϕ_Y, and h be as defined in the inductive construction. Since the index sets are the same, the homeomorphisms $h_i : B_i \to C_i$ combine to form a single homeomorphism $h : X \to Y$. Since $\phi_X \circ h^{-1} = f^{-1} \circ \phi_Y$ as functions, ϕ_Y is a quotient map by Lemma 1.16, and $\phi_X \circ h^{-1}$ is continuous, the function f^{-1} must be continuous, by Lemma 1.1. Thus both f and f^{-1} are continuous, and f is a homeomorphism. □

1.4 Coverings of Conical CW Complexes

In the next lemma, the structure of conical CW complexes is shown to lift through covering maps. Although the proof mimics the proof of the result for traditional CW complexes, there are enough differences to warrant a complete proof.

Lemma 1.21 *Let B be a topological space which has the structure of a conical CW complex, and suppose that all of the cones B_i in B are topologically equivalent to traditional CW complexes. If $f : C \to B$ is a covering map, then f induces a conical CW complex structure on C which is canonically defined.*

Proof: Let B_i, ∂B_i, and ϕ_i be a set of topological spaces, subspaces, and characteristic maps indexed by a graded index set I which provide B with a conical CW complex structure. In particular, this means that these data satisfy the conditions listed immediately prior to Lemma 1.19. After establishing suitable notations, a graded set of spaces, subspaces, and maps for C will be defined from the given data for B, and it will be shown that the new data provide C with a conical CW complex structure.

For every $i \in I$ let v_i be the point in B which is the image under ϕ_i of the vertex of B_i. Next let V_B be the set of all points v_i for $i \in I$ and let $V_C = f^{-1}(V_B)$. By Lemma 1.17 the set V_B is a closed and discrete subset of B since it contains exactly one point in every open cone. Notice that the set V_C is also closed since f is continuous, and the set is discrete since f is a covering map. Let J be a set in 1 to 1 correspondence with V_C, and let v_j be the point in V_C which corresponds to $j \in J$. For every $j \in J$ there is a unique $i \in I$ with

$f(u_j) \in \phi_i(B_i \setminus \partial B_i)$, since B is the disjoint union of the images of the interiors of the cones B_i. This serves to define a function $p : J \to I$, and if J_n is defined as $p^{-1}(I_n)$ then J becomes a graded index set and p becomes a graded function. If for a particular $j \in J$, $p(j) = i$, then define $C_j = B_i$, define $\partial C_j = \partial B_i$, and define $\phi_j : C_j \to C$ by lifting $\phi_i : B_i \to B$ through f to C in such a way that the image of the vertex of C_j is v_j. Since by Lemma 1.10 B_i is contractible and B_i has the topology of a traditional CW complex by assumption, such a lift is always possible by Lemma 1.13, and by Lemma 1.14 it is unique.

The spaces C_j, the subspaces ∂C_j, and the maps ϕ_j indexed by the graded set J will provide C with a conical CW complex structure. First, notice that the spaces C_j are either points or cones, and the subspaces ∂C_j are either empty or compact subspaces of finite-dimensional Euclidean spaces as appropriate, since each C_j and each ∂C_j is equal to a space B_i or ∂B_i, and the corresponding spaces have the same rank. Next, if there are indices $j \in J_n$ and $j' \in J_m$ such that $\partial\phi_j(C_j) \cap \phi_{j'}(C_{j'}) \neq \emptyset$, then letting $i = p(j) \in I_n$ and $i' = p(j') \in I_m$, and recalling that $f \circ \phi_j = \phi_{p(j)}$, it is also true that $\partial\phi_i(\partial B_i) \cap \phi_{i'}(B_{i'}) \neq \emptyset$. Thus $n > m$, and consequently the attaching map sends the boundary of the cone C_i to the $(n-1)$-skeleton.

Let $c \in C$ be an arbitrary point, and let i be the unique index such that $f(c) \in \phi_i(B_i \setminus \partial B_i)$. Next let b be a point in $B_i \setminus \partial B_i$ such that $\phi_i(b) = f(c)$. Using the contraction of B_i to its vertex, there is a path in $B_i \setminus \partial B_i$ from b to the vertex. When this path is sent to B by ϕ_i, it becomes a path from $f(c)$ to v_i, and the unique lift of ϕ_i which lifts $f(c)$ to c must lift this path to a path in C which starts at c and ends at a lift of v_i. The lift of v_i must be v_j for some j by definition, and this shows that c is in $\phi_j(C_j \setminus \partial C_j)$ for this j. Since lifts are unique once a single point has been lifted, the j described above is the unique $j \in J$ for which the condition is true, and this shows that C is the disjoint union of the images of the interiors of the cones C_j as required. In symbols, $C = \sqcup_{j\in J}\phi_j(C_j \setminus \partial C_j)$.

It only remains to show that the covering space C has the weak topology with respect to the maps ϕ_j. If U is an open subset of C it is clear that $\phi_j^{-1}(U)$ is open in C_j since each ϕ_j is continuous. So assume instead that $\phi_j^{-1}(U)$ is open in C_j for all $j \in J$, and for the moment, assume that U is contained in an open subset of C which is mapped homeomorphically onto its image under f. The set $\phi_i^{-1}(f(U)) = \cap\{\phi_j^{-1}(U) | p(j) = i\}$ for each $i \in I$ since, if $b \in \phi_i^{-1}(f(U))$ then $\phi_i(b) = f(u)$ for some $u \in U$, there is a lift of ϕ_i to a map ϕ_j for some $j \in J$ which extends the lift of $\phi_i(b)$ to u, and thus b is in $\phi_j^{-1}(U)$ for this j. Conversely, if b is in $\phi_j^{-1}(U)$ then $\phi_j(b) = u$ is in U, $f(\phi_j(b)) = \phi_{p(j)}(b) = \phi_i(b)$ is in $f(U)$, and b is in $\phi_i^{-1}(f(U))$. Since by assumption the sets $\phi_j^{-1}(U)$ are open for all $j \in J$, and each set $\phi_i^{-1}(f(U))$ is the union of such sets, the sets $\phi_i^{-1}(f(U))$ must be open sets for all $i \in I$. By Lemma 1.16 and the fact that B has the structure of a conical CW complex, the set $f(U)$ is open in B. Since by assumption U is contained in an open subset which is mapped homeomorphically onto its image under f, U is open in C.

Let U' be an arbitrary subset of C with $\phi_j^{-1}(U')$ open for all $j \in J$, let u

be a point in U', and let U be the intersection of U' with an open set of C containing U which is mapped homeomorphically onto its image by f. Such a subset exists since f is a covering map. Since $\phi_j^{-1}(U)$ is the intersection of $\phi_j^{-1}(U')$ with an open set for each $j \in J$, U is a set satisfying the conditions described above, and thus U is an open subset of C. The union of all such sets U, one for each $u \in U'$, shows that U' is also an open set. $\Box$

Lemma 1.22 *Let B be a topological space which has the structure of a conical CW complex, and suppose that all of the cones B_i in B are topologically equivalent to traditional CW complexes. If $f : C \to B$ is a covering map, and C is given the conical CW structure induced by B through f, then f is a cone map. In particular, all of the deck transformations of C relative to f are also cone maps.*

Proof: Since in the proof of Lemma 1.21 the characteristic maps of C were created by lifting the characteristic maps of B through f, it is immediate that the composition of a characteristic map of C with f yields a characteristic map of B. Thus f is a cone map between the conical CW complexes C and B. Since all deck transformations are trivial covering maps of C in their own right, they are cone maps by the above reasoning. $\Box$

2 Cone Complexes

Under certain conditions, all of the information needed to construct a conical CW complex can be summarized in a single algebraic structure. In particular, any partially ordered set in which all principal ideals are finite can be used to construct a unique conical CW complex directly from the poset. Moreover, simplicial complexes and cell complexes can be viewed as special cases of this construction. At the end of the section it is shown that all of the conical CW complexes created in this way are in fact polyhedra.

2.1 Posets and Cone Complexes

A partially ordered set, or poset for short, is a set P with a reflexive, antisymmetric, and transitive binary operation $\leq$. If P is a poset with $p \in P$, then define $[p, q] = \{r \in P | p \leq r \leq q\}$, and $(p, q) = \{r \in P | p < r < q\}$, etc. A principal ideal P/p is defined as $P/p = (-, p] = \{q \in P | q \leq p\}$. Principal ideals are denoted P/p because of their close connections with slice categories as described below. An ideal Q of P is a subset of P such that $p \in Q$ implies $P/p \subset Q$. The height of an element p is given by $\text{height}(p) = \sup\{n | \exists p_0 < p_1 < p_2 \ldots < p_n = p\}$. The elements $p_0 < p_1 < p_2 \ldots < p_n$ are called a chain of length n, and a subset of the p_i's form a subchain. Notice that according to this definition minimal elements have height 0, not 1. The height of an ideal, principal ideal or poset P is the supremum of the heights of its elements.

A poset C in which all of the principal ideals are finite is called a cone complex. The fact that the principal ideals are finite guarantees that every

$$\begin{tikzcd}
B/b \arrow[r, "h_b"] \arrow[d, "\phi_b"'] & C/f(b) \arrow[d, "\phi_{f(b)}"'] \\
B \arrow[r, "f"] & C
\end{tikzcd}$$

Figure 5: A morphism between cone complexes

element in C has a well-defined and finite height. In particular, every cone complex C can be assumed to be equipped with a rank function r from the elements of C to $\mathbf{N}$ such that $r(p) = 0 \Leftrightarrow p$ is a minimal element, and $p > q \Rightarrow r(p) > r(q)$ for all p, q in C without loss of generality, since the height function shows that such a function always exists. The elements of a cone complex are called open cones, the principal ideals are called closed cones, if $b < c$ then b is called a face of c, and the poset of all faces of c, $\partial c = \{b \in C | b < c\}$, is called the boundary of c. Notice that the boundary of c is the difference between the closed cone c and the open cone c.

The rank of a cone complex or of an ideal of a cone complex is defined as the supremum of the ranks of its elements. A cone of rank n is called an n-cone, and the ideal of all i-cones with $i \leq n$ is called the n-skeleton. The 0-skeleton is also called the vertex set and the 0-cones are called vertices. Notice that the vertices are the only elements which are both open and closed and consequently have no boundary. A subcomplex of a cone complex is any ideal of the poset. A morphism between two cone complexes is an order-preserving function which is an isomorphism when restricted to corresponding closed cones. More explicitly, a morphism between cone complexes is a poset morphism $f : B \to C$ and a set of isomorphisms $h_b : B/b \to C/f(b)$ for all $b \in B$ such that the diagram in Figure 5 commutes. That is, $f \circ \phi_b = \phi_{f(b)} \circ h_b$, where ϕ_b and $\phi_{f(b)}$ represent the natural inclusions.

Lemma 2.1 *If $f : B \to C$ is a morphism between cone complexes B and C and B and C are also closed cones, then the rank of B is less than or equal to the rank of C, and the ranks are equal if and only if f is an isomorphism of cone complexes.*

Proof: Let b and c be elements of B and C respectively such that $B = B/b$ and $C = C/c$. Since posets are antisymmetric, the elements b and c are unique. Moreover, they clearly have the maximum rank in B and C, respectively, so that $\text{rank}(B) = \text{rank}(b)$ and $\text{rank}(C) = \text{rank}(c)$. Next, since f is an isomorphism whenever the domain and range are restricted to corresponding closed cones and since B itself is a closed cone, $f(B)$ is an isomorphic copy of B situated as a subcomplex of C. It follows that, $\text{rank}(B) = \text{rank}(b) = \text{rank}(f(b)) \leq$

rank(c) = rank(C). If, in addition, rank$(f(b))$ = rank(c) then $f(b) = c$ since c is the unique element of maximum rank in C. Thus $B = B/b$ is isomorphic to $C/f(b) = C/c = C$ since f is a morphism between cone complexes. Conversely, if f is an isomorphism then B and C clearly have the same rank. $\square$

Let C be a cone complex and let I be its graded index set. More specifically, let c_i be the element of C which corresponds to $i \in I$, and let the rank of c_i equal the rank of i. Over the course of the next several lemmas a conical CW complex corresponding to C will be defined and will be shown to be unique. Initially, the resemblance will be assumed to be only slight. A geometric realization of the cone complex C is a conical CW complex B with the same index set I, such that the characteristic map ϕ_i is a homeomorphism, and $\partial\phi_i(\partial B_i) = \cup\{\phi_j(B_j)|c_j \in \partial c_i\}$ for all $i \in I$. If B is a geometric realization of the cone complex C, then the space $\phi_i(B_i \setminus \partial B_i)$ in B will be called the topological open cone c_i to distinguish it from the open cone c_i which is an element in the poset C. Once the equivalence of the two notions has been shown, the distinction will be less important.

Lemma 2.2 *If B is a geometric realization of a cone complex C, then the union of a set of topological open cones forms a closed subspace of B iff the set of corresponding elements in C forms an ideal.*

Proof: Let D be an ideal of C, and let B_i be one of the cones of B. The image of B_i is the subspace C/c_i. Since the intersection of the ideals D and C/c_i is a finite ideal, the intersection of the subspaces D and C/c_i is the finite union of open cones in B. Because $D \cap C/c_i$ is an ideal, $\partial\phi_j(\partial B_j)$ is contained whenever $\phi_j(B_j \setminus \partial B_j)$ is contained, so that $D \cap C/c_i$ is the finite union of sets of the form $\phi_j(B_j)$. Thus $D \cap C/c_i$ is closed, $\phi_i^{-1}(D \cap C/c_i) = \phi_i^{-1}(D)$ is closed for all $i \in I$, and by Lemma 1.16 D is closed in B.

Conversely, let D be the union of a set of open cones in B and assume that D is closed. By Lemma 1.16, the sets $\phi_i^{-1}(D)$ are closed for all $i \in I$. Since every point in the boundary of a topological cone such as B_i is the limit of points in its interior, any closed set which contains the interior $B_i \setminus \partial B_i$ must contain the boundary ∂B_i as well. Thus if D contains the topological open cone c_i it must also contain all of the topological open cones c_j where $c_j < c_i$ in C. This implies that the subset of C corresponding to the space D is an ideal. $\square$

If B is a geometric realization of a cone complex C and D is an ideal of C, then the union of the topological open cones in B corresponding to the elements of D is called the subspace D. Thus, it is possible to speak of the subspace C/c or the subspace ∂c of B for each element $c \in C$.

Lemma 2.3 *Every cone complex has a geometric realization as a conical CW complex.*

Proof: Let C be an arbitrary cone complex and let I be the graded index set determined by the rank function of C. A conical CW complex B which is a geometric realization of C will be constructed inductively. Let $B^{(0)}$ be a discrete set of points in 1 to 1 correspondence with the elements in I_0, let B_i be a single point, let ∂B_i be the empty set, and the let $\phi_i : B_i \to B^{(0)}$ be the map which sends the point in B_i to the point in $B^{(0)}$ corresponding to the element i, for all $i \in I_0$. Notice that $B^{(0)}$ is a geometric realization of the 0-skeleton of C. Next suppose that $B^{(n)}$ has already been constructed and is a geometric realization of the n-skeleton of C. For every element $c_i \in C$ of rank $n+1$, the fact that $B^{(n)}$ is a geometric realization of the n-skeleton means that by Lemma 2.2 there is a well-defined closed subspace ∂c_i in $B^{(n)}$. Since ϕ_i is supposed to be a homeomorphism, the space ∂B_i can be defined to be this closed subspace with $B_i = \text{Cone}(\partial B_i)$ and $\partial \phi_i : \partial B_i \to B^{(n)}$ the natural inclusion map. These attaching maps are sufficient to construct the $(n+1)$-skeleton, and more generally, the spaces B_i and the attaching maps $\partial \phi_i : \partial B_i \to B^{(n)}$ indexed by the set I completely determine the inductive construction of a conical CW complex B which satisfies the lemma. $\Box$

For every cone complex C there is a standard geometric realization of C which can be described as a subspace of a real vector space whose basis is in 1 to 1 correspondence with the elements of C. Notice that even though the result may not be finite-dimensional, every finitely generated subspace is a Euclidean space. Let v_c be the basis element which corresponds with $c \in C$. The construction proceeds by induction on the skeleta of C. To begin the minimal elements $c \in C$ are placed at the points v_c and this is trivially a geometric realization of the 0-skeleton. Next, suppose that the k-skeleton has been realized geometrically. For every element c of rank $k+1$ in C a new vertex v_c is added, and then the straight line segments connecting v_c with each of the points in the subspace ∂c are added. The points added by the construction of this cone, namely, the point v_c together with the interiors of all of the line segments, form the subset which can be called the topological open cone c since a routine check shows that this construction satisfies the necessary conditions, and that the result is a geometric realization of the $k+1$-skeleton. Continuing in this way yields a geometric realization of the cone complex C in which every space C/c is a standard cone over its base in the Euclidean space spanned by the basis elements corresponding to the elements of C/c. The next lemma shows that every geometric realization of a cone complex is homeomorphic to the standard realization.

Lemma 2.4 *The geometric realization of a cone complex as a conical CW complex is unique up to homeomorphism. In particular, if B and B' are both geometric realizations of a cone complex C, then there is a homeomorphism $f : B \to B'$ which is also a cone map which sends the topological open cone c in B to the topological open cone c in B' for all $c \in C$.*

Proof: The proof is by induction through the skeleta. Any two realizations of a cone complex of rank 0 consist solely of isolated points and these realizations clearly possess the requisite homeomorphism. Next suppose that every cone complex of rank at most n satisfies the lemma, and let C be a cone complex of rank at least $n+1$. If B and B' are two geometric realizations of C, then by induction there is a homeomorphism between their n-skeletons which satisfies the lemma. If c_i is an element of C of rank $n+1$, then the homeomorphism of the n-skeletons restricts to a homeomorphism between the subspaces of B and B' called ∂c_i. Since the characteristic maps g_i are embeddings, this induces a homeomorphism between the subspaces ∂B_i and $\partial B_i'$. By Lemma 1.5 there is a canonical homeomorphism h_i between B_i and B_i' which extends the homeomorphism between the bases. If $X = \sqcup\{B_i | i \in I_{n+1}\}$ and $Y = \sqcup\{B_i' | i \in I_{n+1}\}$, then piecing together the homeomorphisms h_i and the homeomorphism between the n-skeletons yields a homeomorphism $h : B^{(n)} \sqcup X \to B'^{(n)} \sqcup Y$. Since the composition of h with the quotient map from $B'^{(n)} \sqcup Y$ to $B'^{(n+1)}$ is continuous, by Lemma 1.2 there is a continuous map from $B^{(n+1)}$ to $B'^{(n+1)}$. Similarly, composing h^{-1} with the quotient map from $B^{(n)} \sqcup X$ to $B^{(n+1)}$ induces a continuous map in the other direction. It is easy to check that these two continuous maps are inverses of each other, so that they provide the homeomorphism described in the lemma. Thus, by induction, the n-skeleton of B is homeomorphic with the n-skeleton of B' for all n. Since by construction these homeomorphisms agree on their overlaps the union of the homeomorphisms is a bijective function between B and B'. By Lemma 1.16 this function is a homeomorphism. □

Because of the strong correlation between the algebraic properties of a cone complex and the topological properties of its geometric realization, the two constructions will henceforth be denoted by the same letter. Thus the geometric realization of a cone complex C will be called simply the space C.

Lemma 2.5 *If $f : B \to C$ is a morphism between cone complexes B and C, then there is a cone map from the space B to the space C such that the map restricted to an open cone b is a homeomorphism onto the open cone $f(b)$, for all $b \in B$. Moreover, the cone map corresponding to an isomorphism of cone complexes is a homeomorphism between conical CW complexes.*

Proof: The lemma is true for the standard realizations of B and C since the unique $\mathbf{R}$-linear map from $\oplus_{b\in B}\mathbf{R}$ to $\oplus_{c\in C}\mathbf{R}$ which sends the basis element v_b to the basis element $v_{f(b)}$ restricts to a cone map from the standard realization of B to the standard realization of C, which satisfies the conditions of the lemma. The details of the verification are left to the reader. If B and C are given arbitrary geometric realizations, then there are homeomorphisms, which are also cone maps, between these other geometric realizations and the standard realization by Lemma 2.4. The composition of these homeomorphic cone maps with the cone map between the standard realizations described above provides the desired cone map. The final statement follows from Lemma 1.20. □

2.2 Simplicial Complexes and Cell Complexes

A simplicial complex can be completely reconstructed from its poset of faces. A description of simplicial complexes in terms of posets will be given below. This description is equivalent to the usual definition in content, as will be shown in

Lemma 2.7. Besides providing a familiar example of a cone complex, simplicial complexes and their properties, such as the Simplicial Approximation Theorem, will be needed at various times throughout the article.

Let V be a set and let P be the poset of all non-trivial finite subsets of V ordered by subset. The elements of V can be identified with the minimal elements of P. If C is an ideal of P which contains all of V, then C is called a simplicial complex over V. Since every principal ideal of C is finite, C is also a cone complex. Without loss of generality assume that C comes equipped with a rank function $r : C \to \mathbf{N}$ such that r is order-preserving, and $r(p) = 0 \Leftrightarrow p$ is a minimal element in C. This is possible since the height function shows that such a function always exists. Once a simplicial complex C is viewed as a cone complex, the notions of face, boundary, rank of an element, rank of an ideal, n-skeleton, vertex set, vertices, subcomplex and geometric realization are already defined for C. Some terminology becomes more specialized. For instance, an open cone of a simplicial complex is called an open simplex, a closed cone is called a closed simplex, a cone of rank n is called an n-simplex, and a geometric open cone is called a geometric open simplex.

Notice that morphisms between simplicial complexes, as defined here, must preserve the rank of the simplices. Such a definition is much more restrictive than the usual notion of a simplicial map. Finally, notice that the principal ideals of C are finite Boolean lattices minus the 0 element and that each principal ideal is uniquely determined by the vertices it contains. Conversely, any cone complex whose principal ideals satisfy these two conditions is a simplicial complex.

Lemma 2.6 *A cone complex is a simplicial complex iff all principal ideals are closed simplices and every principal ideal is uniquely determined by the vertices it contains.*

Proof: The forward direction is immediate from the definition, so let C be a cone complex satisfying the two conditions above, let V be the set of minimal elements of C, and let P be the poset of all non-trivial finite subsets of V ordered by subset. Because every principal ideal in C is uniquely determined by its minimal elements, the order-preserving morphism from C to P which sends every element to the set of vertices of its principal ideal is injective. Thus C can be considered a substructure of P. Since, in addition, the principal ideals of C are closed simplices, the image of C must actually be an ideal of P. By definition, C is a simplicial complex. □

By Lemma 2.3 every simplicial complex C has a geometric realization. The topological space C is called a polyhedron, and more generally, a polyhedron is any topological space which is homeomorphic to the space of a simplicial complex. This discussion of simplicial complexes concludes with a sketch of a proof that simplicial complexes as defined here correspond with the traditional definition.

Lemma 2.7 *The geometric realization of a simplicial complex as defined here is homeomorphic to the traditional definition of a simplicial complex.*

Proof: (Sketch) If the construction of the standard realization is modified so that the points corresponding to the elements c of rank 1 or more are placed at the average of the coordinates of the vertices in the subspace ∂c, then it can be shown that, because of the special restrictions placed on simplicial complexes, the conditions necessary for the construction to qualify as a geometric realization are still true, and in addition the geometric closed cone c is the convex hull of the vertices it contains. Since these vertices are in general position, the convex hull is the traditional notion of a geometric closed n-simplex. By Lemma 2.4 this geometric construction of C is the geometric realization of C up to homeomorphism, and it corresponds to the traditional definition of a geometric simplicial complex built out of geometric n-simplices. $\square$

A cell complex is a type of cone complex which is more flexible than a simplicial complex and for that reason it is frequently used in topology as a means of efficiently constructing piecewise linear manifolds. A geometric closed cell is the convex hull of a finite set of points in a Euclidean space $\mathbf{R}^n$. The interior of a closed cell is called an open cell, and the boundary of a cell is the difference between the two. It can be shown that every open (closed) cell is homeomorphic to an open (closed) n-dimensional ball for some unique n called the dimension of the cell. An n-dimensional cell is called an n-cell. If c is the interior of the convex hull of a set of points and b is the interior of the convex hull of a subset of these points which is wholly contained in the boundary of c, then b is called a face of c, and we write $b < c$. A cell complex C is a collection of disjoint open cells, c, in a generalized Euclidean space, such that if $c \in C$ and $b < c$ then $b \in C$, and if $b, c \in C$ then $b \cap c$ is a face of both b and c. The n-skeleton is the cellular subcomplex consisting of the cells with dimension less than or equal to n. If c is an open cell of dimension $n+1$, then let ∂c be the set of all faces of the cell c. The set ∂c is always finite and, if c is an n-cell, the union of the open cells in ∂c is homeomorphic to an $(n-1)$-sphere.

Lemma 2.8 *Geometric cell complexes are recoverable up to homeomorphism from their poset of open cells ordered by faces.*

Proof: Once it is realized that the convex hull of a set of points in $\mathbf{R}^n$ can be viewed as a cone over the boundary of the closed cell with the vertex of the cone located at an arbitrary point in the interior, the original geometric cell complex is seen to be a geometric realization of the poset of its faces viewed as a cone complex. By Lemma 2.4, this is the unique geometric realization of the cone complex, and the proof is complete. $\square$

2.3 Quillen's Construction on Posets

A general construction for turning an arbitrary poset into a geometric object was described by D. Quillen in [12]. Starting from a poset P, a new poset called $\mathrm{Chain}(P)$ is constructed whose elements are the nonempty finite chains of elements in P with one chain being less in the ordering than another chain iff the former is a subchain of the latter. The resulting poset is a simplicial complex which can be realized geometrically either in the traditional manner or by the cone construction of Lemma 2.3. If the poset is a cone complex, then the construction of Lemma 2.3 can be used directly on the original poset P. The relationship between the Quillen procedure and the cone construction described above is detailed in the following lemmas.

Lemma 2.9 *If P is an arbitrary poset then $C = Chain(P)$ is a simplicial complex in which the vertices of C correspond to the elements of P.*

Proof: The nonempty chains of minimal length are clearly in 1 to 1 correspondence with the elements of P. Since it is also clear that every chain in a poset is uniquely determined by its elements and that the nonempty subchains of a given chain form a Boolean lattice without the 0 element, it follows from Lemma 2.6 that C is a simplicial complex. $\square$

Lemma 2.10 *If P is a poset, $C = Chain(P)$, and P' is a poset obtained from P by forcibly adding a new maximum element called p, then $C' = Chain(P')$ is a simplicial complex containing C as a subcomplex and geometrically C' is the simplicial cone over the topological space C. The vertex of the cone is the 1-element chain p, and the additional open simplices are precisely the chains in P' which contain p as a largest element.*

Proof: Immediate. $\square$

Lemma 2.11 *If C is a cone complex and $C' = Chain(C)$ then there is a homeomorphism f between the conical CW complexes C and C' such that for every open cone c in C, $f(c)$ is the topological union of all of the open simplices in C' which contain an element c as its largest member. In this sense, the simplicial structure of C' is a refinement of the cone structure of C.*

Proof: The proof proceeds by induction on the height of the cone complex C. The lemma is trivially true for cone complexes of height 0 since the spaces involved are collections of isolated points of the same cardinality. If it is assumed that the lemma is true for all cone complexes of height at most k, then Lemma 2.10 guarantees that the homeomorphism f can be extended to the additional $(k+1)$-cones in such a way that the statement of the lemma is preserved. By induction, the lemma holds for all cone complexes of finite height. Since general cone complexes are constructed inductively from their k-skeletons, this is sufficient to prove the lemma in general. $\square$

In the particular case where P is already a simplicial complex, Chain(P) is its barycentric subdivision. In general, given a cone complex P the existence of a simplicial subdivision such as Chain(P) shows that the space P is a polyhedron.

Corollary 2.12 *The geometric realization of a cone complex is a polyhedron.*

Since the geometric realizations of simplicial complexes are a type of CW complex, all of the lemmas about CW complexes given in section 1 are applicable to the spaces of cone complexes, cell complexes, and circular complexes. Although the cone construction and the Quillen construction are homeomorphic topologically, they create distinct partitions of the space and there are advantages to each. The Quillen construction produces a traditional simplicial complex which allows for the utilization of the tremendous number of results already developed for such spaces. The cone construction on the other hand divides the topological space into skeleta in a significant way. In the case of an originally geometric simplicial or cell complex, the cone construction is able to reconstruct the original complex from its poset of faces, whereas the Quillen construction produces either a barycentric subdivision or a simplicial subdivision of the original. Of more importance in the current context is the fact that the skeleta of the space of a particular type of cone complex called a circular complex correspond nicely to algebraic properties of groups defined using these complexes. As a consequence, the partitioning of the underlying space into skeleta by the cone construction is used later in the article to efficiently organize many of the statements and proofs of the lemmas.

3 Cone Categories

The construction of the previous section can be made more flexible by changing the underlying posets to categories. The resulting structure is called a cone category. In this section cone categories are introduced and their properties investigated in a manner which parallels the previous section.

3.1 Categories and Cone Categories

Familiarity with the basic definitions of category theory, such as that of a category and a functor, is assumed. Let C be a category and let c be an object of C. The slice category, C/c, of all objects over c is given by

$$\text{Objects}(C/c) = \{\text{arrows with codomain } c\}$$

$$\text{Arrows}(C/c) = \{\text{arrows which form commuting triangles}\}$$

For example, if $f : d \to c, f' : d' \to c$, and $h : d \to d'$ are arrows in C with $hg = f$ then f and f' are objects in C/c and h is an arrow in C/c from f to f' (see Figure 6). The functor $\phi_c : C/c \to C$ which sends each object $f : d \to c$ to its domain d is called a characteristic functor of C/c on analogy with the functions used in the construction of a CW complex. The connection with CW complexes is examined in more detail below. See Lemma 3.6. Notice that the slice category C/c has the identity arrow of c as a terminal object.

An important fact in this context is that every poset can be viewed as a category by thinking of the elements of the poset as the objects of the category and imagining a unique arrow $p \to q$ iff $p \leq q$. The notion of a slice category in

$$\begin{tikzcd} d \arrow[rr, "h"] \arrow[dr, "f"'] & & d' \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

Figure 6: Commuting triangle over c

a category generalizes the notion of a principal ideal in a poset in the sense that if a poset P is viewed as a category, then the slice category P/p is isomorphic to the ideal (p) viewed as a category. The attaching map in this case is injective on both objects and arrows. Notice that the usual descriptions of posets and categories use different spatial metaphors. In particular, the ideal P/p can at the same time be described as the elements of the poset 'below' p and, when viewed as a category, as the objects of the category 'over' p.

Lemma 3.1 *A category C is the category corresponding to a poset iff C has singleton hom-sets and contains no non-trivial cycles.*

Proof: The forward direction is immediate, and the other direction is almost immediate. The condition that the hom-sets be singleton sets guarantees that C is the category of a preorder, namely the preorder given by taking the objects with $c \leq d$ iff there is an arrow from c to d. The restriction that there are no non-trivial cycles implies that the preorder is anti-symmetric. $\Box$

A useful result about categories in general is the following.

Lemma 3.2 *If $f : B \to C$ and $g : C \to B$ are functors between categories B and C such that fg is the identity functor on C, and gf is the identity functor on B, then f and g are category isomorphisms.*

Proof: Since fg is the identity functor on C, the functor f is onto, and the functor g is 1 to 1 on both objects and arrows. Similarly, since gf is the identity functor on B, g is onto and f is 1 to 1 on both objects and arrows. Thus both f and g are bijections on objects and arrows. $\Box$

A category C in which every slice category C/c is the category of a finite poset is called a cone category. The fact that the slice categories are finite posets guarantees that every object c in C has a well-defined and finite height based on the height of the poset associated with C/c. In particular, every cone category C can be assumed to be equipped with a rank function r from the objects of C to $\mathbf{N}$ such that $r(c) < r(d)$ whenever there exists a non-identity arrow from c to d, and $r(c) = 0 \Leftrightarrow c$ is not the codomain for any non-identity arrows, without loss of generality since the height function shows that such a function always exists. An object c of a cone category C is called an open cone and the slice category C/c with the characteristic functor $\phi_c : C/c \to C$ implicitly understood is called a closed cone. The full subcategory of C/c on the set of all objects other than its terminal object is called the boundary of c and it is denoted ∂c. The restriction of ϕ_c to the subcategory ∂c is called the attaching functor $\partial\phi_c : \partial c \to C$ of the closed cone C/c. The terminology is justified by the fact, shown below, that every cone category can be realized geometrically as a conical CW complex, in which the functions corresponding to the characteristic functors and the attaching functors are the characteristic maps and the attaching maps of the closed topological cones and their boundaries.

Lemma 3.3 *If C is a cone category then C contains no non-trivial cycles and in the slice category C/c the unique terminal object corresponds to the identity arrow at c.*

Proof: If C contains a non-trivial cycle of arrows and c is an object in the cycle then it is easy to check that C/c also contains a non-trivial cycle. This, however, contradicts the fact that C/c is the category of a poset by Lemma 3.1. For the second statement, simply note that closed cones in a cone complex have unique maximum elements, so that the categories of these posets possess unique terminal objects. Since every arrow in C with codomain c factors trivially through the identity arrow id_c at c, id_c is a terminal object in C/c and thus the unique terminal object. □

Every cone category C determines a graded index set I given by setting I_n in 1 to 1 correspondence with the objects of C which have rank n. Let c_i be the object of C corresponding to $i \in I$. In the same way that the poset ∂c is an ideal in the poset C/c for all c in a cone complex C, the category of the poset ∂c is a full subcategory of the category C/c for all objects c in a cone category C. The restriction of the characteristic functor ϕ_c to the category boundary ∂c, denoted $\partial\phi_c : \partial c \to C$, is called the attaching functor of the closed cone C/c. The n-skeleton is the full subcategory of C on the objects of rank less than or equal to n. A cone functor is a rank-preserving functor $f : B \to C$ between cone categories which induces an isomorphism on slices. Specifically, for every object $b \in B$ there must be a category isomorphism h_b such that the diagram in Figure 5 commutes, and in addition, the rank of b in B must equal the rank of $f(b)$ in C. Notice that the diagram has been reinterpreted in terms of categories and functors, instead of posets and order-preserving maps.

Many of these definitions are identical to the definitions in the previous section. The precise relationship between cone categories and cone complexes is shown in the following lemma.

Lemma 3.4 *A cone category is the category of a cone complex iff all the characteristic functors are injective on objects.*

Proof: The forward direction is immediate. If on the other hand all of the attaching functors are injective on vertices, then all of the hom-sets of C must be singletons since the slice category over the codomain of a non-trivial hom-set produces an attaching map which is not injective on objects. By Lemma 3.3 C contains no non-trivial cycles and thus by Lemma 3.1 C is the category of a poset. In this case the characteristic functors correspond to the embeddings of the principal ideals C/c into the poset C. Since C is a cone category and all of its slices are finite, the poset of C possesses finite principal ideals, making it a cone complex. $\square$

Lemma 3.5 *Let $f : c \to d$ be an arrow in a cone category C, and let C/f be the full subcategory of C/d corresponding to the principal ideal of f in the poset of C/d. There exists a functor $g : C/c \to C/f$ such that $\phi_c = \phi_d \circ g$. The functor g is an category isomorphism which embeds C/c as a subcategory of C/d and sends the terminal object of C/c to the object f in C/d.*

Proof: An object of C/c is an arrow in C with codomain c. Its product with f is then an arrow in C with codomain d and thus an object in C/d. Since the product as an arrow in C factors through f it is also an object in the subcategory C/f. Notice that the identity arrow of c is sent to f under this procedure. Since all of the hom-sets in C/c and C/f are singleton and the ordering of the underlying posets is preserved, there is a unique functor $g : C/c \to C/f$ which extends the function on the objects described above.

The fact that the hom-sets in C/d are singletons implies that if an arrow in C with codomain d factors through f, and thus is an object in C/f, then it must factor uniquely. Since the objects of C/f are precisely the arrows of C with codomain d which factor through f, there is a well-defined function from the objects of C/f to the objects of C/c which is the inverse of the function described above. This second function also preserves the ordering of the underlying posets, and it determines a functor from C/f to C/c. It is easy to check that the second functor is the inverse of the functor g, that g is thus a category isomorphism (by Lemma 3.2), and that $\phi_c = \phi_d \circ g$. $\square$

Recall from the section on conical CW complexes that the topological cones C_i, their bases ∂C_i, and the attaching maps $\partial\phi_i$ from ∂C_i to the appropriate skeleton over some graded index set I are sufficient to reconstruct an entire conical CW complex.

Lemma 3.6 *Every cone category C has a geometric realization as a conical CW complex.*

Proof: Let I be the graded index set determined by the rank function on the objects of the cone category C. For all $i \in I$, let C_i be the geometric realization of the poset of the slice category C/c_i, and let ∂C_i be the conical CW subcomplex of C_i which corresponds to the ideal ∂c_i. These geometric realizations C_i exist by Lemma 2.3. It only remains to describe the attaching maps $\phi_i : \partial C_i \to C^{(n-1)}$ for $i \in I_n$. If $n = 0$ then C_i is a point, ∂C_i is empty, and the attaching maps do not need to be described. So assume that $n > 1$, that the $(n-1)$-skeleton has already been constructed, and that $i \in I_n$ is fixed.

In this case the characteristic maps $\phi_j : C_j \to C^{(n-1)}$ for all $j \in I_m$ with $m < n$ have already been defined.

By Lemma 3.5 every proper principal ideal in C/c_i is isomorphic with a slice category over an element c_j of rank strictly less than n. Since this category isomorphism also provides an isomorphism between the cone complexes, it can be geometrically realized as a homeomorphic cone map between the conical CW complex C_j and the appropriate subcomplex of the conical CW complex C_i by Lemma 2.5. The composition of this homeomorphism with the already defined characteristic map on C_j provides a portion of the attaching map for C_i. If this procedure is followed for all proper principal ideals in C/c_i and the resulting portions of the attaching map are pieced together, a complete attaching map for C_i is formed. Another application of Lemma 3.5 insures that the definitions of the portions of the attaching maps derived from different principal ideals agree on their overlaps, so that the resulting map is well-defined. Since this completes the list of information necessary to create a conical CW complex, the proof is complete. □

Lemma 3.7 *Given any cone functor $f : B \to C$ between cone categories there is a cone map between geometric realizations of B and C as conical CW complexes such that the cone map restricted to the topological open cone b is a homeomorphism onto the open cone $f(b)$ for all objects $b \in B$. Moreover, the cone map corresponding to an isomorphism of cone categories is a homeomorphism between conical CW complexes.*

Proof: Let I (J) be the graded index sets for the cone category B (C), let B_i (C_j) be the geometric realization of the slice categories B/b_i (C/c_j), let X (Y) be the disjoint union of the topological open cones B_i (C_j), and let ϕ_X (ϕ_Y) be the quotient map described in Lemma 1.16. If $f(b_i) = c_j$, then by the definition of a cone functor there is a category isomorphism between the slice categories B/b_i and C/c_j and by Lemma 2.4 there is a canonical homeomorphism between the conical CW complexes B_i and C_j. These homeomorphisms combine to form a map $h : X \to Y$.

Define a map f between the geometric realizations of B and C as follows. By Lemma 1.16 each point in the geometric realization of B is contained in a unique open cone b_i, and since the characteristic map ϕ_i restricted to the interior of B_i is a homeomorphism, it lifts uniquely to B_i in X. Then map it by h and then ϕ_Y to a point in the geometric realization of C. This creates a function f such that $f \circ \phi_X = \phi_Y \circ h$ as functions when restricted to the interiors of the cones B_i in X. See Figure 4. By the construction of the attaching maps given above in Lemma 3.6, these functions are equal on the boundaries of the cones as well. Moreover, since $\phi_Y \circ h$ is continuous and ϕ_X is a quotient map, by Lemma 1.1 f is continuous. Clearly this function is a homeomorphism when f is restricted to an open cone in the domain. Finally, an isomorphism between cone categories will produce a map f which is bijective. Thus by Lemma 1.20 f will be a homeomorphism. □

Corollary 3.8 *The geometric realization of a cone category is unique up to homeomorphism.*

3.2 Simplicial Categories and Cell Categories

A simplicial category is a category in which every slice is a closed simplex. That is, for all $c \in \text{Objects}(C)$, the category C/c is isomorphic to the category of faces of an n-dimensional simplex under inclusion for some n. Note that the dimension of the simplex need not be the rank of the object c unless the rank function corresponds to the height function in this instance. An object of C is called an open simplex, and the attaching map $\phi_c : C/c \to C$ is called a closed simplex. For simplicity, C/c will be referred to as a closed simplex with the attaching functor understood. The rank of an open simplex c is given by the rank function on C, and the n-skeleton is the full subcategory on all objects with rank less than or equal to n. The boundary of c, ∂c, is the poset C/c minus its maximal element together with its attaching map. Every simplicial category has a geometric realization which is constructed inductively via its skeleta. The functors $\phi_c : C/c \to C$ are the categorical equivalents of attaching maps, which describe how to attach the boundary of a closed n-simplex into the $(n-1)$-skeleton, hence the name. A simplicial functor between simplicial categories is a rank-preserving functor which is isomorphic on slices. That is, $f : C \to B$ is a simplicial functor iff for all $c \in \text{Objects}(C)$ there is an isomorphism between C/c and $B/f(c)$, with the rank of c in C equal to the rank of $f(c)$ in B.

Simplicial categories illustrate the type of flexibility which is added by the change from poset to categories. The boundary of a simplex in a simplicial category can be collapsed in some way, and distinct simplices can have identical boundaries without being identical.

Lemma 3.9 *A category is a simplicial complex iff all slice categories are categories of closed simplices, all of the attaching maps are injective on objects and every slice is determined by the image of its initial objects.*

Proof: The forward direction is immediate from the definition. Assume that C is a category which satisfies the latter conditions. If a hom-set in C contained two or more arrows then the attaching map of the slice category over the terminal object would not be injective. Thus C has singleton hom-sets, and C is the category of a preorder. If C contained a non-trivial cycle of arrows or an infinite chain of arrows with no first arrow in the chain, then the same condition would hold in one of the slice categories, contradicting the fact they are each categories of closed simplices. Thus C is the category of a poset. Once this is known the remaining conditions become simply those listed in Lemma 2.6 as necessary and sufficient to show that a poset is a simplicial complex. □

To illustrate the differences between a simplicial complex and a simplicial category, three examples are given below. The first is a 2-dimensional simplicial complex and thus also a simplicial category, while the other two are examples of simplicial categories which are not complexes.

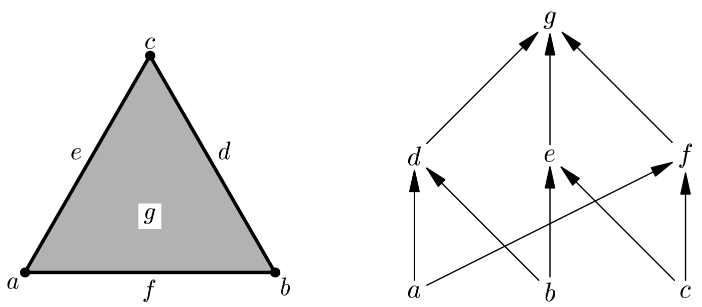

Figure 7: A 2-simplex

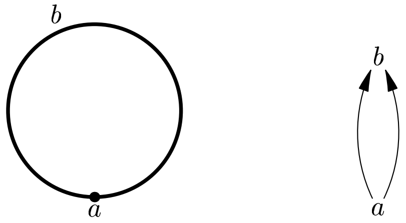

Figure 8: A loop edge

Example 1 In Figure 7 there are three vertices, three edges, and one triangle. These correspond to elements and simplices of height 0,1, and 2 respectively.

Example 2 The simplicial category in Figure 8 contains a single vertex and a loop edge, that is, an edge whose endpoints are identical. It is clearly not a simplicial complex since the category used to define it is not a poset. Notice that the geometric realization of the slice category of objects over b is a straight line segment; it is the attaching map which identifies the two endpoints.

Example 3 In the simplicial category in Figure 9, the three edges all have isomorphic slice categories and are attached to the same vertices, but the attaching maps have distinct images. This is not possible in a simplicial complex since the simplices are determined by the vertices they contain.

When the type of flexibility described above is incorporated into a cell complex the result is called a cell category, or as it is known elsewhere in the literature, a regular CW complex. A cell category is a category in which the geometric realization of every slice category is homeomorphic with a closed n-cell, for some n. That is, for all $c \in \text{Objects}(C)$, the category C/c is isomorphic to the category of faces of an n-cell under inclusion, for some n. The definitions of open cells, closed cells, cell boundaries, characteristic functors, the dimension

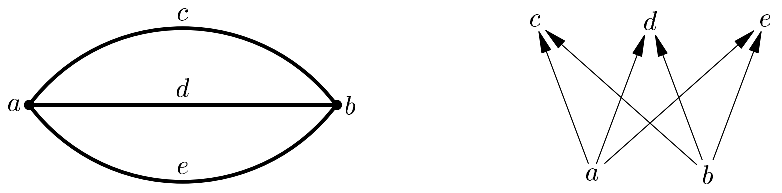

Figure 9: Multiple edges

of an object, the n-skeleton, and cell functors are analogous with those used above for simplicial categories. The Poincaré construction associated with a group presentation is a non-trivial example of a 2-dimensional cell category. It will be shown by Lemma 3.12 that every cell category can be subdivided to form a simplicial complex, and thus topologically they are polyhedra.

3.3 Quillen's Construction on Categories

Let C be a cone category with distinct objects c_i, $i = 0, 1, \ldots, n$, and arrows $f_i : c_{i-1} \to c_i$. The sequence of composable arrows

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \cdots c_{n-1} \xrightarrow{f_n} c_n$$

is called a chain of length n. A chain of length 0 is simply an object of C with no arrows. Since the objects are distinct, the arrows f_i are all non-identity arrows. Moreover, since C is a cone category and thus has no non-trivial cycle of arrows, the product of any composable subsequence of the arrows is also not an identity arrow. A subchain of a chain of arrows in C is an arbitrary subset of the objects, arranged in order, with the appropriate products of arrows from the chain placed between the objects. For any arbitrary cone category C, the poset Chain(C) is defined as the set of all finite chains ordered by subchains.

Lemma 3.10 *If C is a cone category, the category of the poset Chain(C) is a simplicial category in which the attaching functors are injective on objects. Additionally, the minimal elements of Chain(C) are in 1 to 1 correspondence with the objects of C.*

Proof: The principal ideals in the poset Chain(C) are clearly closed simplices since the subchains of a chain are indexed by the objects of the chain which it contains. When the poset is converted into a category, this shows that the resulting category is a simplicial category whose attaching functors are injective on objects. The final statement is a reformulation of the fact that a chain of length 0 is simply an object of C. □

In the original paper by Quillen[12], no restrictions are placed on the objects and arrows. In fact, such restrictions are only possible in categories such as cone categories where there are no non-trivial cycles of arrows. The difference is inconsequential topologically since the resulting constructions are homotopically equivalent, but the construction without the identity arrows is closer in spirit to the poset construction, and the result is cleaner to work with.

Lemma 3.11 *If C is a cone category and C' is the poset Chain(C) then the topological space of C is homeomorphic with that of C' and the homeomorphism carries an open cone c onto the union of the chains (open simplices) of C' which contain c as a terminal object. In this sense, C' is a simplicial subdivision of C.*

Proof: The proof proceeds by induction on the height of the cone category C. The lemma is trivially true for cone categories of height 0 since the spaces involved are collections of isolated points of the same cardinality. If it is assumed that the lemma is true for all cone categories of height at most k, then Lemma 2.10 guarantees that the homeomorphism f can be extended to the additional $(k+1)$-cones in such a way that the statement of the lemma is preserved. By induction, the lemma holds for all cone catgories of finite height. Since general cone caetgories are constructed inductively from their k-skeletons, this is sufficient to prove the lemma in general. □

Lemma 3.12 *If C is a cone category then $C'' = \mathit{Chain}(\mathit{Chain}(C))$ is a simplicial complex whose topological space is homeomorphic to that of C. Thus every cone category is a polyhedron.*

Proof: The result is a combination of Lemma 3.11 and Lemma 2.11. □

Let $f : B \to C$ be a cone functor between cone categories. Since cone functors preserve the height of the objects, the image of a sequence of arrows which form a chain in B will also form a chain in C. This defines a function called $\mathrm{Chain}(f)$ from $\mathrm{Chain}(B)$ to $\mathrm{Chain}(C)$.

Lemma 3.13 *If $f : B \to C$ is a cone functor between cone categories, then $\mathit{Chain}(f) : \mathit{Chain}(B) \to \mathit{Chain}(C)$ is a simplicial functor between simplicial categories.*

Proof: Since subchains are clearly sent to subchains under $\mathrm{Chain}(f)$, it is an order-preserving function, and since the principal ideals of $\mathrm{Chain}(B)$ and $\mathrm{Chain}(C)$ are the poset of all subchains of a given chain, the restriction of $\mathrm{Chain}(f)$ to a principal ideal is clearly an isomorphism. □

3.4 Coverings of Cone Categories

The final result in this section shows that the covers of the geometric realizations of cone categories also have the structure of a cone category. Recall that the same letter is used to denote a cone category, technically a category rather than a topological space, and also to denote its geometric realization as a conical CW complex.

Lemma 3.14 *If B is the geometric realization of a cone category, and $f : C \to B$ is a covering map between a topological space C and the conical CW complex B, then f is the geometric realization of a cone functor $f : C \to B$ between two cone categories. Similarly, all of the deck transformations of C relative to f are geometric realizations of cone functors from the cone category C to itself.*

Proof: Clearly the lemma is true if B has rank 0, so assume that the lemma is true whenever B has rank less than n for some n at least 1. By Lemma 3.12 the conical CW complex B and all of its closed cones B_i are polyhedra and thus are topologically equivalent to traditional CW complexes. Thus Lemma 1.21 can be applied giving C the structure of a conical CW complex and making f a cone map. It only remains to show that the conical CW complex C is also the geometric realization of a cone category. Since the k-skeleton of C is a cover of the k-skeleton of B for all $k < n$, the $(n-1)$-skeleton of C is the geometric realization of a cone category by the inductive hypothesis. Also, since f is a cone map, the closed cones of rank n in C are homeomorphic with closed cones in B, and thus are also geometric realizations of cone categories.

If the category underlying the $(n-1)$-skeleton of C and the categories underlying the n-cones of C are joined together according to the attaching maps the result is a new category of rank n. It is easy to check that the multiplication on this new structure is well-defined, that it is associative, and that the slice categories of the new category correspond to the categories of the closed n-cones of C. Moreover, since the attaching maps of the closed n-cones of C agree with those which would be constructed following the procedure in Lemma 3.6, the n-skeleton of C is the geometric realization of a cone category.

By induction, the k-skeleton of C is the geometric realization of a cone category for all k, and since these categories are compatible, their union is a category whose geometric realization is C. Finally, the second statement of the lemma is an immediate consequence of the first one. □